Radiative constraints on brane quintessence

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Abstract. We investigate the constraints on quintessence arising from both renormalisable and non-renormalisable couplings where the 5d Plank mass is around the TeV scale. The quintessence field vacuum expectation value is typically of order the 4d Planck mass, while non-renormalisable operators are expected to be suppressed by the 5d Planck mass. Non-renormalisable operators are therefore important in computing the 4d effective quintessence potential. We then study the quantum corrections to the quintessence potential due to fermion and graviton loops. The tower of Kaluza–Klein modes competes with the TeV-scale cut-off, altering the graviton contribution to the vacuum polarization of quintessence. Nevertheless we show that, as in four dimensions, the classical potential is stable to such radiative corrections.

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1. Introduction

The case for the existence of a Λ-like component dominating the currently observable universe is now compelling [41]. In the simplest models such a component has two quite independent properties. On the one hand, it does not cluster on scales much smaller than the Hubble scale, $H^{-1}$, and on the other it influences the background evolution of the cosmos, causing acceleration at very recent redshifts and giving rise to the coincidence problem – why do we appear to be living at a special time in the Universe’s history? While it is possible to construct models which exhibit only one of these properties, eg. [6, 13], both now have observational support. In particular, the recent detection of cross-correlations between the Wilkinson Microwave Anisotropy Probe (WMAP) cosmic microwave background (CMB) anisotropies and various tracers of large scale structure [9, 41, 40, 17], which are consistent with the decay of perturbations on large scales due to an accelerating background, make construction of convincing non-accelerating models difficult.

However, despite its success at the purely phenomenological level, the standard ΛCDM model has almost no deep understanding to back it up. We are in the age of precision book-keeping in cosmology, but despite many attempts we do not yet have even a well-founded theoretical order-of-magnitude estimate of the size of the cosmological constant: most naïve field theory calculations disagree by $O(10^{120})$ with observations, yielding perhaps the worst estimate in the history of physics. One can improve the situation somewhat by invoking supersymmetry, but it proves generically quite hard to construct supergravity vacua with positive cosmological constant [44]. The string theory case is even harder [26]. In the absence of any theoretical control over Λ itself, there is a strong temptation to explain the observations by invoking some other mechanism. Some proposals utilise the large number of possible string theory vacua, either by appealing to the anthropic principle [42] or other quantum effects [27], but a more moderate approach is simply to include, among the matter inventory of the universe, some tensile matter with appropriate equation of state whose behaviour is under good control. This allows us to set $Λ = 0$ by supposing that one or more of the string theory proposals for cancelling Λ applies, and then to exclude the complexities of Λ itself and deal instead with the relatively well-understood properties of matter. We will argue that, at least in TeV-scale models of quintessence, control is not manifest even in this case.

Quintessence consists of a scalar field $Q$, which drives a late-time accelerated cosmological expansion via its vacuum expectation value in a rather similar way to scalar-field driven inflation. If $Q$ is still rolling today then it must be very light in order to satisfy the standard slow-roll conditions and hence its Compton wavelength, $\lambda_c \approx V_Q^{-1/2}$, is very large (we denote first, second, ..., $Q$-derivatives of $V$ as $V_Q$, $V_{QQ}$, ..., etc.) As a result it only clusters on very large scales, typically greater than 100 Mpc.

This is not necessary. In models involving the Albrecht–Skordis potentials, where the dark energy reaches a minimum of the potential at non-zero energy, the mass and
expectation value of the quintessence is arbitrary, and so the dark energy may cluster on all scales after reaching the minimum. Such models are attractive for another important reason, for if the quintessence is very light (it typically has a mass \( m_Q \sim 10^{-33} \text{ eV} \)), then we must find a way to protect this mass from radiative corrections which will otherwise spoil the flatness of the potential [28] (see also, eg., Refs. [11, 37]). This has been studied at the one-loop level [15]. The result depends on which particle species one includes in the loops. One typically finds that couplings to bosons are benign [15], but couplings to fermions are severely constrained. The bounds found by the authors of Ref. [15] are extremely stringent and give rise to concern that gravitational couplings alone might be strong enough to violate them. Estimates presented in Ref. [15] show that quintessence is safe, but this safety is model-dependent and must be assessed carefully.

In addition, \( Q \) must be extremely weakly coupled to standard model fields, otherwise it is difficult to see how it could have evaded detection via particle physics or cosmological interactions. Despite their overall weakness, such couplings can alter standard cosmology in an interesting way [43, 3, 33], but obtaining them requires significant fine-tuning of the renormalisable couplings.

A more worrying problem is provided by constraints from 4d non-renormalisable couplings between the standard model and the quintessence field. Such couplings are generically expected from supergravity and string theory, and are problematic in ‘tracking’ quintessence models which generally have Planckian vacuum expectation values (vevs). Couplings such as \( \beta Q F^2/M \), where \( F^2 \) is the usual Maxwell Lagrangian and \( M \) is the mass-scale at which we expect supergravity to fail as an effective theory cause variation of the fine-structure constant, and because of the large \( Q \)-vev require fine-tuning of the dimensionless coupling \( \beta \) of order \( \beta < 10^{-5} \). Since we have no reason to expect \( \beta \) to differ significantly from order unity, this unexplained fine-tuning is unsettling. Carroll [12] has argued that such dimension-five operators may be excluded by the existence of a discrete \( Z_2 \) symmetry in the fundamental description, which acts on the extra dimension as \( \phi \rightarrow -\phi \), but even in this case such fine-tuning persists with higher-order operators of the form \( Q^n F^2/M^n \) [36]. One of our aims is to consider the effect of such non-renormalisable couplings in models with a low-scale of quantum gravity.

There are many constraints one can consider. Despite arising from a variety of different physics, these bounds and the constraints on fermion couplings arising from stability of the classical potential share a common feature: they are sensitive to some power of the ratio \( M/M_{\text{cut-off}} \), where \( M \) is some energy scale characteristic of the process in question, and \( M_{\text{cut-off}} \) is an energy scale which controls the details of heavy physics, which we consider to have been integrated out in our effective description.

