RELATIVISTIC ACCELERATION OF MAGNETICALLY DRIVEN JETS

AKIRA TOMIMATSU

Department of Physics, University of Nagoya, Nagoya 464-8602, Japan

atomi@allegro.phys.nagoya-u.ac.jp

and

MASAAKI TAKAHASHI

Department of Physics and Astronomy, Aichi University of Education, Kariya, Aichi 448-8542, Japan

takahasi@phyas.aichi-edu.ac.jp

ABSTRACT

We present an analytical model for describing highly relativistic acceleration of magnetically driven jets, within the framework of ideal MHD for cold, stationary and axisymmetric outflows. Our novel procedure is to treat the wind equation as an algebraic relation between the relativistic Alfvén Mach-number and the poloidal electric to toroidal magnetic field amplitudes ratio $\xi$. This allows us to obtain easily the wind solutions for trans-fast-magnetosonic flows, together with the required range of $\xi$. Then, to determine the spatial variation of $\xi$, we solve approximately the Grad-Shafranov equation applied to a jet flow ejected with a very large total specific energy $E$ and confined within a very small opening angle. Our trans-fast-magnetosonic model provides a closed-form expression for the transition from a magnetically dominated flow to a kinetic-energy dominated one, which occurs in the sub-asymptotic region far beyond the light cylinder of the radius $R_L$. Importantly, we find that the equipartition between magnetic and kinetic energies is realized at a cylindrical radius of order of $R_L E/c^2$, and confirm that the further conversion of magnetic to kinetic energy proceeds logarithmically with distance in the asymptotic region. Finally, we discuss briefly the astrophysical implications of our model for jets originating from active galactic nuclei.

Subject headings: MHD—relativity—galaxies: jets
1. INTRODUCTION

1.1. Collimated Relativistic Jets

Highly relativistic jet motion is one of the most interesting phenomena observed in active galactic nuclei (AGN), microquasars and possibly γ-ray bursts (GRB), and many works have been devoted to numerical and analytical investigations of the mechanisms for producing, collimating and accelerating matter to relativistic speeds (Sauty et al. 2001; Ghisellini & Celotti 2002). At present the most promising approach to the jet phenomena seems to consider magnetically driven outflows within the framework of relativistic magnetohydrodynamics (MHD).

Based on the MHD scenario, the jet ejection is expected to be realized under the magnetic energy-dominated state (i.e., the Poynting jet) in the vicinity of a central source. This initial Poynting flux will be originated by electromagnetic extraction of rotational energy from a spinning black hole or/and an accretion disk, as was first discovered by Blandford & Znajek (1977) and numerically studied by the time-dependent MHD simulations in Kerr geometry (see Komissarov 2001 for the magnetically dominated regime; Koide et al. 2002 for the full MHD regime). Then, a significant fraction of the huge energy in the outflow will be converted from the Poynting flux into the fluid kinetic energy of bulk motion. Such an efficient energy conversion may be able to occur unsteadily in the inner magnetosphere close to the black hole as a result of the MHD interaction with infalling matter. Koide et al. (2000) discussed this problem and found a magnetically driven jet inside a gas pressure-driven jet in the counter-rotating black hole case against the disk rotation. However, the poloidal velocity of the jet is only sub-relativistic.

Otherwise, the steady magneto-centrifugal acceleration in the propagation over a large enough distance should become important. It is well-known that the ideal stationary axisymmetric MHD equations reduce to a set of two equations describing the local force-balance along the field and across the field, and called the poloidal wind equation and the Grad-Shafranov (trans-field) equation (for reference, see, e.g., Okamoto 1992; Beskin 1997). Self-similar solutions of magnetosphere, where the magnetic field lines are anchored to a thin accretion disk, were discussed for jet collimation and acceleration (Blandford & Payne 1986; Li, Chiu & Begelman 1992; Contopoulos 1994). Unfortunately, the outflows given by the self-similar solutions (Contopoulos 1994) reach some maximum radius and recollimate afterwards. The self-similar scaling will not be valid in the asymptotic region. Using the stationary axisymmetric model, the asymptotic flow structure has been found to vary logarithmically with radius, and the kinetic energy-dominated solutions describing collimating jet magnetospheres have been presented (Li, Chiu & Begelman 1992; Eichler 1993; Begel-
man & Li 1994; Tomimatsu 1994), while the decollimation of magnetic field lines has been claimed to become significant in the equatorial (current-closure) region (Beskin & Okamoto 2000). Note that the asymptotic analysis fails to check whether the obtained structure is due to trans-magnetosonic flows satisfying the critical conditions. To assure the validity of the asymptotic structure, we should consider the connection of the plasma source to the asymptotic region (and to the black hole). However, the full MHD description of the global magnetospheric structure in general relativity is a task still too complex even for the stationary axisymmetric system. For example, the available general relativistic models to give both closed and open field lines connecting a disk with a black hole and a distant region are limited to magnetic-energy dominated cases, such as a disk-current field (Tomimatsu & Takahashi 2001) and a force-free field (Fendt 1997).

The jet-type outflows given by the one-dimensional or two-dimensional numerical non-self-similar solutions, on the other hand, were also constructed (Appl & Camenzind 1993a,b; Li 1993; Fendt & Camenzind 1996; Fendt & Memola 2001). By solving the Grad-Shafranov and poloidal equations self-consistently, disk-jet connected flow solutions cylindrically confined by some external pressure were obtained. To compare the MHD predictions with jet observations in more detail (e.g., the terminal velocity and width of the jets, the size of the active disk region in a central engine, which would correspond to jet’s foot point, and the gravitational radius), the efficiency of relativistic acceleration should be discussed in more general asymptotic structures of outflows (depending on the boundary conditions; e.g., the shape of the innermost flux surface, or an external pressure distribution at the outermost flux surface, etc), and the propagation distance necessary for a rough equipartition between magnetic and kinetic energies should be clarified. In this paper we would like to focus on such problems of steady acceleration at large distances from the central source, by developing a new analytical method without any self-similar assumption. Most of the previous works discussed the field structure under the force-free assumption, in which the contributions by the plasma inertia are ignored. However, our new treatment allows us to study more easily trans-fast magnetosonic outflow solutions and to include the plasma inertia effects on the magnetic field structure and the plasma acceleration process. Hence, by neglecting the effect of gravity and gas pressure, we can have some more self-consistent MHD solutions, showing the evolution of the bulk Lorentz factor $\gamma$ of outgoing flows to $\gamma \sim 10$ in accordance with AGN jets.
1.2. New Analytic Procedure for Trans-Magnetosonic Jets

In this paper, for the ideal stationary axisymmetric MHD outflows, we follow the relativistic formulation given by Camenzind (1986, 1987). To avoid a very troublesome task to solve the Grad-Shafranov equation, one may discuss the fluid motion by assuming a magnetic field configuration and solving only the poloidal wind equation (Takahashi & Shibata 1998; Fendt & Greiner 2001). Though this procedure does not always assure the consistency with the Grad-Shafranov equation, the analysis of the poloidal wind equation is useful for studying the critical conditions for MHD flows. For a cold plasma considered here, the key problem is to derive the trans-fast-magnetosonic solutions from the poloidal wind equation. If the poloidal wind equation is algebraically solved under a fixed field configuration in a conventional manner, however, it allows the existence of many unphysical solutions describing a flow which does not smoothly pass through the fast-magnetosonic critical point, unless the slightly complicated critical conditions are required for the four integrals of motion which are constant along a field line. Usually this prevents us from understanding clearly the efficiency of acceleration along a field line in simple analytic computations.

