Caianiello’s Maximal Acceleration. Recent Developments

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A quantum mechanical upper limit on the value of particle accelerations is consistent with the behaviour of a class of superconductors and well known particle decay rates. It also sets limits on the mass of the Higgs boson and affects the stability of compact stars. In particular, type-I superconductors in static conditions offer an example of a dynamics in which acceleration has an upper limit.

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I. INTRODUCTION

In 1984 Caianiello gave a direct proof that, under appropriate conditions to be discussed below, Heisenberg uncertainty relations place an upper limit \( A_m = \frac{2mc^3}{\hbar} \) on the value that the acceleration can take along a particle worldline [1]. This limit, referred to as maximal acceleration (MA), is determined by the particle’s mass itself. With some modifications [2], Caianiello’s argument is the following.

If two observables \( \hat{f} \) and \( \hat{g} \) obey the commutation relation

\[
\left[ \hat{f}, \hat{g} \right] = -i\hbar \hat{\alpha},
\]

where \( \hat{\alpha} \) is a Hermitian operator, then their uncertainties

\[
(\Delta f)^2 = < \Phi | (f - < f >)^2 | \Phi > \]

\[
(\Delta g)^2 = < \Phi | (g - < g >)^2 | \Phi >
\]

also satisfy the inequality

\[
(\Delta f)^2 \cdot (\Delta g)^2 \geq \frac{\hbar^2}{4} < \Phi | \hat{\alpha} | \Phi >^2,
\]

or

\[
\Delta f \cdot \Delta g \geq \frac{\hbar}{2} | < \Phi | \hat{\alpha} | \Phi > |
\]

Using Dirac’s analogy between the classical Poisson bracket \( \{ f, g \} \) and the quantum commutator [3]

\[
\{ f, g \} \rightarrow \frac{1}{i\hbar} \left[ \hat{f}, \hat{g} \right],
\]

one can take \( \hat{\alpha} = \{ f, g \} \hat{1} \). With this substitution (I.1) then yields the usual momentum-position commutation relations. If in particular \( \hat{f} = \hat{H} \), then (I.1) becomes

\[
\left[ \hat{H}, \hat{g} \right] = -i\hbar \{ \hat{H}, \hat{g} \} \hat{1},
\]

(I.6)

(I.4) gives [3]

\[
\Delta E \cdot \Delta g \geq \frac{\hbar}{2} | \{ \hat{H}, \hat{g} \} |
\]

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\[
\Delta E \cdot \Delta g \geq \frac{\hbar}{2} \left| \frac{dg}{dt} \right|, \quad (I.8)
\]

when \(\frac{\partial g}{\partial t} = 0\). Equations (I.7) and (I.8) are re-statements of Ehrenfest’s theorem. Criteria for its validity are discussed at length in the literature [3–5]. If \(g \equiv v(t)\) is the velocity expectation value of a particle whose energy is \(E = mc^2 \gamma\) and it is assumed with Caianiello [1] that \(\Delta E \leq E\), then (I.8) gives

\[
\left| \frac{dv}{dt} \right| \leq \frac{2}{\hbar} mc^2 \gamma \Delta v(t), \quad (I.9)
\]

where \(\gamma = \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}}\). In general and with rigour

\[
\Delta v = \left( < v^2 > - < v >^2 \right)^{\frac{1}{2}} \leq v_{\text{max}} \leq c. \quad (I.10)
\]

An essential point of Caianiello’s argument is that the acceleration is largest in the rest frame of the particle. This follows from the transformations that link the three-acceleration \(\vec{a}_p\) in the instantaneous rest frame of the particle to the particle’s acceleration \(\vec{a}'\) in another frame with instantaneous velocity \(\vec{v}\) [6]

\[
\vec{a}' = \frac{1}{\gamma^2} \left[ \vec{a}_p - \frac{(1-\gamma)(\vec{v} \cdot \vec{a}_p) \vec{v}}{v^2} - \gamma (\vec{v} \cdot \vec{a}_p) \vec{v} \right]. \quad (I.11)
\]

The equation

\[
a'^2 = \frac{1}{\gamma^4} \left( a^2_p - \frac{(\vec{a}_p \cdot \vec{v})^2}{c^2} \right), \quad (I.12)
\]

where \(a^2 = \vec{a} \cdot \vec{a}\), follows from (I.11) and shows that \(a' \leq a_p\) for all \(\vec{v} \neq 0\) and that \(a' \to 0\) as \(\mid \vec{a}_p \cdot \vec{v} \mid \to a_pc\). In addition, in the instantaneous rest frame of the particle, \(E \leq mc^2\) and \(\Delta E \leq mc^2\) if negative rest energies must be avoided as nonphysical. Then (I.9) gives

\[
\left| \frac{dv}{dt} \right| \leq \frac{2mc^3}{\hbar} \equiv A_m. \quad (I.13)
\]