In the conventional kind of four-dimensional cosmology one would usually take \( M_{\text{cut-off}} \) to be of order the Planck mass \( M_P = G^{-2} \approx 10^{19} \text{ GeV} \), although there are other natural candidates: the GUT scale \( M_{\text{GUT}} \) around \( 10^{15} \text{ GeV} \); the string scale, possibly a few orders of magnitude less than the Planck mass; or, more speculatively, the SUSY scale, which may be as low as \( M_{\text{SUSY}} \sim 100 \text{ GeV} \). In recent years, inspired by ideas
from the strongly coupled limit of the heterotic string [25, 30, 31, 30, 38] an alternative scenario for cosmology has become popular, in which the various gauge and matter fields which comprise our universe are affixed to a hypersurface in a larger, five-dimensional bulk spacetime. This spacetime is generically a patch of Schwarzschild–anti de Sitter (SAdS) space. In these models, the fundamental scale $M_P$ of quantum gravity might be much lower, perhaps only of order a TeV ($10^{-16} M_P = 10^{12}$ eV) or so, in which case one would expect to obtain very different constraints on quintessence. (However one should note that in these models, one cannot really be dealing with the heterotic string, since in such theories the string scale—which amounts to the quantum gravity scale—is fixed to roughly coincide with the four-dimensional Planck scale and there is not much freedom to move it. On the other hand, theories such as Type I string theory can acceptably accommodate a low string scale.) Because the troubling bounds and fine-tunings outlined above depend sensitively on the details of the cutoff scale, one should carefully recalculate the constraints they impose in models with quantum gravity at low energies, but this is not sufficient. There are other effects which one should also take into account, arising from new physics associated with the branes. Most notably, for example, these models contain a tower of Kaluza–Klein modes in addition to a massless four-dimensional graviton. These Kaluza–Klein modes can be considered to arise from gravity in the bulk. These modes are typically massive, with masses $m > 3H/2$ where $H$ is the brane Hubble parameter [29, 21, 18]. The presence of these modes introduces processes, absent in four dimensions, where quintessence can interact with gravity off the brane, or with the Kaluza–Klein hierarchy.

In this paper, we apply all these ideas to constrain quintessence couplings and energy scales in TeV scale Planck mass models. Higher dimensional and brane-world models offer interesting new insights into quintessence cosmology [34, 35, 10, 23, 1], but before adopting these models wholesale it is important to consider potential constraints and compare them with the corresponding constraints on 4d quintessence [15, 24, 12].

The plan of this paper is as follows. In the next section we discuss the issue of non-renormalisable couplings. Then we briefly review the bounds on quintessence couplings in four dimensions, paying particular attention to bounds on couplings to fermions. In Section 4 we calculate probability amplitudes for some representative gravitational processes, where bulk gravitons mediate quintessence couplings to fermions. We also estimate the lowest-order contribution of virtual graviton exchange to the vacuum polarization $\Pi^\ast(p)$ of the quintessence field. In Section 5 we state our conclusions. In an appendix, we give a brief derivation of the gravitational propagator in the brane world, using the Fadeev–Popov technique. This has appeared in the literature before [20] but we present this alternative derivation for simplicity and to make our account self-contained.

Throughout we work in units where $\hbar = 1$, but the gravitational coupling in $D$ dimensions is $\kappa_d^2 = 8\pi/M_P^{D-2}$, where $M_P$ is the $D$-dimensional Planck mass. We use eV as units of dimensionful quantities everywhere.
2. Non-renormalisable couplings between quintessence and the standard model

One of the original motivations for TeV-scale quantum gravity was its ability to obviate the need for low-energy supersymmetry by removing the huge hierarchy between the weak and Planck scales. Such a low scale for quantum gravity can be achieved rather elegantly in models of large extra dimensions or in brane-world models with a non-compact extra dimension.

Viewed from this perspective, the four-dimensional theory on the brane is simply an effective theory arising from the dimensional reduction of a more fundamental, higher-dimensional theory by integrating out physics at scales above which the extra dimensions become visible. As such, we must expect our effective four-dimensional theory to be burdened with a potentially infinite number of non-renormalisable interactions, suppressed by powers of the mass scale at which the effective theory breaks down or at which new physics enters the problem. In this case it is natural to assume the cutoff scale $M_{\text{cut-off}}$ to be the Planck scale of the higher-dimensional theory, $M_{\text{cut-off}} \sim M_P \sim \text{TeV}$ provided that this is lower than the effective 4d Planck scale. Above the scale $M_{\text{cut-off}}$ the theory is no longer well-approximated by a four-dimensional theory.

In the discussion that follows we will leave the cutoff mass scale $M_{\text{cut-off}}$ arbitrary. To minimise clutter, we write this scale as $M$. Our conclusions are strongest in models where $M \sim \text{TeV}$ and weaken as $M$ increases towards the four dimensional value $M_P \sim 10^{16} \text{ TeV}$.

Let us begin by considering non-renormalisable Lagrangian operators of the form (see eg. [12])

$$\beta_n \frac{Q^n}{M^n} \mathcal{L}_4$$

where $\mathcal{L}_4$ is any dimension-four standard-model operator such as $F^2$ or $G^2$. We are seeking constraints on the dimensionless coupling $\beta$, and in particular its dependence on the cut-off scale, as discussed above. The case $\mathcal{L}_4 = F^2$ leads to cosmic variation of the fine-structure constant $\alpha$ [12, 36] which, assuming slow variation of $Q$, gives

$$\Delta \alpha \simeq -n \beta_n Q^{n-1} \bigg| \frac{\Delta Q}{Q} \bigg|_{M^n}$$

where $Q = Q(0)$; the symbol $\Delta Q$ abbreviates the field interval $\Delta Q(z) = Q(z) - Q(0)$; and $z$ denotes redshift. The Webb et al results suggest a variation of $\alpha$ at the level $\frac{\Delta \alpha}{\alpha} = (-0.543 \pm 0.116) \times 10^{-5}$, ie. evidence for variation of $\alpha$ at the 4.7σ level [45]. For $n \geq 1$ the ratio $Q/M$ clearly plays a key role. We have argued that in TeV-scale higher dimensional models $M$ may be close to the TeV scale. The estimate of the vev $\langle Q \rangle$ is model dependent; however, we can obtain constraints on standard tracker quintessence models where the dark energy equation of state tracks that of the dominant energy component of the Universe until a low redshift $z \sim 1$. In such models the quintessence field has been rolling since very early times and it is natural that it has a large vev today, whether the fundamental theory be four- or higher dimensional.
We can get a rough lower-bound on the vev of $Q$ today by the following argument. To ensure that the Universe accelerates, the field $Q$ must satisfy the standard slow-roll conditions familiar from early universe inflation. That is, we demand $\varepsilon, \eta \ll 1$ where $\varepsilon \equiv M_P^2(V_Q/V)^2$ and $\eta \equiv M_P^2(V_{QQ}/V)$. At this point it is appropriate to comment on the mass scale appearing in $\varepsilon$ and $\eta$ and whether we really should be using the four-dimensional Planck mass $M_P$, or the fundamental scale $M$. To answer this one can examine the effective four-dimensional equation of motion for $Q$. Slow-roll requires that the potential term $V_Q$ be sub-dominant with respect to the friction term coming from the Hubble expansion. At least at low energies, the effective Friedmann equation giving the Hubble expansion typically contains $M_P$, not $M$, and hence the effective four-dimensional scale $M_P$ is the appropriate scale.