One of our purposes in this paper is to propose a new analytic procedure useful for discussing general properties of the trans-MHD flows within the poloidal wind equation and without assuming a field configuration. We treat the poloidal wind equation as an algebraic relation between the relativistic Alfvén Mach-number $M$ and the poloidal electric to toroidal magnetic field amplitudes ratio $\xi$ (instead of the poloidal field amplitude). Though it is not difficult to contain the effect of gravity, our analysis given here is limited to special relativistic flows for showing explicitly the conversion of magnetic to kinetic energy at large distances. Then, it is easily found that only if $\xi$ is supposed to be a function smooth along a field line at the critical points, the solution (depending on $\xi$) for the poloidal wind equation becomes trans-critical. We would like to emphasize that no special critical conditions are needed for the four integrals of motion.

The spatial variation of the ratio $\xi$ newly introduced into the poloidal wind equation must be determined by the Grad-Shafranov equation. In the case of AGN (or possibly GRB) jets a high degree of collimation as well as a ultrarelativistic bulk motion is observationally indicated. Hence, we consider jet flows with a very small opening angle and a very large asymptotic Lorentz factor to solve approximately the Grad-Shafranov equation, under the condition of a cold plasma and no gravity. Hereafter we use the cylindrical coordinates $R$ and $Z$ with $c = 1$ unit and denote the flux function by $\Psi(R, Z)$. For efficient acceleration of bulk motion a huge magnetic energy should be stored in the outflow near the central source. Then, we assume the total specific energy $E(\Psi)$ to be much larger than unity. (In this paper, we use the term “specific” as a quantity per the rest-mass energy of one plasma
particle.) The specific energy $E(\Psi)$ is conserved along a field line $\Psi = \text{constant}$. The angular velocity of a field line $\Omega_F(\Psi)$ is also an integral of motion, and the light cylinder radius is given by $R_L = 1/\Omega_F$. By virtue of the very large value of $E$, we can define the intermediate region $R_L \ll R \leq R_L E$ between the inner region $R \leq R_L$ and the asymptotic region $R \gg R_L E$. We derive a class of general solutions corresponding to an arbitrary (i.e., cylindrical, paraboloidal or radial) boundary configuration for the outermost flux surface of jet flows with the opening angle smaller than $1/E$, by solving the approximated Grad-Shafranov equation, which remains valid in the spatial range from the intermediate region to the asymptotic region. The total specific energy $E$ may be decomposed into the magnetic part $E_m$ and the plasma kinetic part $E_k$. The importance of the energy conversion from the magnetic part to the kinetic part in a relativistic wind was originally pointed out by Michel (1969), and he showed that for a radial magnetic field the energy conversion from Poynting flux to particle energy flux (or the plasma acceleration) is extremely inefficient. For a flux tube diverging even slightly faster than radially, the significant energy conversion occur beyond the fast magnetosonic point; at the fast magnetosonic point the total energy flux is still dominated by the Poynting flux (e.g., Begelman & Li (1994); Takahashi & Shibata (1998)). The main purpose of this paper is to give the evolution of the ratio $E_k/E_m$ in a simple closed-form expression, and our new result is to show the conversion of $E_m$ to $E_k$ in the intermediate region. The fast-magnetosonic critical point is found to exist in the intermediate region, where the flow is still magnetic-dominated ($E_m \gg E_k$) in accordance with the previous works. Importantly, we can conclude that the rough equipartition $E_m \sim E_k$ is realized at a radius $R \sim R_L E$. The further energy conversion goes on logarithmically with the increase of $R$ toward the kinetic-energy dominated asymptotic state, as was previously pointed out (Eichler 1993; Begelman & Li 1994; Tomimatsu 1994). The complete energy conversion becomes possible, if the outermost flux surface of jet flows extends to an infinite radius.

This paper is organized as follows: In §2, we analyze the poloidal wind equation by introducing the poloidal electric to toroidal magnetic field amplitudes ratio $\xi$. The general properties of the trans-fast-magnetosonic solutions are studied, in particular, for outflows which are initially ejected under a magnetic-energy dominated state and a very large total specific energy $E$. In §3, we give the approximated form of the Grad-Shafranov equation for jet flows propagating beyond the light cylinder surface with a very small opening angle. In §4, we derive a class of general solutions for the Grad-Shafranov equation to determine the Alfvén Mach number $M$ and the flux function $\Psi$ in the intermediate and asymptotic regions. Finally we summarize our results of relativistic acceleration and discuss the implications for AGN jets in §5.
2. TRANS-CRITICAL MHD OUTFLOWS

2.1. Non-singular MHD flow equation

First let us give a brief review (see, e.g., Camenzind 1986) of ideal stationary axisymmetric MHD outflows for a cold plasma in Minkowski spacetime background with the cylindrical line element written by $c = 1$ unit as follows,

$$ds^2 = dt^2 - dR^2 - dZ^2 - R^2 d\phi^2 .$$

(1)

As was mentioned in §1, we have the four integrals of motion which are constant along a field line given by $\Psi(R, Z) = \text{constant}$. The total specific energy $E(\Psi)$ and angular momentum $L(\Psi)$ are decomposed into the magnetic and kinetic parts as follows,

$$E = E_m + E_k , \quad E_m = -\frac{R\Omega_F B_\phi}{4\pi k} , \quad E_k = \gamma ,$$

(2)

$$L = L_m + L_k , \quad L_m = -\frac{RB_\phi}{4\pi k} , \quad L_k = \gamma R^2 \Omega ,$$

(3)

where the angular velocity of a field line $\Omega_F(\Psi)$ and the rest-mass energy loading rate per unit magnetic flux $k(\Psi)$ are also the integrals of motion. (These four integrals of motion for outflows are assumed to be positive in the following.) We call $B_\phi$ the toroidal field, and the poloidal field strength is denoted by $B_p$. If the flux function $\Psi(R, Z)$ is determined, the poloidal field strength is given by

$$B_p^2 = (B^R)^2 + (B^Z)^2 , \quad B^R = -\frac{1}{R} \frac{\partial \Psi}{\partial Z} , \quad B^Z = \frac{1}{R} \frac{\partial \Psi}{\partial R} ,$$

(4)

and the toroidal field can be expressed in terms of the poloidal quantities. Further, the Lorentz factor $\gamma$ equal to $E_k$ corresponds to bulk plasma motion involving both poloidal and rotational components. The angular velocity of plasma is denoted by $\Omega$ in $L_k$. The poloidal 4-velocity $u_p = \gamma v_p$ is used instead of the 3-velocity $v_p$. Then, the integral of motion $k$ is given by $k = \rho u_p / B_p$, where $\rho$ is the proper mass density of plasma.

