Unitary local invariance

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Abstract

We address unitary local invariance of bipartite pure states. Given a bipartite state \( |\Psi\rangle\rangle = \sum_{ij} \psi_{ij} |i_1\rangle \otimes |j_2\rangle \) the complete characterization of the class of local unitaries \( U_1 \otimes U_2 \) for which \( U_1 \otimes U_2 |\Psi\rangle\rangle = |\Psi\rangle\rangle \) is obtained in terms of the singular values of the matrix \( \Psi = [\psi_{ij}] \).

Suppose you are given a bipartite pure state \( |\Psi\rangle\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \) and asked for which (pairs of) unitaries the state is locally invariant \( i.e. \)

\[
U_1 \otimes U_2 |\Psi\rangle\rangle = |\Psi\rangle\rangle .
\] (1)

This kind of invariance is closely related to the so-called environment-assisted invariance (envariance) which has been recently introduced \[1\] \[2\] to understand the origin of Born rule. More generally, unitary local (UL) invariance naturally arises whenever one investigates the possibility of undoing a local operation performed on a subsystem of a multipartite state by acting, yet locally, on another subsystem. The somehow related concept of twin observables has been also investigated in order to account for the invariance that can be observed in measurements performed on correlated systems \[3\] \[4\].

As we will see any state \( |\Psi\rangle\rangle \) is UL invariant for some pairs of unitaries and, as one may expect, UL invariance and entanglement properties of \( |\Psi\rangle\rangle \) are somehow related. However, there are separable UL invariant states and, overall, the characterization of the class of unitaries leading to UL invariance is not immediate. In this paper a complete characterization of the pairs of unitaries for which a given state \( |\Psi\rangle\rangle \) is UL invariant is achieved in terms of the Schmidt decomposition of \( |\Psi\rangle\rangle \) \( i.e. \) of the singular value decomposition

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of the matrix $\Psi$. We will show that the relevant parameter is the number of terms in the Schmidt decomposition of $|\Psi\rangle\rangle$ and, in particular, the number of (nonzero) equal Schmidt coefficients.

Let us start by establishing notation. Given a bases $\{i_1 \otimes j_2\}$ for the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ (with $\mathcal{H}_1$ and $\mathcal{H}_2$ generally not isomorphic), we can write any vector $|\Psi\rangle\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as

$$|\Psi\rangle\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \psi_{ij} |i_1 \otimes j_2\rangle,$$

where $\psi_{ij}$ are the elements of the matrix $\Psi$. The above notation induces a bijection among states $|\Psi\rangle\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and Hilbert-Schmidt operators

$$A = \sum_{ij} a_{ij} |i\rangle_1 \langle j|$$

from $\mathcal{H}_1$ to $\mathcal{H}_2$. The following relations are an immediate consequence of the definitions (2) and (3)

$$A \otimes B |\Psi\rangle\rangle = |A \Psi B^\top\rangle\rangle, \quad \langle\langle A |B\rangle\rangle = \text{Tr}[A^\dagger B],$$

$$\text{Tr}_2 [\langle\langle A \rangle\rangle \langle B\rangle\rangle] = AB^\dagger, \quad \text{Tr}_1 [\langle\langle A \rangle\rangle \langle B\rangle\rangle] = A^T B^*$$

where $A^T, A^\dagger, A^*$ denote transpose, conjugate and hermitian conjugate respectively of the matrix $A$ (and of the operator $A$ with respect to the chosen basis). $\text{Tr}_j[...]$ denotes the partial trace over the Hilbert space $\mathcal{H}_j$ whereas $AB^\dagger$ and $A^T B^*$ in Eq. (5) should be meant as operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

Using Eq. (4) the condition (1) for UL invariance can be rewritten as $|U_1 \Psi U_2^\top\rangle\rangle = |\Psi\rangle\rangle$, thus leading to the matrix relation

$$U_1 \Psi = \Psi U_2^*.$$  

The singular value decomposition of $\Psi$ is given by $\Psi = R^\top \Sigma S$ where $R$ and $S$ are unitary matrices of suitable dimension and $\Sigma$ is the diagonal matrix $\Sigma = \text{Diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots)$ where $\sigma_j$ are the singular values of $\Psi$ i.e. the square roots of the eigenvalues of $\Psi^\dagger \Psi$; $r$ is the rank of the matrix $\Psi$. The singular value decomposition of $\Psi$ corresponds to the Schmidt decomposition of $|\Psi\rangle\rangle$

$$|\Psi\rangle\rangle = |R^\top \Sigma S\rangle\rangle = \sum_{ij} (R^\top \Sigma S)_{ij} |i\rangle_1 \otimes |j\rangle_2$$

$$= \sum_{ij} \sum_{kl} R_{ki} |\varphi_k\rangle_1 \otimes |\theta_k\rangle_2.$$
where \( |\varphi_k\rangle_1 = \sum_i R_{ki} |i\rangle_1 \) and \( |\theta_k\rangle_2 = \sum_l S_{lj} |j\rangle_2 \) are the Schmidt basis in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively.