It is at times argued that the uncertainty relation

\[
\Delta E \cdot \Delta t \geq \frac{\hbar}{2} \quad (I.14)
\]

implies that, given a fixed average energy \(E\), a state can be constructed with arbitrarily large \(\Delta E\), contrary to Caianiello’s assumption that \(\Delta E \leq E\). This conclusion is erroneous. The correct interpretation of (I.14) is that a quantum state with spread in energy \(\Delta E\) takes a time \(\Delta t \geq \frac{\hbar}{2\Delta E}\) to evolve to a distinguishable (orthogonal) state. This evolution time has a lower bound. Margolus and Levitin have in fact shown [7] that the evolution time of a quantum system with fixed average energy \(E\) must satisfy the more stringent limit

\[
\Delta t \geq \frac{\hbar}{2E}, \quad (I.15)
\]

which determines a maximum speed of orthogonality evolution [8,9]. Obviously, both limits (I.14) and (I.15) can be achieved only for \(\Delta E = E\), while spreads \(\Delta E > E\), that would make \(\Delta t\) smaller, are precluded by (I.15). This effectively restricts \(\Delta E\) to values \(\Delta E \leq E\), as conjectured by Caianiello [10].

Known transformations now ensure that the limit (I.13) remains unchanged. It follows, in fact, that in the rest frame of the particle the absolute value of the proper acceleration is

\[
\left( \frac{d^2x^\mu}{ds^2} \frac{d^2x^\nu}{ds^2} \right)^{\frac{1}{2}} = \left( \frac{1}{c^4} \frac{d^2x^i}{dt^2} \right)^{\frac{1}{2}} \leq A_m. \quad (I.16)
\]

Equation (I.16) is a Lorentz invariant. The validity of (I.16) under Lorentz transformations is therefore assured.
The uncertainty relation (I.14) can then be used to extend (I.13) to include the average length of the acceleration \(< a >\). If, in fact, \(v(t)\) is differentiable, then fluctuations about its mean are given by

\[
\Delta v \equiv v - < v > \simeq \left( \frac{dv}{dt} \right)_0 \Delta t + \left( \frac{d^2 v}{dt^2} \right)_0 (\Delta t)^2 + \ldots
\]  

Equation (I.17) reduces to \(\Delta v \simeq \left| \frac{dv}{dt} \right| \cdot \Delta t = < a > \cdot \Delta t\) for sufficiently small values of \(\Delta t\), or when \(\left| \frac{dv}{dt} \right|\) remains constant over \(\Delta t\). The inequalities (I.14) and (I.10) then yield

\[
< a > \leq \frac{2\epsilon \Delta E}{\hbar}
\]  

and again (I.13) follows.

The notion of MA delves into a number of issues and is connected to the extended nature of particles. In fact, the inconsistency of the point particle concept for a relativistic quantum particle is discussed by Hegerfeldt [11] who shows that the localization of the particle at a given point at a given time conflicts with causality.

Classical and quantum arguments supporting the existence of MA have been frequently discussed in the literature [12–21]. MA would eliminate divergence difficulties affecting the mathematical foundations of quantum field theory [22]. It would also free black hole entropy of ultraviolet divergences [23–25]. MA plays a fundamental role in Caianiello’s geometrical formulations of quantum mechanics [26] and in the context of Weyl space [27]. A limit on the acceleration also occurs in string theory. Here the upper limit appears in the guise of Jeans-like instabilities that develop when the acceleration induced by the background gravitational field is larger than a critical value for which the string extremities become casually disconnected [28–30]. Frolov and Sanchez [31] have also found that a universal critical acceleration must be a general property of strings.