Now we can define the relativistic Alfvén Mach number $M$ by the equation

$$M^2 \equiv 4\pi \rho u_p^2 / B_p^2 = 4\pi k u_p / B_p ,$$

(5)

which is used to give the toroidal field as

$$B_\phi = \frac{4\pi k}{M^2 + x^2} \left( \frac{L\Omega_F}{x} - x E \right) ,$$

(6)
and \( x \equiv R\Omega_F = R/R_L \). Because \( \gamma \) and \( R\Omega \) can be also represented by the four integrals of
motion and \( M^2 \), the condition \( \gamma^2 (1 - R^2 \Omega^2) - u_p^2 = 1 \) for normalization of 4-velocity leads to the poloidal wind equation
\[
(1 + u_p^2)(1 - x^2 - M^2)^2 = \varepsilon^2 (1 - x^2 - 2M^2) + \left( E^2 - \frac{L^2 \Omega_F^2}{x^2} \right) M^4 ,
\]
where \( e = E - L\Omega_F = \gamma (1 - R^2 \Omega \Omega_F) \) is denoted by the plasma parts of the conserved energy and angular momentum. The angular velocity of plasma \( \Omega \) is given by
\[
R\Omega = \frac{(\Omega_F L/E) M^2 - x^2 (1 - \Omega_F L/E)}{x (M^2 - 1 + \Omega_F L/E) .}
\]

Let us rewrite equation (5) to give the poloidal 4-velocity \( u_p \) in equation (7) as follows,
\[
u_p = B_p M^2/4\pi k .
\]

Then, by assuming a specific flux function \( \Psi(R, Z) \) (giving the field strength \( B_p \)), one may solve the algebraic equation (7) for \( M^2 \) to determine the evolution along a fixed field line. In general, however, such solutions giving \( M^2 \) as a function of \( x \) and \( \Psi \) would not be trans-fast-
magnetosonic: If \( u_p \) becomes equal to the relativistic fast-magnetosonic wave speed at a point on a field line of \( \Psi = \) constant, the partial derivative \( \partial M^2 / \partial x \) diverges there. (Otherwise, the flow remains sub-fast-magnetosonic or super-fast-magnetosonic everywhere on a field line.) It becomes necessary to find a special class of solutions satisfying the critical condition to keep a finite acceleration of \( u_p \) at the fast-magnetosonic point (see, e.g., Takahashi & Shibata 1998). In this paper we do not adhere to such an analysis of the critical condition relating the four integrals of motion with \( B_p \) at the fast-magnetosonic point. We rather turn our attention to a different useful information concerning plasma acceleration of trans-fast-
magnetosonic flows, which is derived from the poloidal wind equation without assuming any details of a flux function \( \Psi(R, Z) \).

For this purpose we introduce the poloidal electric to toroidal magnetic field amplitudes ratio \( \xi \) as follows,
\[
\xi \equiv \frac{E_p}{|B\phi|} = x \frac{B_p}{|B\phi|} ,
\]
where we have used the relation \( E_p = x B_p \) between the poloidal field amplitudes. If equation (10) by the help of equations (6) and (9) is used, the poloidal velocity \( u_p \) (and the Lorentz factor \( \gamma \)) is given by \( M^2 \) and \( \xi \) (instead of \( B_p \)). Then, the poloidal wind equation (7) can be reduced to the quadratic equation for \( M^2 \)
\[
AM^4 - 2BM^2 + C = 0 ,
\]
where the coefficients $A$, $B$ and $C$ are given by

$$
A = E^2 - 1 - \xi^2 \left( E - \frac{L\Omega F}{x^2} \right)^2 - \frac{L^2\Omega_F^2}{x^2},
$$

(12)

$$
B = x^2 + \varepsilon^2 - 1,
$$

(13)

$$
C = -(x^2 - 1)B.
$$

(14)

If the ratio $\xi$ is given as a smooth function of $x$ under a constant $\Psi$, the evolution of $M^2$ along a field line can be derived from equation (11). Then, we can determine the toroidal field $B_\phi$ from equation (6) and also the poloidal field $B_p$ from the relation $B_p = \xi|B_\phi|x$. As will be briefly explained later, this field amplitude $B_p$ obtained through this procedure automatically satisfies the critical condition at the fast-magnetosonic point. This means that the quadratic equation (11) is not singular even at the fast-magnetosonic point. Hence, our necessary task in this section is just to solve explicitly equation (11) and to discuss the parametric range of $\xi$ for allowing the existence of highly relativistic trans-fast-magnetosonic solutions.

After some algebraic manipulation, it is easy to show that equation (11) gives the solution of the form

$$
M^2 = \frac{x^2 - 1}{[E - (L\Omega F/x^2)]\sqrt{f/B} - 1},
$$

(15)

where the function $f$ depending on $\xi$ and $x$ is defined by $f \equiv (1 - \xi^2)x^2 + \xi^2$. It is clear that the function $B$ involved in equation (15) cannot be negative, because $\varepsilon$ is estimated to be

$$
\varepsilon^2 \geq \frac{(1 - R^2\Omega^2\Gamma)^2}{1 - R^2\Omega^2}.
$$

(16)

Then, we must require the condition $f \geq 0$, under which we obtain the allowed range of $\xi$ for outflows propagating at a radius $x > 1$ beyond the light cylinder as follows,

$$
\xi^2 \leq \frac{x^2}{x^2 - 1}.
$$

(17)

A more stringent constraint is obtained if we consider $M^2$ at the asymptotic radius $x \to \infty$, where we have

$$
\frac{M^2}{x^2} \to \frac{1}{E\sqrt{1 - \xi^2} - 1}.
$$

(18)

This leads to the result that if the inequality $\xi^2 > \xi_c^2 \equiv 1 - (1/E^2)$ always holds on a field line, $M^2$ becomes infinitely large at the radius $x = x_c$ given by

$$
x_c = \frac{L\Omega_F}{1 + E^2(\xi^2 - 1)} \left[ \xi^2E - \frac{L\Omega_F}{2} + \sqrt{\xi^2(Ee - 1) + \frac{L^2\Omega_F^2}{4}} \right].
$$

(19)
To avoid the divergence of $M^2$ at a finite radius, the field lines with $\xi > \xi_c$ should become a closed loop or be asymptotically cylindrical within the radius $x = x_c$. Note that outside the light surface the closed loop configuration should be forbidden because of the inertia effect of plasma. So the flow streaming along the closed loop solution should transit to another open field solution by making a MHD shock. Then, we can expect that the asymptotically cylindrical configuration is a more plausible one. (For the closed loop aligned flows without the shock formation, the ideal MHD approximation may be broken at least near the equatorial plane, because non-ideal MHD flows can stream across the magnetic field lines.)

The Alfvén point on a field line is known to be present at $M^2 = M^2_{AW} \equiv 1 - x^2$, from which under the condition $E > L\Omega_F$ the solution (15) gives the position $x^2 = x^2_A \equiv L\Omega_F/E$ and the Alfvén Mach number $M^2 = M^2_A \equiv M^2_{AW}(x_A) = e/E$. In general, the differential form of the poloidal equation contains a singular term (see, e.g., Takahashi & Shibata 1998). However, with the above condition, apparently the sub-Alfvénic outflows given by the wind solution (15) can smoothly pass through the Alfvén point and propagate to the super-Alfvénic region $x > x_A$. After passing through the light cylinder $x = x_L = 1$, which is not a singular point for physical flows, the super-Alfvénic outflows would reach distant regions. Of course, $M^2$ remains finite also at the light cylinder, where we have

$$M^2_L \equiv M^2(x_L) = \frac{M^2_A}{1 - w}, \quad w \equiv \frac{1 + (\xi^2 + 1) e^2}{2 e E},$$

(20)

The Alfvén Mach number at the light cylinder becomes larger if compared with the value $M_A$ at the Alfvén point.

