Semiclassical Expansions, the Strong Quantum Limit, and Duality

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Abstract
We show that Feynman’s exponential of the action exhibits a $\mathbb{Z}_2$ duality symmetry that illustrates a relativity principle for the notion of quantum versus classical.

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1 Introduction
The motivation for this letter is taken from ref. [1], section 6, from which we quote: *The notion of quantum versus classical is relative to which theory we measure from.* In what follows we develop a formalism that implements such a relativity principle between classical and quantum. We begin by considering the exponential

$$\exp\left(\frac{i}{\hbar} S\right)$$

of a certain action $S$. Such a function may arise, e.g., as the integrand of a field–theoretic partition or correlation function, as the phase of the wavefunction in a quantum–mechanical WKB approximation, etc. The precise nature of $S$ will be immaterial for our purposes.
2 Expanding exponentials in terms of Bessel functions

The generating function for the Bessel functions \( J_n(x) \) of integer order \( n \) is \( e^{\frac{x}{z} (-z)} \) [2]. Since \( J_{-n}(x) = (-1)^n J_n(x) \),

\[
e^{\frac{x}{z} (-z)} = \sum_{n=-\infty}^{\infty} z^n J_n(x) = J_0(x) + \sum_{n=1}^{\infty} \left[ z^n + \left( -\frac{1}{z} \right)^n \right] J_n(x). \tag{2}
\]

This generating function is symmetric under the exchange of \( z \) and \(-1/z\). For small values of \( z \) this reduces to

\[
e^{-\frac{x}{z}} \simeq J_0(x) + \sum_{n=1}^\infty \left( -\frac{1}{z} \right)^n J_n(x), \quad z \to 0, \tag{3}
\]

while for large values of \( z \) one has

\[
e^{\frac{x}{z}} \simeq J_0(x) + \sum_{n=1}^{\infty} z^n J_n(x), \quad z \to \infty. \tag{4}
\]

Thus, in the expansions (3) and (4), large \( z \) or small \( z \) are essentially the same limit, the map between them being \( z \to -1/z \). For small \( x \), \( J_n(x) \) behaves as

\[
J_n(x) \simeq \frac{1}{2^n n!} x^n, \quad n = 0, 1, 2, \ldots, \quad x \to 0, \tag{5}
\]

while for large \( x \)

\[
J_n(x) \simeq \sqrt{\frac{2}{\pi x}} \cos \left( x \frac{\pi}{4} - \frac{n\pi}{2} \right), \quad x \to \infty. \tag{6}
\]

We can now combine eqns. (3), (4), (5) and (6) in 4 different ways, according to the different possible regimes for the variables \( z \) and \( x \). For simplicity we will from now on write all \( \simeq \) signs as equalities, followed by the range of variables where the corresponding approximations hold. Eqns. (3) and (5) together yield the usual Taylor series for \( e^{-\frac{x}{z}} \), while eqns. (4) and (5) together give that for \( e^{\frac{x}{z}} \). Nontrivial expansions are obtained from eqns. (3) and (6) combined,

\[
e^{-\frac{x}{z}} = \sum_{n=0}^{\infty} \left( -\frac{1}{z} \right)^n \sqrt{\frac{2}{\pi x}} \cos \left( x \frac{\pi}{4} - \frac{n\pi}{2} \right), \quad z \to 0, x \to \infty, \tag{7}
\]

as well as from eqns. (4) and (6) combined,

\[
e^{\frac{x}{z}} = \sum_{n=0}^{\infty} z^n \sqrt{\frac{2}{\pi x}} \cos \left( x \frac{\pi}{4} - \frac{n\pi}{2} \right), \quad z \to \infty, x \to \infty. \tag{8}
\]
3 The semiclassical expansion

Under the identifications
\[ z = -\hbar, \quad x = 2iS, \]  
the left-hand side of eqn. (3) equals the exponential (1). Thus, by eqn. (3),
\[ e^{iS} = \sum_{n=0}^{\infty} \frac{1}{\hbar^n} J_n(2iS), \quad \hbar \to 0. \]  
(10)

In the limit when the action \( S \) is large compared with Planck’s constant \( \hbar \), eqn. (10) becomes
\[ e^{iS} = \sum_{n=0}^{\infty} \frac{1}{\hbar^n} \frac{1}{\sqrt{i\pi S}} \cos \left( \frac{2iS - \pi}{4} - \frac{n\pi}{2} \right), \quad \hbar \to 0, \ S \to -i\infty. \]  
(11)

The summands in eqn. (11) are reminiscent of (though not quite identical with) the WKB expression for the phase of the wavefunction at first nontrivial order in \( \hbar \) [3]. The fundamental difference with the WKB method, however, lies in the fact that eqn. (11) contains an infinite power series in \( 1/\hbar \). This is so because, rather than expanding \( S \) in powers of \( \hbar \), as in the WKB method, we have expanded the exponential (1) itself.