Incorporating MA in a theory that takes into account the limits (I.13) or (I.16) in a meaningful way from inception is a very important question. In Caianiello’s reasoning the usual Minkowski line element

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu,
\]  

must be replaced with the infinitesimal element of distance in the eight-dimensional space-time tangent bundle \(TM\)

\[
d\tau^2 = \eta_{AB} dX^A dX^B,
\]  

where \(A, B = 0, \ldots, 7, \eta_{AB} = \eta_{\mu\nu} \otimes \eta_{\mu\nu}, X^A = \left( x^\mu, \frac{x^2}{m^2} \frac{dx^\mu}{dx} \right), x^\mu = (ct, \vec{x})\) and \(dx^\mu / ds \equiv \dot{x}^\mu\) is the four-velocity. The invariant line element (I.20) can therefore be written in the form

\[
d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{A_m^2} \eta_{\mu\nu} dx^\mu \dot{x}^\nu = \left[ 1 + \frac{\dot{x}^\mu \dot{x}^\nu}{A_m^2} \right] \eta_{\mu\nu} dx^\mu dx^\nu,
\]  

where all proper accelerations are normalized by \(A_m\). The effective space-time geometry experienced by accelerated test particles contains therefore mass-dependent corrections which in general induce curvature and give rise to a mass dependent violation of the equivalence principle. In the presence of gravity, we replace \(\eta_{\mu\nu}\) with the corresponding metric tensor \(g_{\mu\nu}\), a natural choice which preserves the full structure introduced in the case of flat space. In the classical limit \(A_m^{-1} = \frac{h}{k m c} \to 0\) the terms contributing to the modification of the geometry vanish and one returns to the ordinary space-time geometry.

The model of Ref. [26] has led to interesting results that range from particle physics to astrophysics and cosmology [32,33].

A second, equally fundamental problem stems from Caianiello’s original paper “Is there a maximal acceleration?” [34]. In particular, is it possible to find physical conditions likely to lead to a MA? An approach to this problem is illustrated below by means of three examples that are offered here as a tribute to the memory of the great master and to the ever challenging vitality of his thoughts.

The limits (I.13) and (I.16) are very high for most particles (for an electron \(A_m \sim 4.7 \times 10^{31} \text{cm} \text{s}^{-2}\) and likely to occur only in exceptional physical circumstances. The examples considered in this work involve superfluids, type-I superconductors in particular, that are intrinsically non-relativistic quantum systems in which \(\Delta E\) and \(\Delta v\) can be lower than the uncertainties leading to the limit (I.13). Use of (I.8) is here warranted because the de Broglie wavelengths of the superfluid particles vary little over distances of the same order of magnitude [3,5]. Superfluids are particular types of quantum systems whose existence is predicated upon the formation of fermion pairs. They behave, in a sense, as universes in themselves until the conditions for pairing are satisfied. The dynamics of the resulting bosons differs in essential ways from that of “normal” particles. The upper limits of the corresponding dynamical
variables like velocity and acceleration must first of all be compatible with pairing and may in principle differ from $c$ and $A_m$. If the limit on $v$ is lower, then according to (I.8) the possibility of observing the effects of MA is greater. It is shown below that indeed type-I superconductors in static conditions offer an example of a dynamics with a MA.

Lepton-lepton interactions with the final production of gauge bosons also seem appropriate choices because of the high energies, and presumably high accelerations, that are normally reached in the laboratory. The application of (I.8) is justified by the high momenta of the leptons that satisfy the inequality $\Delta x\Delta p \gg h$ [3,5].

A last example regards the highly unusual conditions of matter in the interior of white dwarfs and neutron stars. In this case the legitimacy of (I.8) is assured by the inequality $N^3 v_\lambda^3 T \ll 1$, where $N$ is the particle density in the star and $\lambda = \frac{2\hbar^2}{mkT}$ is of the same order of magnitude of the de Broglie wavelength of a particle with a kinetic energy of the order of $kT$ [3,5].

II. TYPE-I SUPERCONDUCTORS

The static behavior of superconductors of the first kind is adequately described by London’s theory [35]. The fields and currents involved are weak and vary slowly in space. The equations of motion of the superelectrons are in this case [36]

$$\frac{D\vec{v}}{Dt} = \frac{e}{m} \left[ \vec{E} + \left( \frac{\vec{v}}{c} \times \vec{B} \right) \right] = \frac{\partial\vec{v}}{\partial t} + \left( \vec{v} \cdot \vec{\nabla} \vec{v} \right),$$

(II.22)

On applying (I.8) to (II.22), one finds

$$\sqrt{\frac{1}{2} \vec{\nabla} v^2 - \vec{v} \times \left( \vec{\nabla} \times \vec{v} \right)} \leq \frac{2}{h} \Delta E \cdot \Delta v,$$

(II.23)

and again

$$\sqrt{\frac{1}{4} \left( \nabla v^2 \right)^2 + \frac{e}{mc} \epsilon_{ijk} \vec{v} B^k + \left( \frac{e}{mc} \right)^2 [v^2B^2 - (v_iB^i)^2]} \leq \frac{2}{h} \Delta E \cdot \Delta v,$$