Now let us give the field amplitude $B_p$, using $M^2$ obtained by equation (15). From the calculation of $B_\phi$ and the definition of $\xi$ we find the simple relation

$$B_p = \frac{4\pi k \xi}{M^2} \sqrt{\frac{B}{J}}.$$  

(21)

This allows us to estimate the field amplitude $B_p$ at the fast-magnetosonic point $x = x_F$ with $M^2 = M^2_F \equiv M^2_{FW}(x_F)$, where we have

$$M^2_{FW} \equiv \frac{f}{\xi^2}$$

(22)

for the Alfvén Mach number related to the fast magnetosonic wave speed. From equation (15) it is easy to find that the relation $M^2 = M^2_F$ is satisfied at the point determined by the equation

$$\left(E - \frac{L\Omega_F}{x^2}\right)^2 f^3 = x^4 B.$$  

(23)
We must remark that the light cylinder radius $x = 1$ satisfying equation (23) is not the fast-magnetosonic point. In spite of the existence of such a redundant solution, we use this concise form (23) for mathematical simplicity. In fact, equation (23) is helpful for calculating the partial differentiation of equation (21) with respect to $x$ along a fixed field line at the fast-magnetosonic point, and we can straightforwardly check that the terms involving $\partial \xi / \partial x$ automatically cancel out in the derivative $\partial B_p / \partial x$ at this critical point, and the critical condition for the value of the partial derivative of $B_p$ holds, irrespective of any choice of $\xi$. (If the derivative $\partial \xi / \partial x$ diverges at the fast-magnetosonic point, the critical condition for $\partial B_p / \partial x$ is not satisfied. This means that a smooth change of $\xi$, namely $B_\phi$, at the critical point is the essential requirement for a trans-magnetosonic flow.) Thus, we can obtain a trans-fast magnetosonic flow solution without the regularity condition at the fast magnetosonic point. The behavior of the acceleration in the outgoing flow is determined by $\xi = \xi(x)$, and the asymptotic feature depends on the value of $\xi$; that is, $\xi > \xi_c$ or $\xi < \xi_c$. The typical solutions for outgoing trans-fast magnetosonic flows are demonstrated in Figures 1a and 1b, for both cases of $\xi < \xi_c$ and $\xi > \xi_c$. The outflows can start from the plasma source ($x \ll 1$), and after passing through the Alfvén point (A) and the fast magnetosonic point (F) they reaches distant regions. In the case of $\xi < \xi_c$, the flow reaches $x \gg 1$ region with a finite Mach number. This is the standard picture discussed by lots of previous wind models. On the other hand, in the case of $\xi > \xi_c$ the flow confines within $x < x_c$. We can expect that asymptotically the magnetic field line becomes cylindrical and the flow streams toward $Z$-direction. In the distant $Z$-region ($Z \Omega_F \gg 1$), the flow becomes to have a very high Alfvén Mach number.

2.2. Acceleration and Energy conversion

Though we have constructed the basic formulae for discussing the evolution of $M^2$, in this paper we are particularly interested in highly relativistic acceleration of bulk motion through a conversion of magnetic to kinetic energy. In a sub-Alfvénic region the outflows are expected to be injected under a magnetic-energy dominated state ($E \simeq L \Omega_F$) with a very large value of $E$. This means that we can analyze the evolution of $M^2$ in more details, using the approximations

$$\frac{e}{E} = \frac{E - L \Omega_F}{E} \ll 1 , \quad E \gg 1 . \quad (24)$$

For example, equation (23) claims that the value of $x_F^2$ at the fast-magnetosonic point becomes very large and is approximately given by $E^{2/3} f_F$, in which the exact form of $f = f(x)$ is necessary because $\xi_F \equiv \xi(x_F)$ may be very close to unity. Hence, we can write the position
of the fast-magnetosonic point as follows,

\[ x_F^2 = \frac{\xi_F^2 E^{2/3}}{1 + (\xi_F^2 - 1)E^{2/3}}. \]  

(25)

We note that this critical point exists only if \( \xi_F^2 > 1 - (1/E^{2/3}) \), while the allowed range is \( 1 < x_F < \infty \), depending crucially on the difference \( \xi_F^2 - 1 \). On the other hand, for the value of \( M_F^2 \) at the fast-magnetosonic point we obtain

\[ \frac{M_F^2}{x_F^2} = \frac{1}{E^{2/3}} \ll 1. \]  

(26)

To show the physical implication of equation (26), we consider the specific magnetic energy \( E_m \) at a radius \( x \gg 1 \) beyond the light cylinder. From equations (2) and (6) we obtain approximately

\[ E_m = \frac{E}{1 + \tilde{M}^2}, \]  

(27)

where we have introduced the quantity \( \tilde{M}^2 \) defined by \( \tilde{M}^2 \equiv M^2/x^2 \). Then the ratio of the specific kinetic energy \( E_k = E - E_m \) to \( E_m \) is given by

\[ \frac{E_k}{E_m} = \tilde{M}^2, \]  

(28)

which implies that \( \tilde{M}^2 \) is an indicator of the conversion of the huge magnetic energy into the kinetic energy for outflows propagating to a radius \( x \gg 1 \). (The term \( \tilde{M}^2 \) corresponds to the inverse of the magnetization parameter \( \sigma \) in some wind models; see, e.g, Michel (1969) and Camenzind (1986) for the radial wind, and Begelman & Li (1994) and Takahashi & Shibata (1998) for non-radial winds.) At the fast-magnetosonic point, however, we obtain from equation (26)

\[ \left( \frac{E_k}{E_m} \right)_F = \frac{1}{E^{2/3}} \ll 1, \]  

(29)

which is a generic result independent of a field configuration \( \Psi(R, Z) \) of highly relativistic outflows. It is clear that the energy conversion is still inefficient at the fast-magnetosonic point.

Then, we consider plasma acceleration of trans-fast-magnetosonic outflows propagating at a radius \( x \gg 1 \) beyond the critical point, where we have approximately

\[ \tilde{M}^2 = \frac{1}{\sqrt{(1 - \xi^2)E^2 + (\xi^2 E^2/x^2)} - 1}. \]  

(30)
As was previously mentioned, $M^2$ becomes infinitely large at $x = x_c$ if $\xi > \xi_c$. For highly relativistic outflows this critical radius is approximately given by

$$x_c^2 = \frac{\xi^2 E^2}{1 + (\xi^2 - 1)E^2}.$$  \hspace{1cm} (31)

A possible configuration of a field line given by the parameter $\xi$ larger than $\xi_c$ would be asymptotically cylindrical, because it can be confined within a radius smaller than $x_c$ even in the limit $Z \to \infty$. Unless $\xi$ is very close to unity, the value of $x_c$ is of order of unity, and such a cylindrical jet becomes very narrow. In the limit $x \to x_c$, the energy ratio $\dot{M}^2$ is estimated to be

$$\dot{M}^2 = \frac{1}{1 + (\xi^2 - 1)E^2} \times \frac{1}{1 - (x/x_c)}.$$  \hspace{1cm} (32)

The narrow jet corresponding to $\xi_c$ of order of unity can be kinetic-energy dominated only if the asymptotic radius $x$ is very close to $x_c$, i.e., $1 - (x/x_c) = O(1/E^2)$. This is the fine-tuning problem required for the cylindrical field to realize highly relativistic acceleration of bulk motion. On the other hand for a cylindrical field line with the value of $\xi$ such that $|\xi^2 - 1| = O(1/E^2)$ the maximum radius $x$ may become of order of $E^2$, and from equation (30) we find that at least a rough equipartition $E_k \sim E_m$ will be realized at the scale of $x \sim E$ without the fine-tuning of $x \to x_c$. In particular, if $\xi \leq \xi_c$, namely, $\xi^2 \leq 1 - (1/E^2)$, the field line may be extending to an infinite radius $x \to \infty$, where $\dot{M}^2$ becomes equal to $(E \sqrt{1 - \xi^2} - 1)^{-1}$. Such a field line configuration may be asymptotically paraboloidal or conical, and the efficient energy conversion into the state $E_k \geq E_m$ becomes possible at $x \geq E$ for field lines with $1 - \xi^2 = O(1/E^2)$. This is another fine-tuning problem required for the poloidal electric and toroidal magnetic field amplitudes. The precise fine-tuning of $\xi^2 = 1 - (1/E^2)$ means that the magnetic energy $E_m$ can be completely transported into the kinetic energy $E_k$ as outflows propagate to an infinite radius.