To quantify the constraints coming from requiring $\varepsilon, \eta \ll 1$, consider a standard tracking potential $V(Q) \propto Q^{-\gamma}$. In this case the field evolves as $Q \propto t^{2/(2+\gamma)}$ and the slow-roll parameter $\eta$ is

$$\varepsilon = \gamma^2 \frac{M_P^2}{Q^2} \implies Q \gg \gamma M_P$$

with a similar constraint arising from $\eta \ll 1$. Clearly one can make the vev as small as one likes by fine-tuning the exponent $\gamma$ to be sufficiently small but then one loses the attractiveness of the model, since one is simply converging to a cosmological constant. Even then the fine-tuning required is significant. To have $\eta \leq 1$ with $Q \leq \text{TeV}$ requires $\gamma \leq 10^{-16}$. This is quite undesirable.

With this restriction on the form of the potential in mind, let us return to the dimensionless couplings $\beta$. If we use the standard result that tracking models typically require vevs of order the 4d Planck scale, $Q \sim M_P$, then this implies that the ratio $Q/M$ can be as large as $M_P/\text{TeV} \sim 10^{16}$. Matching the Webb et al data for $n = 1$ is possible with either a fine-tuning of $\beta \sim 10^{-5}$ (the result of [12]) or is compatible with $\beta \sim 1$ by requiring $\Delta Q \sim 10^{-5}M = 10 \text{ MeV}$. Such a slow variation of the field is not consistent with the assumption of a rolling quintessence. However it is consistent with the Albrecht–Skordis [2] model where $Q$ becomes trapped at the minimum of a potential well. Interestingly, a more detailed analysis shows that the union of constraints on $\alpha$ at various redshifts favour very little variation of $Q$ at low redshifts $z < 2$ even when $M = M_P$ [36].

For $n > 1$ the situation is much worse, since the large ratio $(Q/M)$ appears. The required fine-tuning on the dimensionless couplings $\beta_n$, or variation $\Delta Q$, become enormous. For $n = 2$ we have $\beta_2 \Delta Q \sim 10^{-9} \text{ eV}$, or if we conservatively set $\Delta Q \sim M$ we find $\beta_2 \sim 10^{-21}$. Such a coupling is quite inexplicably small. As $n$ increases the fine-tuning on the dimensionless couplings $\beta_n$ rapidly increases. Of course this simply underlines a more fundamental point: since $(Q/M) \gg 1$ we have absolutely no control over the effective potential of the quintessence field. The potential should be computed from the higher-dimensional theory. This is a standard argument against chaotic inflation which typically requires super-Planckian initial conditions to obtain sufficient inflation and the correct amplitude of density perturbations.
Of course, one may simply argue that the non-perturbative potential for $Q$ is completely well-behaved and only gives rise to small couplings to the standard model fields. This is reminiscent of the runaway dilaton model of quintessence in which a massless dilaton runs to infinity where it decouples from all matter, as in the proposal of Damour & Polyakov. If the quintessence field is a radion, representing the distance between two branes, driven apart, for example, by a repulsive Casimir force, then we expect the $Q/M \to \infty$ limit to be trivial: the potential should vanish. This occurs because the $Q$ field dynamics effectively vanish in this limit: it is equivalent to the degenerate case where the branes approach each other. Couplings to gauge fields localised on the brane world could also reasonably be expected to vanish in that limit. However, to appreciate this one requires the full five-dimensional picture, and in other scenarios the corrections may not be so harmless.

3. Quintessence couplings in four dimensions

In the above analysis, we obtained restrictions on a given set of coupling constants and shape parameters for some popular, rather generic potentials. In doing so we assumed that the form of the potential could simply be given as an Ansatz, so strictly speaking we were dealing with renormalized quantities, and the potential was the quantum effective potential. A more subtle question is to ask how a given tree-level potential may be modified when quantum effects are taken into account. This involves the study of loop corrections to the quintessence potential and the couplings of $Q$ to other fields.

For scalar quintessence and fermions, this was first done by Doran & Jäckel and by Horvat, who considered couplings to neutrinos. We review their arguments as applied in four dimensions, and explain how this generalizes to the brane world. Many of the bounds described in Ref. do not depend on the scale of gravity and are not strongly modified in the brane world, so we focus on the gravitational couplings described in Ref. In particular, we are interested in the coupling of quintessence to fermions, for which the strongest bounds apply.

In the quintessence sector we work with the Euclidean action

$$S = \int d^4 x \sqrt{-g} \left( \frac{1}{2} \partial_a Q \partial^a Q + V(Q) + \bar{\psi}(\partial + m(Q)) \psi \right)$$

where $Q$ is the quintessence field, $V(Q)$ is its classical potential, $\psi$ is a Dirac fermion, and $m(Q)$ is a possibly $Q$-dependent fermion mass. The leading correction to the quintessence potential in the fermionic sector comes from the diagram of Fig. [1]

Let $m$ be given by a large field independent mass $m_0$ plus some correction $c$ generated by couplings to other fields. Then the condition that the classical potential dominates becomes

$$\frac{V^L}{V} = \frac{1}{4\pi^2} \frac{\Lambda^2 m_0 c}{V} \ll 1,$$

where we have discarded the $c$-independent piece $m_0^2$ which does not affect dynamics, and we are assuming that $c \ll m_0$. One can estimate $V$ by supposing that $Q$
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currently dominates the energy density of the universe, so that $V$ must be comparable to $\rho_{\text{crit}} = 8.1 \times 10^{-11} \ h^2 \ eV^4$. Setting $\Lambda$ at around the GUT scale $\Lambda = 10^{-3} \ MP$, and taking the field-independent fermion mass to be around the supersymmetry breaking scale, perhaps of order a TeV, or $m_0 = 10^{-16} \ MP$, that gives

$$c \ll 10^{-71} \ eV. \quad (6)$$

This calculation only depends on the details of quantum field theory in the four-dimensional world, so it is valid on the world volume of a brane universe provided that we take the effective cut-off $\Lambda$ to be sized appropriately. Since the bound on $c$ scales with $\Lambda^{-2}$, this means that a reduced cut-off will weaken any constraint. For example, in a model where $\Lambda$ should be $\mathcal{O}$(TeV) $\sim 10^{12} \ eV$, one finds $c \ll 10^{-44} \ eV$. This weakening is a mixed blessing. It is harder to rule out any given quintessence model, but it may make the construction of a viable phenomenological model easier.