(II.24)

where use has been made of London’s equation

$$\vec{\nabla} \times \vec{v} = -\frac{e}{mc} \vec{B},$$

(II.25)

and $\epsilon_{ijk}$ is the Levi-Civita tensor. Static conditions, $\frac{\partial\vec{v}}{\partial t} = 0$, make (II.23) and (II.24) simpler. It is also useful, for the sake of numerical comparisons, to apply (II.24) to the case of a sphere of radius $R$ in an external magnetic field of magnitude $B_0$ parallel to the polar axis. This problem has an obvious symmetry and can be solved exactly. The exact solutions of London’s equations for $r \leq R$ are well-known [37] and are reported here for completeness. They are

$$B_r = \frac{4\pi}{\beta^2 c} \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta j_\varphi),$$

(II.26)

$$B_\theta = -\frac{4\pi}{\beta^2 c} \frac{\partial}{\partial r} (r j_\varphi),$$

(II.27)

$$j_\varphi = nev_\varphi = \frac{A}{r^2} (\sinh \beta r - \beta r \cosh \beta r) \sin \theta,$$

(II.28)

where $A = -\frac{eB_0 R}{4\pi \hbar \sin \theta \cdot n}$, $n$ is the density of superelectrons and $\beta = \left( \frac{4\pi n e^2}{mc^2} \right)^{\frac{1}{2}}$ represents the reciprocal of the penetration length. On using (II.22), the inequality (II.23) can be transformed into

$$|E_r| \leq \left| \frac{v_\varphi B_\theta}{c} \right| + \sqrt{\left( \frac{2mc}{e\hbar} \right)^2 (\Delta E)^2 (\Delta v)^2 - \left( \frac{v_\varphi}{c} B_r \right)^2}.$$
For a gas of fermions in thermal equilibrium $\Delta E \sim \frac{3}{2} \mu$, $\Delta v \sim \frac{3}{2} \sqrt{\frac{\epsilon}{\mu}}$ and the chemical potential behaves as $\mu \approx \epsilon_F - \frac{(\epsilon k T)^2}{16 \pi^2} \approx \epsilon_F \sim 4.5 \times 10^{-12} \text{erg}$ for $T$ close to the transition temperatures of type-I superconductors. The reality of (II.29) requires that $\Delta E \geq \mu B_c$, where $\mu_B = \frac{e h}{2 \pi m c}$ is the Bohr magneton, or that $\frac{3}{5} \epsilon_F \geq \mu B_c$. This condition is certainly satisfied for values of $B_c$, where $B_c$ is the critical value of the magnetic field applied to the superconductor. From (II.29) one also obtains

$$|E_r| \leq \frac{3}{2m} \left( \frac{\epsilon_F}{2} \right)^{\frac{1}{2}} \left[ \frac{|B_0|}{c} + \sqrt{\left( \frac{3\epsilon_F}{5\mu_B} \right)^2 - \left( \frac{|B_c|}{c} \right)^2} \right], \tag{II.30}$$

which is verified by the experimental work of Bok and Klein [38]. More restrictive values for $\Delta E$ and $\Delta v$ can be obtained from $B_c$. The highest value of the velocity of the superelectrons must, in fact, be compatible with $B_c$ itself, lest the superconductor revert to the normal state. This value is approximately a factor $10^3$ smaller than that obtained by statistical analysis. The upper value $v_0$ of $v_c$ is at the surface. From $\Delta E \leq \frac{1}{2} m v_0^2$, $\Delta v \leq v_0$ and (II.29) one finds that at the equator, where $B_r = 0$, $E_r$ satisfies the inequality

$$|E_r| \leq \frac{v_0}{c} \left( \frac{|B_0|}{c} + \frac{v_0^2}{2\mu_B} \right). \tag{II.31}$$

For a sphere of radius $R = 1 \text{cm}$ one finds $v_0 \simeq 4.4 \times 10^4 \text{cm s}^{-1}$ and $E_r \leq 69 N/C$. If no magnetic field is present, then (II.31) gives $E_r \leq 4.2 N/C$. On the other hand London’s equation gives

$$E_r = \frac{m}{2e} \frac{\partial v_r^2}{\partial r} \simeq 0.32 N/C. \tag{II.32}$$

The inequality (II.32) agrees with the experimental data [38]. The MA limits (II.30) and (II.31) are therefore consistent with (II.32) and its experimental verification.