To claim that the kinetic energy can asymptotically become larger than the magnetic energy for injection of magnetic-energy dominated outflows with very large $E$, we must solve the fine-tuning problem such that $1 - (x_c/x) = O(1/E^2)$ or $|\xi^2 - 1| = O(1/E^2)$ as a result of MHD interaction described by the Grad-Shafranov equation. In this paper we consider a jet ejection with a very small opening angle such that $R/Z \leq 1/E$, and we study the spatial variation of $\xi$ to show the dynamical fine-tuning of $\xi$ in jet flows.

3. THE APPROXIMATED GRAD-SHAFRANOV EQUATION

The asymptotic analysis of relativistic outflows has been developed in previous works (Li, Chiueh & Begelman 1992; Appl & Camenzind 1993a,b; Eichler 1993; Begelman & Li
1994; Tomimatsu 1994), and the logarithmic dependence of the asymptotic structure on \(x\) has been pointed out. However, the transition from magnetic-energy dominated state into kinetic-energy dominated one is an important unsolved problem which is beyond the usual scheme of the asymptotic analysis based on the naive approximation \(x \gg 1\) in the poloidal wind and Grad-Shafranov equations. In our approach presented in the previous section, such an approximation corresponds to

\[
f \approx (1 - \xi^2)x^2 \gg 1 ,
\]

for which we have

\[
\hat{M}^2 = \frac{1}{E\sqrt{1 - \xi^2} - 1}
\]

from equation (30). (This should be the case of \(\xi \leq \xi_c\).) The key point missed in this calculation is that the fine-tuning of \(1 - \xi^2 = O(1/E^2)\) to realize the energy equipartition \((\hat{M}^2 \sim 1)\) may occur at a radius \(x\) in the range \(1 \ll x \leq E\). Then, to discuss the energy conversion in the range \(1 \ll x \leq E\), which is called the “intermediate” range of \(x\) in this paper, we must analyze the evolution of \(\hat{M}^2\) without assuming \((1 - \xi^2)x^2\) to be very large.

The existence of the intermediate range of \(x\) is a main feature of highly relativistic outflows with a very large specific energy \((E \gg 1)\). Recalling that we obtain \(\hat{M}^2 \approx c/E\) at the light cylinder surface \(x = 1\), we expect \(\hat{M}^2\) to increase from a sub-fast-magnetosonic value in the range \(1/E \ll \hat{M}^2 \ll 1/E^{2/3}\) to a rough equipartition value of \(\hat{M}^2 \sim 1\) as outflows propagate in the intermediate region. The fast-magnetosonic point at which we have the value of \(\hat{M}^2 = 1/E^{2/3}\) can be involved in this region. The usual asymptotic analysis becomes valid only in the region \(x \gg E\), where \(\hat{M}^2\) may increase logarithmically with \(x\), and our purpose here is to give the more precise treatment valid both in the intermediate and asymptotic regions.

The Grad-Shafranov equation can be written in the form

\[
\frac{M^2 + x^2 - 1}{8\pi^2} \nabla \left( \frac{M^2 + x^2 - 1}{R^2} \nabla \Psi \right) = s_1 + s_2 ,
\]

if no gravity is included (see, e.g., Tomimatsu 1994). The source terms \(s_1\) and \(s_2\) are given by

\[
s_1 = \frac{d}{d\Psi}(Ek)^2 - \frac{1}{R^2} \frac{d}{d\Psi}(Lk)^2 - \frac{1}{M^2} \frac{d}{d\Psi}(ek)^2 ,
\]

and

\[
s_2 = -\frac{M^2 + x^2 - 1}{M^2} \frac{d}{d\Psi} k^2 - \frac{R^2 k^2}{M^4} (M^2 + x^2 - 1 + e^2) \frac{d}{d\Psi} \Omega_F^2 .
\]

Here we regard \(M^2, \partial \Psi/\partial R \equiv \Psi_R\) and \(\partial \Psi/\partial Z \equiv \Psi_Z\) in equation (35) as functions of the variables \(R\) and \(\Psi\), instead of \(R\) and \(Z\). Further, by virtue of the introduction of \(\xi\), from
equations (15) and (21) we find the relation

$$M^2 + x^2 - 1 = 4\pi k \left( E - \frac{L\Omega_F}{x^2} \right) \frac{\xi}{B_p},$$

(38)

which is substituted into the left-hand side of equation (35). Then, after some manipulation we obtain

$$\frac{1}{\Psi_R} \frac{\partial}{\partial R} \left[ \xi^2 k^2 \left( E - \frac{L\Omega_F}{x^2} \right)^2 \right] = \tilde{s}_1 + s_2,$$

(39)

where \( q \equiv -\Psi_Z/\Psi_R \) means the slope of a poloidal magnetic field line, and the modified source term \( \tilde{s}_1 \) is

$$\tilde{s}_1 = -\frac{\partial}{\partial \Psi} \left[ \left( E - \frac{L\Omega_F}{x^2} \right)^2 k^2 \xi^2 \right] + s_1.$$

(40)

Now we derive the approximated form of the Grad-Shafranov equation valid for highly relativistic outflows (with \( E \gg e \geq 1 \)) propagating in the intermediate and asymptotic regions, where we obtain

$$1 \gg 1 - \xi^2 + \frac{\xi^2}{x^2} \simeq 1 - \frac{\xi^2}{x^2} \simeq \frac{1}{E^2} \left( 1 + \frac{1}{M^2} \right)^2.$$

(41)

as a result of the approximation \( M^2 + x^2 - 1 \simeq (M^2 + 1)x^2 \) in equations (38) and (21). It is interesting to note that the source terms are reduced to the compact form

$$\tilde{s}_1 + s_2 \simeq (1 + \dot{M}^2)\Omega_F^2 \frac{\partial}{\partial \Psi} \left( \frac{k^2}{\Omega_F^2 \dot{M}^4} \right).$$

(42)

To see clearly the fine-tuning of \( \xi \), we rewrite it as follows

$$\xi^2 \simeq 1 - \frac{\beta}{E^2},$$

(43)

where we define

$$\beta \equiv \left( 1 + \frac{1}{M^2} \right)^2 - \frac{1}{x^2}.$$

(44)

The renormalized variable \( \tilde{x} \equiv x/E \) is also useful to discuss the evolution of \( \dot{M}^2 \) in the regions considered here.