4. Gravitationally mediated couplings in the brane world

This bound Eq. (6), in brane or four-dimensional form, is rather stringent and could conceivably be violated by gravitational couplings. The low order diagrams showing the gravitational coupling of quintessence to fermions are shown in Fig. 2. In four dimensions, both of these diagrams involve the classical quintessence potential, so in fact the bound Eq. (6) does not apply. This happens because one can absorb the corrections into a renormalization of $V(Q)$ [15]. We will compute the brane world
diagrams equivalent to Fig. 2.

The principal result we shall require is the propagator for gravitons in the bulk. As a first approximation, we work in the original Randall–Sundrum scenario where the branes are exactly flat. The metric is

\[ ds^2 = \frac{1}{\ell^2 z^2} \left( -dt^2 + dx^2 + dz^2 \right). \]

The propagator has been derived by Giddings, Katz & Randall [20], and is given by

\[ \Delta_{\text{on-brane}}^{r s m n} = \frac{2 \kappa^2}{\ell R \rho} K_{2}(k R) K_{1}(k R) \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \]

where \( K_{\nu} \) is the Macdonald or Basset function

\[ K_{\nu}(z) = \frac{1}{2\pi i^{\nu+1}} H^{(1)}_{\nu}(iz), \]

the brane is fixed at \( z = R \) (for RS branes, \( R = \ell^{-1} \)), and \( \rho^{r s m n} \) satisfies

\[ \rho^{r s m n} = \delta^{r m} \delta^{s n} - \frac{1}{d-2} \delta^{rs} \delta^{mn}, \]

where \( d = \delta_{i}^{i} \) is the trace of the \( \mathbb{R}^{3} \) Kronecker delta. This is 3 in the present case but changes if one sends \( \mathbb{R}^{3} \) to \( \mathbb{R}^{n} \). We give an alternative derivation of this result in an appendix.

It is convenient to use the Feynman rules in configuration space, rather than momentum space. This is because the brane matter theory has support only on the brane and its couplings naturally include a term \( \delta(z-R) \) which is most easily accommodated in the configuration space formulation. The presence of the brane breaks bulk translational isometries in the transverse direction, and there is no conserved Noether charge to play the role of a conserved momentum in the \( z \) direction. For this reason, loop diagrams will still involve an integration over four-momenta on slices \( z = \) constant, and not full five-momenta in the bulk.

The on-brane gravitational propagator is as given in Eq. (A.18), and the fermion and scalar propagators are as usual. We introduce a matter theory on the brane described by the analogue of the four-dimensional quintessence–fermion system

\[ S_{\text{brane}} = \int_{z=R} d^{4}x \sqrt{-\det h} \left[ \frac{1}{2} \partial_a Q \partial^a Q + V(Q) + \bar{\psi}(-i\tilde{\partial} + m)\psi \right] \]

where, \( h_{ab} \) is the pull-back of the five dimensional metric \( g_{ab} \) to the brane, \( Q \) is the quintessence field and \( \psi \) is a four-dimensional (not five-dimensional) Dirac fermion. We are ignoring any gravitational coupling to \( \psi \) via the spin connexion. In the Randall–Sundum case, \( h_{ab} \) is just four-dimensional Minkowski space plus the tensor perturbation \( e_{ij} \) evaluated at \( z = R \), so

\[ \sqrt{-\det h} = 1 - \frac{\text{tr } e}{2} + \frac{\text{tr } e}{4} + \frac{(\text{tr } e)^2}{8} + \cdots \]

The vertices for this theory are shown in Fig. 3 (See also, eg., Ref. [22].)

We can now proceed to evaluate the diagrams in Fig. 2 with the brane world graviton propagator.
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\[ V(q) = -im \left( \frac{\delta_{ij}\delta_{rs}}{8} - \frac{\delta_{i(r}\delta_{s)}j}{4} \right) \delta(x^z - R) \]

**Figure 3.** Some vertices in the quantum field theory corresponding to Eq. (11), where Dirac indices are suppressed. Fermions are solid lines. Gravitons are indicated by wavy lines, and quintessence by dashes. The vertices are written in configuration space and taken to occur at a spacetime point \( x \). Indices \( ij, rs \) on graviton lines refer to the \( \mathbb{R}^3 \) index structure. Fermions entering the diagram at a point \( x \) with momentum \( p \) carry a coefficient function \((2\pi)^{-3/2}u(p)e^{ip \cdot x}\), where \( \cdot \) denotes the flat, Euclidean inner product on the brane. Fermions leaving the diagram from a point \( x \) with momentum \( p \) carry \((2\pi)^{3/2}\bar{u}(p)e^{-ip \cdot x}\). Here, the constant functions \( u, \bar{u} \) are Dirac spinors with indices suppressed: our conventions for spinors and \( \gamma \)-matrices match Weinberg [46]. Quintessence particles entering or leaving the diagram with momentum \( p \) carry \((2\pi)^{-3/2}u(p)e^{ip \cdot x}\). To find amplitudes, one integrates over the coordinates \( x_1, x_2, \ldots \) of all interaction points. These integrals should include the appropriate volume measure \( \sqrt{-\det g} \), which is unity on the brane. In addition, scalar products are taken in the metric \( h_{ab} \) with is the pull-back of the spacetime metric \( g_{ab} \) to the slice \( z = \text{constant} \) over which they are evaluated; this reduces to the flat, Euclidean scalar product on the brane.

### 4.1. Loop diagram

Consider the first diagram in Fig. 2. The amplitude for this process is

\[ A_L = \frac{5i}{2} \frac{mR^2}{\ell^2} V(Q_{cl}) \frac{\bar{u}(p_1)u(p_2)}{(2\pi)^{3/2}} \delta(\sum p) \int d^4k_E \frac{1}{k} \frac{1}{k'} \frac{K_2(kR)}{K_1(kR)} \frac{K_2(k'R)}{K_1(k'R)} \bigg|_{k' = p_1 - p_2 - k}. \] (13)

where \( \sum p = p_1 - p_2 - p_{out} \) and \( d^4k_E \) is the four-dimensional Euclidean volume measure.