### III. HIGH ENERGY LEPTON-LEPTON INTERACTIONS

Consider the process $A + B \to D$ by which two particles $A$ and $B$ of identical mass $m$ give rise to particle $D$. $A$, or $B$, or both particles may also be charged. Assume moreover that $D$ is produced at rest in its proper frame. The width of $D$ is $\Gamma(AB)$ and the time during which the process takes place in the center of mass frame of $A$ and $B$ is $\Delta t = \frac{h}{\Gamma(AB)}$. The acceleration of $A$ and $B$ is thought to remain close to zero until $D$ is produced. If this were not the case, radiation would be produced and the process would acquire different characteristics. The root mean square acceleration of the reduced mass $m_r$ over $\Delta t$ will in general be $a_r \simeq \frac{v_0^2}{\gamma m_r}$, where $\gamma$ refers to the velocity $v$ of $m_r$ [39]. The MA limit (I.16), applied to $a_r$, gives $\frac{e}{\gamma m_r^2} \leq \frac{m_r c^2}{h}$, or

$$\Gamma(AB) \leq m c^2 \gamma, \tag{III.33}$$

and $v$ obeys the condition $2m c^2 \gamma = m_D c^2$, or $\gamma = \frac{m_D}{2m}$. The limit (III.33) can also be written in the form

$$\Gamma(AB) = \frac{p_f}{32\pi^2 m_D} \int |M(AB)|^2 \leq \frac{m_D c^2}{2}, \tag{III.34}$$

where $M(AB)$ is the invariant matrix element for the process. Several processes can now be considered. For the process $e^+ + e^- \to Z^0$ one has (in units $\hbar = c = 1$) $\Gamma(e^+ e^- \to Z^0) \simeq 0.08391 \text{GeV}$, $m_{Z_0} = 91.188 \text{GeV}$ and the inequality (III.34) is certainly satisfied.

Similarly, $\Gamma(W \to ee) \simeq 0.22599 \text{GeV} < \frac{80.419}{2} \text{GeV}$. One expects the limit (III.34) to be less restrictive at lower energies. In the case of $e^+ + e^- \to J/\psi$ one fact finds $\Gamma(e^- + J/\psi) \simeq 5.2 \text{KeV}$, while the r.h.s. of (III.34) gives $\frac{m_D}{2} \text{MeV}$. Alternatively, (III.34) can be used to find an upper limit to the value of $m_D$. These values depend, of course, on theoretical estimates of the corresponding decay widths [40]. From

$$\Gamma(Z^0 \to ee) \simeq \frac{G_F m_D^3}{12\pi \sqrt{2}} \leq \frac{m_{Z_0}^2}{2}, \tag{III.35}$$
one obtains $m_{Z^0} \leq (\frac{6\pi\sqrt{2}}{G_F})^\frac{1}{2} \simeq 1512\text{ GeV}$ which is a factor 16.6 larger than the experimental value of the mass of $Z^0$. Analogously

$$\Gamma(W \rightarrow d\bar{u}) \simeq \frac{G_F 4.15 m_W^3}{6\sqrt{2}\pi} \leq \frac{m_W}{2},$$

(III.36)

yields $m_W \leq 525\text{ GeV}$ which is $\sim 6.53$ times larger than the experimental value of the boson mass. Finally, from

$$\Gamma(J/\psi \rightarrow ee) \simeq \frac{16\pi \alpha^2 0.018}{m_{J/\psi}} \leq \frac{m_{J/\psi}}{2},$$

(III.37)

one finds $m_{J/\psi} \geq 4.6 \times 10^{-2}\text{ GeV}$, a lower limit which is $1.5 \times 10^{-2}$ times smaller than the known value of $m_{J/\psi}$.

For the Higgs boson the inequality

$$\Gamma(H^0 \rightarrow ee) \simeq \frac{G_F m_H^2 m_H}{4\sqrt{2}\pi} \left(1 - \frac{4m_e^2}{m_H^2}\right)^\frac{3}{2} \leq \frac{m_H}{2},$$

(III.38)

is always satisfied for $m_H \geq 2m_e$. Finally, from [40]

$$\Gamma(H^0 \rightarrow ZZ) \simeq \frac{G_F m_Z^2 m_H}{16\pi \sqrt{2}x_Z} (1 - x_Z)^\frac{1}{2} (3x_Z^2 - 4x_Z + 4) \leq \frac{m_H}{2},$$

(III.39)

where $x_Z \equiv 4m_e^2/m_H^2$, one finds

$$2m_Z \leq m_H \leq 1760\text{ GeV}.$$  

(III.40)

Both lower and upper limits are compatible with the results of experimental searches. The upper limit also agrees with Kuwata’s analogous attempt [41] to derive a bound on $m_H$ from the MA constraint, when account is taken of the fact that in Ref. [41] MA is $\lambda m^2/2$ and therefore $m_H \leq 500\text{ GeV}$.