4 Changing variables

With the new choice of variables
\[ z = 2iS, \quad x = \frac{1}{\hbar}, \]  
eqn. (4) becomes
\[ e^{iS} = \sum_{n=0}^{\infty} (2iS)^n J_n(\hbar^{-1}), \quad S \to -i\infty. \]  
(13)

Comparing the right-hand sides of eqns. (13) and (10), we observe that the roles of \( 2iS \) and \( 1/\hbar \) are exchanged. With the choice of variables made in eqn. (12), the measurable action \( S \) becomes the (inverse of the) fundamental quantum of action \( \hbar \). Conversely, the variable \( \hbar \) that in eqn. (9) played the role of the quantum of action now becomes the (inverse of the) variable on which the Bessel functions on the right-hand side of our expansions depend.

Let us recapitulate. In section 3 we have a quantum of action \( \hbar \) and a measurable action \( S \). The expansion (10) is in powers of \( 1/\hbar \), with coefficients given by \( J_n(2iS) \). It is applicable only in the limit \( \hbar \to 0 \), while the measurable action \( S \) is unconstrained.

In section 4 the picture is dual to the previous one: we have a quantum of action \( 2iS \) and a measurable action \( \hbar^{-1} \). The relevant expansion now, eqn. (13), holds only when
\[ S \to -i \infty. \] It is an expansion in powers of \( 2iS \), with coefficients \( J_n(\hbar^{-1}) \). However, now \( \hbar \) is free. Letting \( \hbar \to 0 \), where eqn. (8) applies, leads to

\[
e^{\frac{\hbar}{S}} = \sum_{n=0}^{\infty} (2iS)^n \sqrt{\frac{2\hbar}{\pi}} \cos \left( \frac{1}{\hbar} - \frac{\pi}{4} - \frac{n\pi}{2} \right), \quad S \to -i \infty, \hbar \to 0. \tag{14}
\]

The expansion (14) is valid in the same range of variables as that in eqn. (11).

Alternatively we can let \( \hbar \to \infty \) in eqn. (13). Hence, as seen from the variables (9), we can reach the strong quantum regime, which was inaccessible in section 3. In the limit \( \hbar \to \infty \) we apply eqn. (5) to the right-hand side of (13) and obtain the usual Taylor series for the exponential,

\[
e^{\frac{\hbar}{S}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{iS}{\hbar} \right)^n, \quad S \to -i \infty, \hbar \to \infty. \tag{15}
\]

Admittedly, eqn. (15) does not tell us much new; it also holds true for any values of \( S \) and \( \hbar \), i.e., outside its stated range of applicability. However, the conceptual value of the previous arguments lies in showing that the notion of classical versus quantum does not possess an absolute meaning but only a relative one, i.e., one that depends on our choice of variables. This is established by the existence of the two dual expansions (11) and (14), in which the inverse of the quantum of action gets exchanged with the measurable action.

Transformations between different regimes of the parameters of a given theory, leading to apparently different descriptions of the same physics, are called dualities [1]. M-theory appears as a prototype for dualities. However, as suggested in ref. [1] and explicitly illustrated here, one can implement dualities already within the realm of a finite number of degrees of freedom, before moving on to field theory, strings and branes. The duality illustrated in the previous example is of a very simple nature: it is just a \( \mathbb{Z}_2 \) duality exchanging the inverse of the quantum of action with the measurable action. This duality could be made manifest from the beginning in a theory whose starting point were some completion of eqn. (1) such as, e.g.,

\[
\exp \left( \frac{i}{\hbar} \left( \frac{S}{\hbar} + \frac{\hbar}{S} \right) \right). \tag{16}
\]

The term \( e^{i\frac{S}{\hbar}} \) becomes unity as \( \hbar \to 0 \) and \( S \to -i \infty \), thus reducing to the usual exponential (1). On the contrary, in the strong quantum regime \( \hbar \to \infty, S \to 0 \), the exponential \( e^{i\frac{\hbar}{S}} \) survives while \( e^{i\frac{S}{\hbar}} \) tends to unity. Eqn. (16) is manifestly selfdual, but is it physically correct? This question can be recast as follows: can the \( \mathbb{Z}_2 \) duality presented above be raised to the category of a physical principle? If so, then eqn. (16) will be physically correct. As long as only the usual exponential (1) is considered, this \( \mathbb{Z}_2 \) duality cannot be realised explicitly. It can only be observed through a change of variables such as the one from eqn. (9) to eqn. (12). In this latter case this duality is reminiscent, mutatis mutandi, of de Broglie’s wave/particle duality or Bohr’s complementarity principle [3].
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