From now on we switch to Schmidt basis, which will be employed for the rest of the paper. Bipartite pure states are thus represented by kets \( |\Sigma\rangle\rangle \) where \( \Sigma \) is a diagonal matrix and. The UL invariance relation (6) rewrites as follows

\[
U_1 \Sigma = \Sigma U_2^* ,
\]

where, with a slight abuse of notation, we have denoted by the same symbol \( U_j \), \( j = 1, 2 \) the matrices corresponding to the unitaries transformations in the new (Schmidt) basis.

From Eq. (8) we can already draw some conclusions about the UL invariance properties of some particular class of quantum states. Let us first consider separable states. These states correspond to rank one matrices \( \Sigma = \sigma_1 \oplus 0 \) and therefore are UL invariant under transformation \( U_1 \otimes U_2 \) if

\[
U_1 = e^{i\phi} \oplus V_1 \quad U_2 = e^{-i\phi} \oplus V_2 ,
\]

where \( \phi \) is an arbitrary phase, and \( V_j \), \( j = 1, 2 \) are arbitrary unitaries, each acting on the \((d_j - 1)\)-dimensional null subspace of \( \mathcal{H}_j \), corresponding to zero singular values. More generally, Eq. (8) indicates that each matrix \( U_j \) should be written as \( U_j = W_j \oplus V_j \) where \( W_j \) acts on the \( r \)-dimensional subspace of \( \mathcal{H}_j \) corresponding to the support of \( \Sigma \), and \( V_j \) on the complementary \((d_j - r)\)-dimensional null subspace. The rest of the paper is devoted to investigate the structure of \( W_j \).

We first note that if \( |\Sigma\rangle\rangle \) is a maximally entangled state then \( \Sigma = \frac{1}{\sqrt{d}} \mathbb{I}_d \), \( d = \min(d_1, d_2) \) and therefore \( \Sigma \) implies that \( U_1 = U_2^* \), i.e. \( |\Sigma\rangle\rangle \) is UL invariant for any transformation of the form \( U \otimes U^* \) with \( U \) arbitrary unitary. This relation in turn expresses isotropy of maximally entangled states \( \mathbf{6} \). If \( |\Sigma\rangle\rangle \) has the form of a maximally entangled states immersed in a larger Hilbert space then the same conclusion holds on the support of \( \Sigma \). The two statements can be summarized by the following lemma:

**Lemma 1**: Let \( |\Sigma\rangle\rangle \) be a rank \( r \) bipartite pure state in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) with \( \Sigma = \frac{1}{\sqrt{r}} \mathbb{I}_r \) then \( U_1 \otimes U_2 |\Sigma\rangle\rangle = |\Sigma\rangle\rangle \) for \( U_1, U_2 \) satisfying \( U_1 = U_2^* \) on the support of \( \Sigma \) i.e. for

\[
U_1 = W \oplus V_1 \quad U_2 = W^* \oplus V_2 ,
\]

where \( W, V_1 \) and \( V_2 \) are arbitrary unitaries on the corresponding \( r \)-dimensional, \((d_1 - r)\)-dimensional and \((d_2 - r)\)-dimensional subspaces.
Proof: An immediate consequence of Eq. (8).

In order to exploit this observation to find the general structure of the $W_j$, $j = 1, 2$ we need the following Lemma.

**Lemma 2:** Let $U_j, j = 1, 2$ be unitaries in $\mathcal{H}_j$ and $|\Psi\rangle\rangle$ a bipartite state on $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $U_1 \otimes U_2 |\Psi\rangle\rangle = |\Psi\rangle\rangle$ then $[U_j , g_j ] = 0$ where $g_j$ are the partial traces of $|\Psi\rangle\rangle$ i.e. $g_1 = \text{Tr}_2[|\Psi\rangle\rangle \langle\langle \Psi|] = \Psi \Psi^\dagger$ and $g_2 = \text{Tr}_1[|\Psi\rangle\rangle \langle\langle \Psi|] = (\Psi^\dagger \Psi)^T$.