**IV. WHITE DWARFS AND NEUTRON STARS**

The standard expression for the ground state energy of an ideal fermion gas inside a star of radius $R$ and volume $V$ is

$$E_0(r) = \frac{m^4c^5}{\pi^2\hbar^3} V f(x_F),$$

(IV.41)

where $x_F \equiv p_F/(mc)$, $p_F \equiv (3\pi^2 N/\sqrt{2})^\frac{1}{3}$ $\hbar$ is the Fermi momentum, $0 \leq r \leq R$, $N$ is the total number of fermions and

$$f(x_F) = \int_0^{x_F} dx x^2 \sqrt{1 + x^2}.$$  

(IV.42)

The integral in (IV.42) is approximated by

$$f(x_F) \simeq \frac{1}{3} x_F^3 \left(1 + \frac{3}{10} x_F^2 + \ldots\right), x_F \ll 1$$

(IV.43)

in the non-relativistic (NR) case, or by

$$f(x_F) \simeq \frac{1}{4} x_F^4 \left(1 + \frac{1}{2} x_F^2 + \ldots\right), x_F \gg 1$$

(IV.44)

in the extreme relativistic (ER) case. The average force exerted by the fermions a distance $r$ from the center of the star is [42]

$$< F_0 > = \frac{\partial E_0}{\partial r} \simeq \frac{4m^4c^5}{\pi\hbar^3} r^2 f(x_F).$$

(IV.45)
From (IV.45) and the expression

\[ N(r) = \frac{4}{9\pi h^3} r^3 p_F^3, \quad (IV.46) \]

that gives the number of fermions in the ground state at \( r \), one can estimate the average acceleration per fermion as a function of \( r \)

\[ <a(r)> = \frac{9e^2}{x_F^3 r} f(x_F). \quad (IV.47) \]

By using (IV.43) and (IV.44) one finds to second order

\[ <a(r)>_{NR} = \frac{3\lambda}{r} \left( 1 + \frac{3}{10} x_F^2 \right) \quad (IV.48) \]

\[ <a(r)>_{ER} = \frac{9e^2}{4r} \left( x_F + \frac{1}{x_F} \right). \]

The MA limit (I.13) applied to \(<a(r)>\) now yields

\[ r \geq (r_0)_{NR} \equiv \frac{3\lambda}{4\pi} \quad (IV.49) \]

\[ r \geq (r_0)_{ER} \equiv \frac{9}{16\pi} \frac{\lambda p_F}{mc}, \]

where \( \lambda \equiv \frac{h}{mc} \) is the Compton wavelength of \( m \). For a white dwarf, \( N/V \approx 4.6 \times 10^{29} \text{cm}^{-3} \) gives \( (r_0)_{NR} \approx 5.8 \times 10^{-11} \text{cm} \) and \( (r_0)_{ER} \approx 4 \times 10^{-11} \text{cm} \).

In order to have at least one state with particles reaching MA values, one must have

\[ Q((r_0)_{NR}) = \frac{4}{9\pi} \left( \frac{(r_0)_{NR} p_F}{h} \right)^3 \sim 1 \quad (IV.50) \]

\[ Q((r_0)_{ER}) = \frac{4}{9\pi} \left( \frac{(r_0)_{ER} p_F}{h} \right)^3 \sim 1. \]

In the case of a typical white dwarf, the first of (IV.50) gives \( (N/V)_{NR} \approx 1.2 \times 10^{30} \text{cm}^{-3} \) and the second one \( (N/V)_{ER} \approx 1.3 \times 10^{30} \text{cm}^{-3} \). On the other hand, the condition \( x_F \ll 1 \) requires \( (N/V)_{NR} \ll 6 \times 10^{29} \text{cm}^{-3} \), whereas \( x_F \gg 1 \) yields \( (N/V)_{ER} \gg 6 \times 10^{29} \text{cm}^{-3} \). It therefore follows that the NR approximation does not lead to electron densities sufficient to produce states with MA electrons. The possibility to have states with MA electrons is not ruled out entirely in the ER case.