This diagram is difficult to evaluate for finite momenta \( p_1 \) and \( p_2 \), so we shall work in an approximation where all external three-momenta vanish: that is, \( p_1 = p_2 = p_{out} = 0 \). This approximation matches Ref. [15]. In this case \( k'^2 = k^2 \), and the loop integral becomes somewhat more tractable. In addition, in our conventions, the product of spinor coefficient functions \( \bar{u}(p')u(p) \) evaluates to 2 when summed over spins at zero momentum. Temporarily ignoring the various numerical pre-factors, one has to evaluate the integral

\[ \int_{\mu}^{\Lambda} k \, dk \left( \frac{K_2(kR)}{K_1(kR)} \right)^2. \] (14)
We have explicitly written in an upper cut-off at Euclidean momenta $k \sim \Lambda$ and a lower cut-off at $k \sim \mu$. This extremely simple regularization has the advantage that it is easy to apply in the present context.

The Macdonald functions $K_\nu$ have asymptotics governed by

$$K_\nu(z) \rightarrow \frac{\Gamma(\nu)}{2} \left( \frac{\nu}{2} \right)^{-\nu} \quad \text{as } z \rightarrow 0; \quad K_\nu(z) \rightarrow \sqrt{\frac{\nu}{2z}} e^{-z} \quad \text{as } z \rightarrow +\infty. \quad (15)$$

In the infra-red, aside from numerical factors, the ratio $K_2(kR)/K_1(kR)$, behaves as a function of $k$ like $k^{-1}$. Combining this behaviour with the factor of $k^{-1}$ already present in the propagator Eq. (A.18), one can see that any infra-red divergence ought to be the same as in four dimensions. However, the large-$z$ asymptotics of $K_\nu(z)$ changes the divergent behaviour in the ultra-violet. To make an estimate of Eq. (14), we write $f_\mu^\Lambda = f_\mu^{1/R} + f_1^{\Lambda/R}$ and approximate the integrand using its asymptotic form in both regions (after changing variable to $z = kR$),

$$\int_{1}^{\Lambda} k \, dk \left( \frac{K_2(kR)}{K_1(kR)} \right)^2 \approx \frac{4}{R^2} \int_{\mu R}^{1} \frac{dz}{z} + \frac{1}{R^2} \int_{1}^{\Lambda R} z \, dz \sim -\frac{4}{R^2} \ln \mu R + \frac{1}{2} \Lambda^2. \quad (16)$$

We have discarded a term of order $O(1/R^2)$, which should be a good approximation provided $\Lambda^2 \gg 1/R^2$. For an extra dimension of order 1 mm, $R^{-1} \sim 1.97 \times 10^{-4}$ eV, so this is abundantly satisfied. Eq. (16) lets us pick out the leading order divergence in the ultra-violet and infra-red by making a small- or large-$k$ approximation in the integrand, as appropriate.

One sees that the ultra-violet divergence, which is logarithmic in the four-dimensional case, is modified to a considerably worse quadratic divergence. It is natural to interpret this modification as due to interactions with the Kaluza–Klein tower. Despite this, the induced coupling remains proportional to the classical quintessence potential $V(Q)$, so this correction term does not destroy properties of the classical dynamics. This is entirely analogous to the situation in four dimensions.

### 4.2. Triangle diagram

The amplitude for the second (`triangle') diagram of Fig. 2 is

$$A_T = -\frac{m^2k^4V(Q_{cl})}{\ell^2 R^2 (2\pi)^{9/2}} \delta(\sum p) \int d^4k_E \frac{\bar{u}(p_2)[-i(p_1 - \not{k}) + m]u(p_1)}{(p_1 - k)^2 + m^2} \frac{1}{k} \frac{K_2(kR)}{K_1(kR)} \frac{1}{k'} \frac{K_2(k'R)}{K_1(k'R)}, \quad (17)$$

which is to be evaluated at $k' = p_1 - p_2 - k$. Using the Dirac equation, which says $-i\not{\partial}_1 u(p_1) = mu(p_1)$, the numerator can be rewritten $\bar{u}(p_2)[i\not{k} + 2m]u(p_1)$. At zero external momentum, $p_1 - p_2 \approx 0$ and when summed over spins $\bar{u}(0)u(0) = 2$. One can also approximate $ik_a \bar{u}(0)\gamma^a u(0) = ik_0\bar{u}(0)\gamma^0 u(0) = k_0$, with a further factor of two arising from a sum over spins. Therefore, the numerator is just $2k_0 + 4m$, before Euclidean continuation. At this point, one would typically complete the square in the denominator and drop terms which are odd in $k_a$, because the integral ought to be rotationally invariant. However this procedure is inconvenient in the present case, since
one wishes to keep the argument of the Macdonald functions simple. Keeping track of factors of \( i \) gives

\[
\mathcal{A}_T = -16\pi m^3 k^4 V(Q_{cl}) \delta(\sum p) \int_{\mu R}^{\Lambda R} \text{d}z \, z \left( \frac{K_2(z)}{K_1(z)} \right)^2 \int_0^\pi \text{d}\theta \, \frac{\sin^2 \theta (1 - \cos \theta)}{z^2 + 4m^2 R^2 \cos^2 \theta},
\]

where we have changed variable to \( z = kR \). This can be approximated using the same technique applied above for the loop diagram. We find,

\[
\int_{\mu R}^{\Lambda R} \text{d}z \, z \left( \frac{K_2(z)}{K_1(z)} \right)^2 \int_0^\pi \text{d}\theta \, \frac{\sin^2 \theta (1 - \cos \theta)}{z^2 + 4m^2 R^2 \cos^2 \theta} \sim \frac{\pi}{2} \ln \frac{\Lambda + \sqrt{\Lambda^2 + 4m^2}}{\Lambda} + \frac{\pi}{m^2 R^2} \ln \frac{\mu (1 + \sqrt{1 + 4m^2})}{\mu + \sqrt{\mu^2 + 4m^2}},
\]

where \( \Lambda \) is a reference energy scale of order \( m \). This exhibits a logarithmic divergence in both the infra-red and the ultra-violet, but remains proportional to \( V(Q_{cl}) \) and so can be absorbed into a redefinition of the potential. Just like the loop-diagram calculated above, it does not destroy the classical potential.