The observations of AGN jets (and possibly GRB jets) show the existence of highly collimated outflows with very narrow opening angles, for which we can assume that \( q^2 \ll 1 \) in equation (39). Hereafter, we consider only the case of \( q < 1/E \) to give

$$R\Psi_R \simeq R^2 B_p \simeq \frac{4\pi k E}{\Omega_F^2 (1 + M^2)}.$$

(45)
This allows us to eliminate $\Psi_R$ from the left-hand side of equation (39), in which we also have $[E - (L\Omega_F/\chi)]^2 \simeq E^2 - (2/\tilde{x}^2)$. Then, we arrive at the final form of the reduced Grad-Shafranov equation

$$ R \frac{\partial \beta}{\partial R} = -\frac{4\pi E}{k} \frac{\partial}{\partial \Psi} \left( \frac{k^2}{\Omega_F^2 M^4} \right) + \frac{4}{\tilde{x}^2}. \tag{46} $$

If equation (44) is substituted into equation (46), we can obtain the partial differential equation of first order for $\hat{M}^2$ as follows,

$$ \frac{R(\hat{M}^2 + 1)}{\hat{M}^6} \frac{\partial \hat{M}^2}{\partial R} + \frac{b}{\hat{M}^6} \frac{\partial \hat{M}^2}{\partial \Psi} = \frac{a}{\hat{M}^4} - \frac{1}{\tilde{x}^2}, \tag{47} $$

where

$$ a = \frac{4\pi E}{\Omega_F} \frac{\partial}{\partial \Psi} \left( \frac{k}{\Omega_F} \right), \tag{48} $$

and

$$ b = \frac{4\pi E k}{\Omega_F^2}. \tag{49} $$

In the intermediate region ($\tilde{x} \leq 1$) the role of the term $1/\tilde{x}^2$ in equation (47) becomes quite important, while it has been completely neglected in the asymptotic analysis corresponding to the case of $\tilde{x} \gg 1$. If a solution $\hat{M}^2(R, \Psi)$ is derived from equation (47), it is straightforward to find the flux function $\Psi(R, Z)$ from equation (45). In the next section we will present the interesting example of a jet solution to discuss the energy conversion and the change of field configuration in the intermediate and asymptotic regions.

### 4. FIELD STRUCTURE OF JET FLOWS

Equation (47) is valid only for jets with small opening angles propagating beyond the light cylinder surface. Considering the limit $\Psi \to 0$ for such outflows, we expect the jet ejection to occur near the polar axis $R\Omega_F \to 0$, where the toroidal field $B_\phi$ should decrease in proportion to $R$. Noting that the small flux function $\Psi$ is also proportional to $R^2$, we estimate to be $E \sim E_m \sim \Psi/k$ in the injection region near the polar axis. Then, to obtain a large specific energy $E$ per one particle, the rest-mass energy loading rate $k$ per unit magnetic flux should also become small in proportion to $\Psi$. This boundary condition near the polar axis motivates us to consider the case such that $E$ and $\Omega_F$ are independent of $\Psi$, while $k = k_0(\Psi/\Psi_0)$, where $k_0$ and $\Psi_0$ are constants and should be given by the boundary conditions at some foot point in a plasma source. Then, we obtain

$$ b = a \Psi, \quad a = \frac{4\pi E k_0}{\Omega_F^2 \Psi_0}, \tag{50} $$
where $a$ is a dimensionless constant.

Further, let us recall that $\hat{M}^2 \sim 1/E$ near the light cylinder surface. Then, we expect $\hat{M}^2$ to be very small in proportion to $\hat{x}$ in the region $1/E \ll \hat{x} \ll 1$, from which equation (45) leads to $R \Psi_R \simeq a \Psi$. This means that for $a = 2$ we have $\Psi \propto R^2$, assuring the smooth matching to the inner solution valid near the polar region. (Of course, we cannot give the condition $a = 2$ to be necessary, because our analysis is limited to the outer solution valid in the range $R \Omega_F \gg 1$.) Fortunately, for the model with $a = 2$ and $b = 2 \Psi$, we can give the general solution of an analytical form for equation (47) as follows,

$$\frac{2\hat{x}^2(1 + \hat{M}^2)}{2\hat{x}^2 + \hat{M}^2} = \ln \left( \frac{2\hat{x}^2}{\hat{M}^2} + 1 \right) + D_1,$$

where $D_1$ is an arbitrary function of $D_2$ defined by

$$D_2 \equiv \frac{\Psi}{\Psi_0} \left( \frac{1}{\hat{M}^2} + \frac{1}{2\hat{x}^2} \right).$$

Applying equation (45) to this general solution, we find that $D_1$ and $D_2$ are arbitrary functions of $Z$, which should be determined by the additional boundary conditions for jet flows. We can study an essential feature of plasma acceleration in jet propagation from this simple model.

First let us discuss the results obtained from equation (51). Considering that $\hat{M}^2 \sim 1/E$ near the light cylinder $\hat{x}_L = 1/E$, we claim that $\hat{x}^2/\hat{M}^2 \sim \hat{x} \ll 1$ in the range $1/E < \hat{x} \ll 1$, from which equation (51) leads to

$$2\hat{x}^2(1 - \hat{x}^2/\hat{M}^4) \simeq D_1(Z).$$

for all $Z$. This equation means that the absolute value of $D_1(Z)$ should be chosen to be at most of order of $1/E^2$ for all $Z$. In fact, if $|D_1| \gg 1/E^2$ for some $Z$, we can easily see that equation (53) breaks down in the range $1/E^2 < \hat{x}^2 \ll |D_1|$. Hence, we can discuss the increase of $\hat{M}^2 \simeq E_k/E_m$ in the range $\hat{x} \gg 1/E$ under the choice of $D_1 = 0$, namely, according to the following equation

$$\frac{2\hat{x}^2(\hat{M}^2 + 1)}{2\hat{x}^2 + \hat{M}^2} = \ln \left( \frac{2\hat{x}^2}{\hat{M}^2} + 1 \right).$$

It is now clear that we have $\hat{M}^2 \simeq \hat{x}$ in the range $1/E \ll \hat{x} \ll 1$, and the outflows can pass through the fast-magnetosonic point (i.e., $\hat{M}^2 = 1/E^{2/3}$) located on the radius $R \Omega_F \simeq E^{1/3}$. If the outflows can arrive at the radius $\hat{x} \simeq 1.4$, the equipartition $E_k = E_m$ between magnetic and kinetic energies is realized. In the asymptotic region $\hat{x} \gg 1$ we can confirm
the logarithmic increase of $\dot{M}^2$ given by $\dot{M}^2 \simeq \ln(2\tilde{x}_L^2/\dot{M}^2)$ toward the complete conversion of magnetic to kinetic energy. The numerical solution of equation (54) is shown in Figure 2a. The corresponding poloidal velocity $u_p(\tilde{x})$ and Lorentz factor $\gamma(\tilde{x})$ are also shown in Figure 2b, where the Lorentz factor is given by $\gamma = E - u_p(\xi \dot{M}^2)$. In this case, we obtain $\xi(\tilde{x}) < 1$ anywhere for the ratio of the poloidal electric to toroidal electric magnetic field amplitude (see Fig.2c). Note that the Lorentz factor includes both the poloidal and toroidal motion. In Fig.2b, the difference between $u_p$-value and $\gamma$-value near the light cylinder is due to the dominated toroidal motion of the plasma. (Just around the light cylinder $\tilde{x} \sim \tilde{x}_L$, the properties of the flow shown in Fig. 2 may be incorrect because of our approximations, but we can expect correct features at least near and outside the fast magnetosonic point, $\tilde{x} \geq \tilde{x}_F$.)