The outlook is however different if one starts from conditions that do not lead necessarily to the formation of canonical white dwarfs or neutron stars [42]. Pressure consists, in fact, of two terms. The first term represents the pressure exerted by the small fraction of particles that can reach accelerations comparable with MA. It is given by

\[ P_{MA} = \frac{2m^2 e^3 Q(r_0)}{h^3}. \quad (IV.51) \]

The second part is the contribution of those fermions in the gas ground state that can not achieve MA

\[ -\frac{\partial E_0'}{\partial V} = -\frac{\partial}{\partial V} \left( 2\tilde{\gamma} V \int_{r_0}^\infty dt \frac{e^{3/2}}{e^{\alpha + \beta} + 1} \right), \quad (IV.52) \]

where \( \tilde{\gamma} \equiv \frac{(2m)^{3/2}}{(2\pi)^3 h^3} \) and \( V = \frac{4}{3} \pi \left( R^3 - r_0^3 \right) \). Since \( r_0 \) is small, one can write

\[ \frac{\partial E_0'}{\partial V} \approx \frac{\partial E_0}{\partial V} \sim \frac{4K \tilde{M}^{5/3}}{5R^5}, \quad (IV.53) \]

where \( \tilde{M} \equiv \frac{2\pi M}{8\pi n_e}, \quad \tilde{R} \equiv \frac{2\pi R}{\lambda}, \quad K \equiv \frac{m^4 \lambda}{12\pi^2 h^3} \) and \( m_p \) is the mass of the proton. In the non relativistic case, the total pressure is obtained by adding (IV.51) to (IV.53) and using the appropriate expression for \( Q(r_0) \) in (IV.50). The hydrostatic equilibrium condition is

\[ \frac{8\pi mc^2 \tilde{M}}{3\lambda^3} \frac{1}{R^3} + \frac{4K \tilde{M}^{5/3}}{5R^5} = K' \tilde{M}^2 \tilde{M}^2 R^4. \quad (IV.54) \]
where $K' \equiv \frac{4\alpha G \pi^6}{64\hbar^2 c^5}$ and $\alpha \approx 1$ is a factor that reflects the details of the model used to describe the hydrostatic equilibrium of the star. It is generally assumed that the configurations taken by the star are polytropes [43]. Solving (IV.54) with respect to $\tilde{R}$ one finds

$$\tilde{R} = \frac{\tilde{M} M_0^2}{8} \left( 1 \mp \sqrt{1 - \frac{64}{5} \left( \frac{M_0}{M} \right)^{4/3}} \right),$$

which $\tilde{M}_0 \equiv (K/K')^{3/2} = \left( \frac{2\pi a_0^2}{\sqrt{3} \pi m_r r_0} \right)^2 \approx \frac{a_0}{\sqrt{3} \pi m_r} M_\odot$. Solutions (IV.55) will be designated by $\tilde{R}_-$ and $\tilde{R}_+$. They are real if $M \geq (\frac{64}{5})^{3/4} M_0 \sim 6.8 M_\odot$. This is a new stability condition. At the reality point the radius of the star already is twice that of a canonical white dwarf. This situation persists for the solution $R_-$ up to mass values of the order of $\sim 10 M_\odot$. The radii of the $R_+$ solutions increase steadily. The corresponding electron densities, calculated from $N/V = \frac{3 M}{\pi a_0^2 M_\odot}$, are $(N/V)_- > \frac{8\pi}{3} (\frac{8}{5})^3 (M_0/M)^2 \sim 6.6 \times 10^{30} cm^{-3}$ and $(N/V)_+ < 6.6 \times 10^{30} cm^{-3}$. NR stars of the $R_-$ type thus appear to be more compact than those of the $R_+$ class. The electron density for $R_+$ is still compatible with that of a canonical NR white dwarf. Equation (IV.54) can also be written in the form

$$\tilde{M}^{1/3} \tilde{R} = \frac{4}{5} \tilde{M}_0^{2/3} \left( 1 + \frac{10\pi m_e^2}{3\lambda^3 K} \frac{\tilde{R}^2}{\tilde{M}^{2/3}} \right),$$

where the second term on the r.h.s. represents the MA contribution to the usual M-R relation for NR white dwarfs. This contribution can be neglected when $\frac{\tilde{R}}{M^{1/3}} < 1/\sqrt{5}$ which requires $N/V > \frac{8\pi^{3/2}}{3\lambda^3} \sim 6.6 \times 10^{29} cm^{-3}$. This condition and the usual M-R relation, $\tilde{M}^{1/3} \tilde{R} = \frac{4\tilde{M}^{2/3}}{3\lambda^3}$, are compatible if $R < \frac{\lambda}{\pi a_0} \left( \frac{2\pi M}{3m_r} \right)^{1/3}$, which leads to the density $N/V > \frac{5^{3/4}}{3\lambda^3} (M/M_0) \sim 2.7 \times 10^{30} (M/M_0) cm^{-3}$.