This result is rather general. Since these amplitudes couple only to the quintessence potential through a vertex factor, which does not change when moving four dimensional cosmology to the brane world, the result is the same, even though the character of the divergences has changed.

### 4.3. Gravitational contribution to quintessence mass

As a final application of the propagator Eq. (A.18), we suppose that the quintessence particle has some bare mass \( m_Q \) and calculate the shift produced by the contribution of the graviton loop in Fig. 4. This vertices in the diagram take the form \((-1)m_Q^2 \delta_{ij}/4\) and come from the mass term \( m_Q^2 Q^2/2 \) in the potential and the single-graviton coupling \( \text{tr} \, \epsilon/2 \) to the quintessence field. This diagram is a contribution to the self-energy \( i(2\pi)^4 \Pi^*(p) \) of the quintessence, at momentum \( p \). That gives, with propagators for the external quintessence particles stripped off,

\[
i(2\pi)^4 \Pi^*(p) = -\frac{4\pi im_Q^4 \kappa^2}{\ell^2 R^2} \int_{\mu R}^{\Lambda R} \text{d}z \, \frac{K_2(z)}{K_1(z)} \int_0^\pi \text{d}\theta \, \frac{z^2 \sin^2 \theta}{z^2 + 4m_Q^2 R^2 \cos^2 \theta}
\]
where we have aligned the polar axis with $p$, $m_Q$ is the quintessence mass, we are assuming that the particle is on shell, and we have made the standard change of variable $z = kR$.  With our conventions, the renormalized propagator becomes
\[
\Delta'(p) \propto \frac{1}{(p^2 + m^2 - \Pi^*(p))^{-1}},
\]
so that a negative contribution to $\Pi^*$ gives a positive $\delta m^2$.  This gives the estimate
\[
\delta m_5^2 \sim \frac{m_Q^2 \kappa^2}{4\pi^2} \frac{1}{\ell R^2} \left( -\frac{2}{m_Q R} \ln \frac{\mu}{2m_Q + \sqrt{\mu^2 + 4m_Q^2}} + \frac{R}{2} \Lambda \right). \tag{21}
\]
That is, the divergence is linear in the ultra-violet and logarithmic at infra-red.  Setting $m_Q$ to be currently of order the Hubble rate, or $m_Q \sim 2.1 \times 10^{-33} h^3 eV$, and the infra-red cutoff $\mu$ to the same, we find
\[
\delta m_5^2 \sim 2.1 \times 10^{-141} h^3 eV^2 \quad (5 \text{ dimensions}). \tag{22}
\]
This is very small, and implies that interactions with new gravitational physics associated with the brane world, such as the Kaluza–Klein hierarchy and graviton transmission through the bulk, do not seriously affect quintessence: its major problems remain its couplings to and interactions with normal matter.  On the other hand, the estimate Eq. (22) is several orders of magnitude smaller than a comparable estimate for a four-dimensional cosmology with Planck scale $M_P = 10^{19} \text{ GeV}$:
\[
\delta m_4^2 = \frac{1}{8\pi^2} \kappa^2 m_Q^3 \Lambda \sim 3.0 \times 10^{-127} h^3 eV^2. \tag{23}
\]
In this case, there is no infra-red divergence so the magnitude of the effect is controlled by the ultra-violet region.  The balance between Eq. (22) and Eq. (23) is controlled by the ratio
\[
\frac{\delta m_5^2}{\delta m_4^2} = 4\alpha \frac{M_4^2 \ell^2}{M_5^2 \Lambda}, \tag{24}
\]
where $M_4$, $M_5$ are the four- and five-dimensional Planck scales, respectively; $\Lambda$ is the four-dimensional ultra-violet cutoff; $\ell$ is the five-dimensional AdS scale; and $\alpha = \ln(1 + \sqrt{5})$ is a constant coming from the five-dimensional infra-red cutoff.  In the brane world, the interpretation of this difference involves interactions with the Kaluza–Klein tower, whereas in the five-dimensional picture one interprets the change as a result of the modified Planck scale.

5. Conclusions

We have studied the constraints that arise on TeV-scale quintessence models from a variety of sources.  Non-renormalisable operators in four dimensions are typically important implying that the quintessence potential needs to be computed from a higher-dimensional framework.  This follows from the fundamental mismatch between the scale $M \sim \text{TeV}$ which determines the scale at which non-renormalisable operators become important and the vacuum expectation value, $Q$, of the quintessence field which is typically of order the 4d Planck mass in tracker models.  Perturbation theory in $Q/M$ fails spectacularly.
In contrast, the gravitational coupling of quintessence to fermionic matter in cosmologies of the Randall–Sundrum brane world type does not yield significant constraints. This is easy to understand since one expects the couplings of quintessence to ordinary matter to be severely constrained and sensitive to the value of the effective ultra-violet cut-off $\Lambda_{\text{UV}}$. The brane world significantly reduces the value of this cut-off, and so one would expect quite radically different constraints on quintessence.

We find that one-loop effects introduce quantum corrections in the effective potential just proportional to the classical potential $V$ and therefore can just be absorbed into a renormalization of $V$. This is exactly the same as in the four-dimensional world and occurs for the same reason: the vertices in the diagram generate factors of $V$, not the propagator, and since this is the only quantity which changes when one moves to the brane world the type and character of the divergences one encounters changes, but the couplings remain the same.

We have also computed the lowest-order contribution from graviton loops to the vacuum polarization of quintessence. In this case one must make a numerical estimate, and we find that the brane universe typically induces a mass shift $\delta m^2$ very much smaller than in four dimensions. This shift is cut-off dependent, and scales with the ratio of the four- and five-dimensional Planck scales, the AdS curvature scale, and the inverse of the ultra-violet cutoff. From the point of view of an observer on the brane, we interpret this as the result of interactions with the Kaluza–Klein hierarchy and with bulk gravitons. The magnitude of this effect would render it undetectable and in practice the dominant contributions to $\delta m^2$ would come from matter fields on the brane.

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Appendix A. The gravitational propagator in the brane world

In this appendix we very briefly derive the graviton propagator in the Randall–Sundrum scenario. Since this calculation has already appeared elsewhere [20] we omit details where they coincide. The authors of Ref. [20] deduced the propagator by solving for the gravitational Green’s function whereas we employ the Fadeev–Popov procedure, but naturally the final answers shall agree. The principal result of this appendix is Eq. (A.18) for the on-brane propagator, which was used in the main text for the diagram calculations in Section 4.