Next, we discuss the field configuration given by equation (52). The jet flows may be confined by an external pressure (see, e.g., Li 1993; Begelman & Li 1994; Fendt 1997). If the shape $Z\Omega_F = H(\tilde{x})$ of the last flux surface $\Psi = \Psi_0$ is determined by the outer boundary condition, the function $D_2(Z)$ in equation (52) is fixed. For example, let us consider the radial last flux surface at an angle $R/Z \equiv \theta_0$ with the pole axis direction. (Because the solution can be applied only to jets with small opening angles, we must require that $\theta_0$ is at most of order of $1/E$.) Using the function $\dot{M}^2 = \dot{M}^2(\tilde{x})$ derived from equation (54), we obtain

$$D_2(\theta_0 \tilde{z}) = \frac{1}{\dot{M}^2(\theta_0 \tilde{z})} + \frac{1}{2(\theta_0 \tilde{z})^2}$$

along a flux function $\Psi(R, Z) = \text{constant}$, where $\tilde{z} \equiv Z\Omega_F/E$. Note that for the value of $\dot{M}^2(\theta_0 \tilde{z})$ in equation (55) we use the function $\dot{M}^2 = \dot{M}^2(\tilde{x})$, where the variable $\tilde{x}$ should be replaced to $\theta_0 \tilde{z}$. Then, in the range $\theta_0 Z\Omega_F/E \ll 1$ of $Z$, equations (52) and (55) leads to the conical field configuration

$$\frac{Z}{R} \simeq \frac{1}{\theta_0 \sqrt{\Psi_0 / \Psi}}.$$ 

This shape of field lines changes as $Z$ increases, and in the asymptotic region we find the leading behavior such that

$$\frac{\theta_0 Z\Omega_F}{E} \simeq \left( \frac{R\Omega_F}{E} \right)^{\Psi_0 / \Psi},$$

which represents a paraboloidal collimation of the inner field lines for $\Psi < \Psi_0$. Figure 3 shows the shape of the magnetic field lines, which can be solved numerically by using equations (55) and (52) with the function $\dot{M}^2 = \dot{M}^2(\tilde{x})$ (or $\dot{M}^2 = \dot{M}^2(\theta_0 \tilde{z})$) derived from equation (54).

From the solution (54) we note that the ratio $\dot{M}^2$ of kinetic to magnetic energy is completely determined by the propagation radius $\tilde{x} = R\Omega_F/E$, irrespective of the field
configuration given by equation (52). If the last flux surface is asymptotically cylindrical with the outermost radius $\tilde{x}_0$, giving the function $D_2(Z)$ approaching a constant value in the limit $Z \to \infty$, the maximum value of $\dot{M}^2$ in the cylindrical jets is crucially dependent on $\tilde{x}_0$. For narrower jets with $\tilde{x}_0 < 1$ the magnetic energy is still dominant even in the asymptotic region.

The outer flux surface corresponds to a larger radius $\tilde{x}$ if compared with a fixed vertical distance $Z$. Thus, we can claim also that the energy conversion along outer flux surfaces of larger $\Psi$ becomes more efficient. This tendency of the efficient energy conversion is due to the condition for the rest-mass energy loading rate per unit magnetic flux such that $k \propto \Psi$. In fact, if we consider the case that $E$, $\Omega_F$ and $k$ are independent of $\Psi$, we find a different dependence of $\dot{M}^2$ on $\tilde{x}$ from the general solution for equation (47) with $a = 0$ and a constant $b$, which is given by

$$\left( \frac{1}{\dot{M}^2} + 1 \right)^2 + \frac{1}{\tilde{x}^2} = F_1^2, \quad (58)$$

and

$$\exp \left( \frac{2\Psi}{b} \right) = \tilde{x}^2 F_2^2 \left( \frac{\dot{M}^2 F_1 - \dot{M}^2 - 1}{\dot{M}^2 F_1 + \dot{M}^2 + 1} \right)^{1/F_1}. \quad (59)$$

From equation (45) we can check that $F_1$ and $F_2$ are arbitrary function of $Z$. If we assume the behavior of these functions in the limit $Z \to \infty$ such that $F_1 \simeq 1 + (1/\ln Z)$ and $F_2$ remains finite, we can confirm the increase of $\dot{M}^2$ in logarithmic scale of $Z$ and the paraboloidal shape of field lines similar with equation (57) in the asymptotic region. However, it is clear from equation (58) that $\dot{M}^2$ decreases as $\tilde{x}$ increases under a fixed $Z$. The energy conversion becomes less efficient on outer flux surfaces. Further we note that the solution (58) fails to give a real value of $\dot{M}^2$ in the range $\tilde{x} \ll 1$ under a fixed $Z$. Such a region should be covered by inner flux surfaces corresponding to small $\Psi$, and the previous solution (54) with $k \propto \Psi$ should be used there. One may consider a more realistic dependence of $k$ on $\Psi$. Then, a smooth change of $\dot{M}^2$ from equation (54) to equation (58) will be allowed as $\Psi$ increases in the range $0 < \Psi < \Psi_0$.

It is sure that according to a choice of the integrals of motion as functions of $\Psi$ we can obtain various evolutionary models of energy conversion in jet flows, which may show more complicated behaviors different from the simplest solution (54). However, we would like to emphasize the key result obtained here that a rough equipartition between magnetic and kinetic energies is realized at the radius $R \sim R_L E$ far beyond the light cylinder radius $R_L$, and the subsequent logarithmic increase of kinetic energy goes on at larger radii $R \gg R_L E$. This potentiality of MHD acceleration will be robust at least under the boundary condition for inner flux surfaces $\Psi \to 0$ such that the rest-mass energy loading rate is given by $k \propto \Psi$.\]
to keep the total specific energy $E$ very large. If so, we can claim that there exists the critical scale given by $R = R_L E$ for conversion of Poynting flux injected in jet flows, which is important for discussing a prompt emission of radiation owing to dissipation of kinetic energy of bulk motion.

5. SUMMARY AND DISCUSSION

We have presented a parametric representation of Alfvén Mach number $M$ by the ratio $\xi$ of poloidal electric to toroidal magnetic field strength, to avoid the troublesome analysis of the critical condition at the fast-magnetosonic point. Then, from the poloidal wind equation we have easily derived trans-fast-magnetosonic solutions including the parameter $\xi$ defined as a smooth function along a field line. If the parametric function $\xi$ is asymptotically larger than the critical value $\xi_c$, the flux surface have been shown to be confined within a finite radius $x = x_c$. Otherwise, the dynamical fine-tuning of $\xi$ have been required for acceleration to highly relativistic bulk speeds. To determine the parametric function $\xi$, we have also given the approximated form of the Grad-Shafranov equation applied to jets ejected with a very large total specific energy $E$ and confined within a very small opening angle of order of $1/E$.