**Proof:**

\begin{align}
U_1^\dagger g_1 U_1 &= \text{Tr}_2 \left[ (U_1^\dagger \otimes \mathbb{I}) |\Psi\rangle\rangle \langle\langle \Psi| (U_1 \otimes \mathbb{I}) \right] \\
&= \text{Tr}_2 \left[ (U_1^\dagger \otimes \mathbb{I}) (U_1 \otimes U_2) |\Psi\rangle\rangle \langle\langle \Psi| (U_1^\dagger \otimes U_2^\dagger) (U_1 \otimes \mathbb{I}) \right] \\
&= \text{Tr}_2 \left[ (\mathbb{I} \otimes U_2) |\Psi\rangle\rangle \langle\langle \Psi| (\mathbb{I} \otimes U_2^\dagger) \right] \\
&= \text{Tr}_2[|\Psi\rangle\rangle \langle\langle \Psi|] = g_1. \tag{11}
\end{align}

The proof that $U_2^\dagger g_2 U_2 = g_2$ and thus that $[U_2, g_2 ] = 0$ goes along the same lines. As a consequence of Lemma 2 $U_j$ and $g_j$ posses a common set of eigenvectors, which coincides with the Schmidt basis of $|\Psi\rangle\rangle$ in each Hilbert space.

We are now ready to state the main result of the paper in the form of the following lemma:

**Lemma 3:** Let $|\Sigma\rangle\rangle$ be a bipartite pure state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\Sigma$ of rank $r$ and let $r_k$ be the number of $k$-tuple of equal singular values, e.g. $r_1$ is the number of distinct singular values, $r_2$ the number of pairs and so on; $r_1 + r_2 + \ldots + r_k + \ldots = r$. Then $|\Sigma\rangle\rangle$ is UL invariant i.e $U_1 \otimes U_2 |\Sigma\rangle\rangle = |\Sigma\rangle\rangle$ for $U_1$, $U_2$ given by

\begin{align}
U_1 &= e^{i\phi_1} \oplus \ldots \oplus e^{i\phi_{r_1}} \oplus D_1 \oplus \ldots \oplus D_{r_2} \\
&\quad \oplus T_1 \oplus \ldots \oplus T_{r_3} \oplus \ldots \oplus V_1, \tag{12} \\
U_2 &= e^{-i\phi_1} \oplus \ldots \oplus e^{-i\phi_{r_1}} \oplus D_1^\dagger \oplus \ldots \oplus D_{r_2}^\dagger \\
&\quad \oplus T_1^\dagger \oplus \ldots \oplus T_{r_3}^\dagger \oplus \ldots \oplus V_2, \tag{13}
\end{align}

where $D_1, \ldots, D_{r_2}$ are arbitrary $2 \times 2$ unitary matrices, $T_1, \ldots, T_{r_3}$ arbitrary $3 \times 3$ unitary matrices and so on. $V_1$ and $V_2$ are arbitrary unitaries in the null subspaces of $\mathcal{H}_j$ corresponding to zero singular values.
**Proof:** According to the considerations made before Lemma 1, we can always write $U_1 = W \oplus V_1$, $U_2 = W^* \oplus V_2$ where $W$ is of rank $r$. Moreover, as a consequence of Lemma 2 $U_j$ can be decomposed into blocks acting on the eigenspaces of $\rho_j$. Inside eigenspaces corresponding to degenerate eigenvalues of $\rho_j$, i.e. if there are $k$-tuple of equal singular values, we apply Lemma 1 thus arriving at

$$W = e^{i\phi_1} \oplus \ldots \oplus e^{i\phi_r} \oplus D_1 \oplus \ldots \oplus D_{r_2} \oplus T_1 \oplus \ldots \oplus D_{r_2} \oplus T_{r_2} \oplus \ldots ,$$  

(14)

from which expressions (12) and (13) immediately follows.

In conclusion, unitary local (UL) invariance of bipartite pure states has been addressed and the complete characterization of the class of local unitaries $U_1 \otimes U_2$ for which $U_1 \otimes U_2 |\Psi\rangle = |\Psi\rangle$ has been obtained in terms of the singular values of the matrix $\Psi$. The explicit expression of the matrices $U_1$ and $U_2$ in the Schmidt basis has been derived. Maximally entangled states are UL invariant under any transformation of the form $U \otimes U^*$ with arbitrary $U$ whereas separable states are UL invariant for unitaries of the form $(e^{i\phi} \oplus V_1) \otimes (e^{-i\phi} \oplus V_2)$ with $V_1$ and $V_2$ acting on the null subspaces of $H_1$ and $H_2$ respectively. In the general case, the two relevant parameters are the rank of $\Psi$ and the number of equal singular values of $\Psi$, which determines the structure of the unitaries on the support. In general, UL invariance is not equivalent to entanglement, though for the pure bidimensional case it may become equivalent when supplemented by a suitable squeezing criterion [7].

**References**