We adopt the conventional line element [8, 7], for a given maximally symmetric three-metric $\gamma_{ij}$,

$$\mathrm{d}s^2 = -n^2(t,y) \mathrm{d}t^2 + a^2(t,y) \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2,$$

(A.1)

where $y$ is a Gaussian normal coordinate transverse to the brane. This metric is taken to be a solution of the five-dimensional Einstein equations with cosmological constant $\Lambda$. 


but vanishing bulk energy–momentum tensor. The brane is considered to be imbedded at \( y = 0 \), and there is a \( \mathbb{Z}_2 \) symmetry which acts on the Gaussian normal coordinate as \( y \mapsto -y \). There is typically a coordinate horizon where the Gaussian normal coordinates used in Eq. (A.1) break down, and we write the location of this horizon as \( y = y_h \).

Gravitational disturbances take the form of small perturbations \( h_{ab} \) to the metric: \( ds^2 = (g_{ab} + h_{ab}) \, dx^a \, dx^b \). In a general \( D \)-dimensional spacetime, \( h_{ab} \) will transform as a representation of the isometry group \( SO(1, D-1) \). This describes the full degrees of freedom of the graviton. Alternatively, one could decompose \( h_{ab} \) into its representations under the brane isometry group, which consists of a tensor (in the \( dx^i \, dx^j \) sector of the metric) and supplementary vector and scalar pieces (respectively, for vectors, in the \( dt \, dx^i \), \( dy \, dx^i \) sectors and for scalars in \( dt \, dy \), \( dt^2 \) and \( dy^2 \) sectors) which must be added in to complete the full degrees of freedom of the graviton. In this paper we will deal only with the case of a flat, Minkowski brane which possesses a larger isometry group and allows us to re-absorb the vector and scalar pieces into the tensor perturbation. For this reason, we only calculate the tensor propagator in this section.

This piece of the perturbation is written \( e_{ij} \) and takes the metric form
\[
dx^i = -n^2(t, y) \, dt^2 + a^2(t, y) (\delta_{ij} + e_{ij}) \, dx^i \, dx^j + dy^2. \tag{A.2}
\]

After integrating by parts and discarding surface terms, the action is
\[
S = -\frac{1}{2\kappa^2} \int d^5 x \, n a^3 \, e^{ij} \left[ 2\delta_{ij} \delta_{rs} (\nabla^2 - \Delta) - \frac{4}{a^2} \delta_{ir} \delta_{js} \right] e^{rs} \tag{A.3}
\]
We have introduced an operator \( \square \) describing \( t \) and \( y \) derivatives,
\[
\square = \frac{1}{n^2} \frac{\partial^2}{dt^2} - \frac{\partial^2}{dy^2} + \frac{1}{n^2} \left( 3\frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \frac{\partial}{\partial t} - \left( 3\frac{\dot{a}}{a} + \frac{\dot{n}}{n} \right) \frac{\partial}{\partial y} \tag{A.4}
\]
and \( \Delta \) is the \( \delta_{ij} \) Laplacian. The brane isometry group \( \mathbb{R}^3 \) now appears as an invariance of this action: Eq. (A.3) is invariant under \( \mathbb{R}^3 \) transformations (spatial rotations and translations).

One passes to the quantum theory by defining correlation functions of the field \( e_{ij} \) using the functional integral prescription, \( \langle e_{ij}(x) \cdots e_{mn}(y) \rangle = \int [de_{rs}] e_{ij}(x) \cdots e_{mn}(y) \exp iS[e_{rs}] \). To render this integral (formally) finite, one divides out by the volume of the gauge group. This is the Fadeev–Popov procedure \[16\], amounting to the inclusion of an extra gauge-fixing contribution in the action,
\[
S \mapsto S + \frac{1}{2\kappa^2} \int d^5 x \, n a^3 \frac{1}{2\xi} (\partial e_b - \alpha \partial_b e)^2 \tag{A.5}
\]
where \( \xi \) and \( \alpha \) are arbitrary numbers. We have suppressed irrelevant indices by writing \( \partial e_b = \partial^a e_{ab} \) and \( e = \text{tr} \, e_{ab} \). Notice that in these formulas we are considering \( e_{ab} \) to be a full five-dimensional tensor which is zero on \( t \) or \( y \) indices. Where derivatives are contracted with \( e_{ab} \) this makes no difference (\( \partial^a e_{ab} = \partial^b e_{ab} \)), but where two derivatives become contracted with themselves as in \( \partial^n \partial_n \) one must include contributions from the \( t \) and \( y \) sectors.
After integrating by parts one obtains the gauge-fixed action,
\[ S = -\frac{1}{2\kappa^2} \int d^5x \, na^3 \, e^{ij} \left\{ \left[ \Box - \frac{\Delta}{a^2} \right] \left[ -\frac{1}{4} \delta_{ir} \delta_{js} + \delta_{ij} \delta_{rs} \left( \frac{1}{4} - \frac{\alpha^2}{2\xi} \right) \right] - \frac{1}{a^2} \delta_{ir} \partial_j \partial_s \left( \frac{1}{2} - \frac{1}{2\xi} \right) + \frac{1}{a^2} \delta_{ij} \partial_r \partial_s \left( \frac{1}{2} - \frac{\alpha}{\xi} \right) \right\} e^{rs}. \] (A.6)

In order to simplify this result, it is convenient to set \( \xi = 1, \alpha = 1/2 \) in which case both terms involving \( \alpha \) disappear and one is left with the reduced action
\[ S = -\frac{1}{2} \int d^5x \, na^3 \, e^{ij} D_{ijrs} e^{rs} \] (A.7)

where
\[ D_{ijrs} = \frac{1}{4\kappa^2} \left( 2\delta_{i(r} \delta_{s)} - \frac{1}{2} \delta_{ij} \delta_{rs} \right) \left( \Box - \frac{\Delta}{a^2} \right). \] (A.8)

These choices for \( \alpha \) and \( \xi \) coincide with the four-dimensional case. The propagator \( \Delta^{rsmn}(x, y) \) satisfies
\[ D_{ijrs} \Delta^{rsmn}(x, y) = -i \delta^5(x - y) D^{m}_{(i} \delta^{n)} \] (A.9)

where \( \delta^5 \) is the covariant \( \delta \)-function.