Acceleration of outflows in generic models with the integrals of motion $E$, $\Omega F$ and $k$ variously dependent on $\Psi$ has not been analyzed in this paper. The comparison of the two models given by equations (54) and (58) suggests a variability of acceleration efficiency depending on various choices of the integrals of motion. Nevertheless, the former solution (54) studied in the previous section will be interesting as a typical model revealing the high potentiality of MHD acceleration, which was also discussed by Okamoto (2002) in relation to pulsar winds (see also Michel (1969); Begelman & Li (1994)). By virtue of the closed-form expression (54) of the model, we can clearly understand the following evolution of jet flows in the intermediate and asymptotic regions far beyond the light cylinder, if the jet radius $R$ extends to an infinite distance: (1) The magnetic-energy dominated and sub-fast-magnetosonic outflows pass through the light cylinder surface ($R = R_L$) with the Alfvén Mach number such that $M^2 \sim 1/E$. (2) Then, the energy ratio $E_k/E_m$ increases to $1/E^{2/3}$ at the fast-magnetosonic point corresponding to the radius $R \sim R_L E^{1/3}$. In the intermediate region ($R_L \ll R \leq R_L E$) the outflows can smoothly become super-fast-magnetosonic. (3) Importantly, the model claims the realization of rough equipartition between kinetic and magnetic energies at the radius of order of $R_L E$. (4) The further energy conversion toward a kinetic-energy dominated state in logarithmic scales of $R$ in the asymptotic region ($R \gg R_L E$) is also confirmed. Figure 4 is the summary of this jet solution.
The full conversion of magnetic energy into kinetic one means that the asymptotic Lorentz factor of bulk motion becomes equal to the total specific energy $E$ of outflows injected near the central source. Hence, for kinetic-energy dominated jets observed with a huge bulk Lorentz factor $\gamma$, we can expect the rough equipartition $E_k \sim E_m$ to occur at the jet radius $R_{\text{jet}} \equiv R_L \gamma$ [Note that the value of $\gamma \sim E_k$ is roughly same order with $E$]. Though we have considered ideal MHD flows in this paper, the observed jet activity, such as a prompt emission of radiation and a ultra relativistic acceleration of electrons, should be due to dissipation of the power of bulk motion, for example, through formation of shocks. Then, the interesting high-energy phenomena of jets will be observed, only after the kinetic energy of bulk motion begins to dominate, namely, the jet radius extends to this critical radius $R_{\text{jet}}$. The shock formed in the energy equipartition region would be distinct from shocks formed in the kinetically dominated asymptotic region by observations. For the kinetically dominated flows, the compression ratio behind the shock is much higher, and flatter synchrotron spectra, higher emissivities, etc. would be observed (Begelman & Li 1994).

In particular, for AGN jets with $\gamma \sim 10$ (see, e.g., Ghisellini et al. 1993), we can estimate the critical radius to be $R_{\text{jet}} \sim 10R_L$. When magnetic fluxes for the jet connect to a rotating geometrically thin disk around a black hole, we can regard $\Omega_F$ as $\Omega_K(r)$, where $\Omega_K(r)$ is the angular velocity for circular equatorial orbit in the Kerr metric, which corresponds to the Keplerian angular velocity in the Newtonian case. If the foot points of most magnetic fluxes distribute near the inner part of the disk $r \sim r_{\text{ms}}$, the angular velocities of the magnetic field lines are roughly $\Omega_F(\Psi) \sim \Omega_K(r_{\text{ms}})$ (see, e.g., Camenzind & Krockenberger 1992). Then, the light cylinder radius $R_L = 1/\Omega_F$ for field lines threading a rotating black hole and the disk’s inner part is possibly equal to several Schwarzschild radii. Hence, our conjecture from the jet model obtained here is that the active region in AGN jets, far from the central source, has the radius as large as a hundred of Schwarzschild radii, corresponding to 0.01 pc for a $10^9 M_\odot$ black hole. This result is consistent with observational results of blazars (Kataoka et al. 2001) and of the M87 jet (Junor, Biretta, & Livio 1999). (For GRB jets expected to have higher Lorentz factors $\gamma > 100$ of bulk motion, the size of the emission region will be larger than a thousand of Schwarzschild radii.) The dependence of the radius of the emission region on the bulk Lorentz factor is an interesting problem to be checked by observations of AGN jets.

Finally let us discuss the vertical distance $Z_{\text{jet}}$ from the central source, which gives the critical radius $R_{\text{jet}}$ on a field line. We have postulated jet flows confined within a small opening angle of order of $1/E$ and derived the solution valid only in the region $Z \geq ER$ from the Grad-Shafranov equation. We can obtain the field configuration according to equation (52) under a suitable boundary condition on the outermost flux surface. As an example, in the previous section, we have considered the case with the radial outermost
flux surface given by $Z = R/\theta_0$. If the opening angle $\theta_0$ is equal to $1/\gamma$, we can claim that the kinetic energy of bulk motion begins to dominate at the vertical distance $Z_{\text{jet}}$ larger than $\gamma R_{\text{jet}} = \gamma^2 R_L$, far from the central source. (The value of $Z_{\text{jet}}$ becomes larger on inner flux surfaces corresponding to smaller $\Psi$.) This will predict a sub-parsec distance to the active region in an AGN jet from a $10^9 M_\odot$ black hole. We expect that at this distant region ($Z > Z_{\text{jet}}$) internal shocks form and make the active region in the jet. The internal shock scenario is generally thought for gamma-ray bursts, radio-loud quasars and blazars (see, e.g., Mezaros & Rees 1993; Spada et al. 2001; Ghisellini & Celotti 2002). In this scenario, it is assumed that the active region powered by collisions among different part of jet itself, moving at different bulk Lorentz factors. Although we consider stationary flows in the magnetosphere, we can also expect some kinds of plasma instability (e.g., the screw instability of plasma in black hole magnetosphere discussed by Tomimatsu, Matsuoka & Takahashi (2001)) to make blobs of plasma in the jet. If the cylindrical collimation of the outermost flux surface develops in the range $R < R_{\text{jet}}$, however, the vertical scale $Z_{\text{jet}}$ should be much larger. Because we have not discussed physical mechanisms (possibly due to external matter) of the confinement of the magnetic flux, the boundary condition on the outermost flux surface to determine the field configuration remains arbitrary. This is a problem beyond the scope of the present paper, which is important for giving a more definite estimation of the vertical distance from the central source to the active region in AGN jets.

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Fig. 1.— Examples of outgoing flow solutions (thick curves) given by equation (15) with (a) \( \xi = 0.8 \) and (b) \( \xi = 1.1 \), where \( E = 10.0 \) and \( x_A = 0.8 \). The curves of the Mach numbers corresponding to the Alfvén wave speed and the fast-magnetosonic wave speed, namely, \( M^2 = M^2_{AW} \) and \( M^2 = M^2_{FW} \), are also plotted, where \( M^2_{AW} = 1 - x^2 \) and \( M^2_{FW} = 1 - x^2 + (x^2/\xi^2) \). The crossings of these curves with the flow solution labeled by “A” and “F” are the Alfvén and fast-magnetosonic points, respectively.
Fig. 2.— A solution of the approximated Grad-Shafranov equation given by equation (47) with $E = 10.0$. (a) The Alfvén Mach number with two limiting curves ($\hat{M}^2 \simeq \tilde{x}$ for $x_L \ll \tilde{x} \ll 1$ and $\hat{M}^2 \simeq \ln(2\tilde{x}^2/\hat{M}^2)$ for $\tilde{x} \gg 1$), (b) the poloidal velocity $u_p(\tilde{x})$ (solid) and the Lorentz factor $\gamma(\tilde{x})$ (dashed), and (c) the ratio of the poloidal electric to toroidal electric magnetic field amplitude $\xi(\tilde{x})$. This solution is valid for $\tilde{x} > \tilde{x}_L = 1/E$. 
Fig. 3.— The magnetic field configuration corresponding to the solution given by Fig.2. It is assumed that the jet flow is confined within a cone of small opening angle $\theta_0$ by an external pressure. For the inner field lined of $\Psi < \Psi_0$, a paraboloidal collimation is obtained.
Fig. 4.— Schematic picture of the jet solution given by Fig.3. The term $\theta_0$ is the opening angle of the last magnetic flux surface. The initial magnetically dominated outflow $E_m \gg E_k$ accelerates along an almost radial field line at the intermediate region $\tilde{x}_L < \tilde{x} < 1$ (region I), and then a rough equitation $E_k \sim E_m$ is realized around the radius $\tilde{x} \sim 1$ far beyond the light cylinder (region II). Subsequent logarithmic increase of $E_k$ goes on at large radii $\tilde{x} \gg 1$ (region III) along a collimating field line.