Similarly, one can calculate the MA contribution to the M-R relation for ER white dwarfs. The equation is

$$\frac{m^4 c^5}{4\pi^2 \hbar^3} + K \left( \frac{\tilde{M}^{4/3}}{R^4} - \frac{\tilde{M}^{2/3}}{R^2} \right) = K' \frac{\tilde{M}^2}{R^4},$$

where the first term on the l.h.s. represents the MA contribution. From (IV.57) one gets

$$\tilde{R} = \tilde{M}^{1/3} \sqrt{4 - \left( \frac{\tilde{M}}{M_0} \right)^{2/3}}$$

and the stability condition becomes $M \leq 8 M_0 \sim 8 M_\odot$. These are very compact objects. For the electron densities determined, the star can still be called a white dwarf. One also finds that for $N/V \sim 3 \times 10^{30} cm^{-3}$ the number of MA states is only $Q(r_0) = \frac{9\lambda^3}{16\pi^2} \frac{N}{V} \simeq 2.2 \frac{M}{M_\odot}$. A few MA electrons could therefore be present at this density. However, interactions involving electrons and protons at short distances may occur before even this small number of electrons reaches the MA.

Analogous conclusions also apply to neutron stars, with minor changes if these can be treated as Newtonian polytropes. This approximation may only be permissible, however, for low-density stars [43]. One finds in particular $Q(r_0) = \frac{9\lambda^3}{16\pi^2} \frac{N}{V} \simeq 4.5 \frac{M}{M_\odot}$. The presence of a few MA neutrons is therefore allowed in this case.

**V. CONCLUSIONS**

A limit on the proper acceleration of particles can be obtained from the uncertainty relations in the following way. Ehrenfest theorem (I.8) is first applied to a particle’s acceleration $\vec{a}$ in the particle’s instantaneous rest frame. The latter is then transformed to a Lorentz frame of instantaneous velocity $\vec{v}$. In any other Lorentz frame the resulting acceleration is $a' \leq a_0$. The absolute value of the proper acceleration satisfies (I.16). No counterexamples are known to the validity of (I.16). For this reason (I.16) has at times been elevated to the status of a principle. It would be only befitting to call it the Caianiello principle.
In most instances the value of MA is so high that it defies direct observation. Nonetheless the role of MA as a universal regulator must not be discounted. It is an intrinsic, first quantization limit that preserves the continuity of space-time and does not require the introduction of a fundamental length, or of arbitrary cutoffs. The challenge is to find situations where MA affects the physics of a system in ways that can be observed.

Though the existence of a MA is intimately linked to the validity of Ehrenfest theorem, it is not entirely subordinate to it and the limit itself may depend on the dynamical characteristics of the particular system considered. This is the case with superconductors of the first kind which are macroscopic, non-relativistic quantum systems with velocities that satisfy the inequality \( \Delta E \Delta v \ll mc^2 \). Superfluid particles in static conditions are known to resist acceleration. The MA constraints (II.23) and (II.24) lead to a limit on the value of the electric field at the surface of the superconductor that agrees with the value (II.31) obtained from London’s equations and with known experimental results. The MA limit in this case is only \( \sim 10 \) times larger than (II.32). If the pairing condition is satisfied, then superfluids in static conditions obey a dynamics for which a MA exists and differs from \( A_m \), as anticipated.

In Section (III) two high-energy particles, typically leptons, produce a third particle at rest. The MA limit applied to the process leads to the constraint (III.34) on the width of the particle produced. The limit is perfectly consistent with available experimental results. When the end product is \( Z^0 \), the acceleration is \( a_r = \frac{2mc}{\hbar m_r} \Gamma(AB) \sim 2.8 \times 10^{26} \text{ cm s}^{-2} \). Even at these energies the value of the acceleration is only a factor \( \sim 6 \times 10^{-6} \) that of the MA for \( m_r \). Equation (III.33) and current estimates of \( \Gamma(ZZ \to H^0) \) can also be used to derive upper and lower limits (III.40) on the mass of the Higgs boson.

The last physical situation considered regards matter in the interior of white dwarfs and neutron stars. For canonical white dwarfs, the possibility that states exist with MA electrons can be ruled out in the NR case, but not so for ER stars. On the other hand, the mere presence of a few MA electrons alters the stability conditions of the white dwarf drastically. Equations (IV.56) and (IV.58) represent, in fact, new stability conditions. The same conclusions also apply to NR neutron stars, with limitations, however, on the choice of the equation of state. In the collapse of stars with masses larger than the Chandrasekhar and Oppenheimer-Volkoff limits from white dwarfs to neutron stars, to more compact objects, conditions favorable to the formation of states with MA fermions may occur before competing processes take place.