In a quite general homogeneous brane world, the metric functions \( n \) and \( a \) are not equal and both depend on \( t \) and \( y \), so one has the three Killing vectors \( \partial/\partial x^i \) which generate translations along the spacelike coordinate axes, but no other translational Killing vectors. For this reason, it is sensible to try and diagonalize \( \Delta^{rsmn} \) as a Fourier transform in the \( x^i \), but one will not be able to deal with the \( t \) and \( y \) dependence in the same way. Therefore,
\[ \Delta^{rsmn} = -i \rho^{rsmn} \int \frac{d^3p}{(2\pi)^3} \, e^{-ip(x-y)} G(p; x_0, y_0; x_5, y_5), \] (A.10)

where \( \rho^{rsmn} \) is the combination
\[ \rho^{rsmn} = \delta^{r(m} \delta^{n)s} - \frac{1}{d-2} \delta^{rs} \delta^{mn}, \] (A.11)

and \( d = \delta_i^j \) is the trace of the \( \mathbb{R}^3 \) Kronecker delta. We are now adopting a convention of writing the coordinates of any point \( x \) in spacetime as \((x^0, x, x^5)\). Substituting Eq. (A.10) into Eq. (A.9) shows that \( G \) must obey the equation
\[ \left( \Box + \frac{p^2}{a^2} \right) G = 2\kappa^2 \delta(x^0 - y^0) \delta(x^5 - y^5). \] (A.12)

Appendix A.1. The Randall–Sundrum propagator

At this point, one can make no further progress without specifying some explicit form for \( a \) and \( n \). The simplest choice is to take the brane to be empty of matter, except for some intrinsic tension \( |39| \) which is tuned to give a Minkowski brane. The line element is
\[ ds^2 = e^{-2\ell |y|} (-dt^2 + \delta_{ij} dx^i dx^j) + dy^2. \] (A.13)
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In this special case, the functions $a$ and $n$ do turn out to be equal, and all quantities are independent of the cosmic time $t$, so one recovers $\partial / \partial t$ as a Killing symmetry. This is a great convenience, because one can now write $G$ as a Fourier integral $G = \int d\omega (2\pi)^{-1} e^{i\omega (x^0 - y^0)} \tilde{G}$ in $x^0 - y^0$, leaving only an ordinary differential equation for the $x^5, y^5$ dependence of the Fourier transform $\tilde{G}$. Changing to a conformal bulk coordinate $z$ defined by $dz^5 = a \, dx^2$, this ordinary differential equation turns out to be just the Bessel equation,

$$
B \tilde{G} = -2\kappa^2 \frac{y^2}{\ell(x^2)} \delta(x^2 - y^2),
$$

(A.14)

where $B$ is the Bessel operator,

$$
B = \frac{\partial^2}{\partial (x^2)^2} + \frac{1}{x^2} \frac{\partial}{\partial x^2} + \left( \beta^2 - \frac{4}{(x^2)^2} \right).
$$

(A.15)

In these coordinates, the location of the brane is $z = \ell^{-1}$, but we will often take its location to be arbitrary and write $z = R$ instead. When making numerical estimates, we restore $\ell R = 1$. We have introduced a new quantity $\beta$ defined by $\beta^2 = \omega^2 - p^2$ and define a four-vector $k = (\omega, p)$ with $k^2 = -\beta^2$ in our signature diag $(-1, 1, 1, 1)$.

From this point, out derivation coincides with the earlier derivation of Giddings, Katz & Randall [20], so we omit further details and merely state the result. The general solution for $\tilde{G}$ is a combination of Bessel functions, and by integrating Eq. (A.14) over a small neighbourhood of $x^2 = y^2$ one obtains a continuity condition on $\tilde{G}$ and a step condition on $\partial \tilde{G} / \partial x^2$. Demanding that the normal derivative of $G$ vanish at the brane, and that positive frequency waves be purely ingoing in the far-field, together with the junction conditions at the brane allows one to solve uniquely for $G$. The result for the entire propagator is

$$
\Delta^{rsmn} = \rho^{rsmn} \int \frac{d^3p \, d\omega}{(2\pi)^4} e^{-ip \cdot (x-y) + i\omega(x^0 - y^0)} W(x^2, y^2; R),
$$

(A.16)

where $W(x^2, y^2; R)$ satisfies

$$
W(x^2, y^2; R) = -i \left( \frac{x^2}{y^2} \right)^2 \frac{2\kappa^2 \pi}{\ell} \frac{H_2^{(1)}(\beta_{z<})}{H_1^{(1)}(\beta R)} \left( J_1(\beta R) H_2^{(1)}(\beta z_{<}) - H_1^{(1)}(\beta R) J_2(\beta z_{<}) \right)
$$

(A.17)

in which $z = R$ is the location of the brane and $z_{<}, z_{>}$ are respectively min$\{x^2, y^2\}$, max$\{x^2, y^2\}$. $H_\nu^{(1)}$ is the Hankel function of order $\nu$ of the first kind. This propagator agrees with the expressions (2.14)–(2.15) given in Ref. 20 once notational differences have been taken into account.

There is an obvious simplification of Eq. (A.16) in the special case that both endpoints $x, y$ are taken on the brane. One finds,

$$
\Delta^{rsmn}|_{on-brane} = \frac{2\kappa^2}{\ell R} \rho^{rsmn} \int \frac{dk}{(2\pi)^4} e^{-ik \cdot (x-y)} \frac{1}{k} \frac{K_2(kR)}{K_1(kR)}
$$

(A.18)

where $K_\nu$ is the Macdonald or Basset function

$$
K_\nu(z) = \frac{1}{2} \pi i^{\nu+1} H_\nu^{(1)}(iz)
$$

(A.19)
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and is entirely real.

In the text, we make use of Wick rotation to Euclidean signature. Since the graviton propagator has acquired a much more complicated space-time dependence than the propagators of Minkowski space fields, there may be more obstructions to this procedure than can be dealt with by moving the $1/k$ poles of Eq. (A.18) off-axis. In particular, if the Macdonald functions $K_\nu$ have singularities anywhere on $\mathbb{C}$, then a Wick rotation could not be justified. To see that this is not so, it is convenient to make use of the following integral representation of the Macdonald function

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{\Gamma(\nu + 1/2)} \int_0^\infty e^{-t \nu - 1/2} \left(1 - \frac{t}{2z}\right)^{-n-1/2} dt$$

(A.20)

From this it is clear that $K_\nu$ has a singularity at $z = 0$ for any $\nu$ but is otherwise analytic everywhere. Moreover it is not zero except at $z = \infty$. There is an essential singularity at $\infty$ stemming from the exponential, but since the contour remains fixed at $z = 0$ and $z = \infty$, this does not interfere with the analytic continuation.

References

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