HYPERFINE SPLITTING IN HEAVY IONS WITH THE NUCLEAR MAGNETIZATION DISTRIBUTION DETERMINED FROM EXPERIMENTS ON MUONIC ATOMS

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Abstract

The hyperfine splitting in hydrogenlike $^{209}$Bi, $^{203}$Tl, and $^{205}$Tl is calculated with the nuclear magnetization determined from experimental data on the hyperfine splitting in the corresponding muonic atoms. The single-particle and configuration-mixing nuclear models are considered. The QED corrections are taken into account for both electronic and muonic atoms. The obtained results are compared with other calculations and with experiment.

Keywords: hyperfine splitting, heavy ions, quantum electrodynamics

1 Introduction

High-precision measurements of the hyperfine splitting (HFS) in heavy hydrogenlike ions [1, 2, 3, 4, 5] have triggered a great interest to calculations of this effect. The main goal of these experiments was to probe the magnetic sector of quantum electrodynamics (QED) in the presence of a strong Coulomb field. The uncertainty of the theoretical results is mainly determined by the uncertainty of the nuclear magnetization distribution correction, the so-called Bohr-Weisskopf (BW) effect. In calculations, based on the single-particle nuclear model [6, 7, 8, 9], which provide a reasonable agreement with the experiments, this uncertainty may amount up to about 100% of the BW effect and is generally few times larger than the total QED contribution. More elaborated calculations, based on many-particle nuclear models [10, 11], do not provide a desirable agreement with the experiments.

In the present paper, we determine the BW correction to the hyperfine splitting of hydrogenlike $^{209}$Bi, $^{203}$Tl, and $^{205}$Tl using experimental data on the hyperfine splitting in the corresponding muonic atoms. We consider the single-particle and configuration-mixing nuclear models. The parameters of the nuclear magnetization distribution are chosen to reproduce the experimental values of the nuclear magnetic moment as well as the BW contribution in muonic atoms extracted from the corresponding experiments. To increase the precision of determining the BW contribution, the QED corrections for the muonic atoms have been evaluated. The obtained results are compared with other calculations and with experiment.

The relativistic units ($\hbar = c = 1$) and the Heaviside charge unit ($\alpha = e^2/(4\pi)$, $e < 0$) are used in the paper.

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The ground-state hyperfine splitting in muonic atoms can be written in the form:

\[ \Delta E = \Delta E_{NS} + \Delta E_{BW} + \Delta E_{QED}, \]  

(2.1)

where \( \Delta E_{NS} \) is the hyperfine splitting value incorporating the relativistic and nuclear charge distribution ("nuclear size") effects, \( \Delta E_{BW} \) is the BW contribution, \( \Delta E_{QED} \) is the QED correction. The \( \Delta E_{NS} \) value can be calculated by the formula:

\[ \Delta E_{NS} = -\alpha \frac{4}{3} \frac{\mu}{\mu_N} \frac{1}{m_p} \frac{(2I + 1)}{2I} \int_0^\infty dr g(r)f(r), \]  

(2.2)

where \( \alpha \) is the fine structure constant, \( \mu \) is the nuclear magnetic moment, \( \mu_N \) is the nuclear magneton, \( m_p \) is the proton mass, and \( I \) is the nuclear spin. \( g(r) \) and \( f(r) \) are the radial parts of the Dirac wave function:

\[ \Psi(r) = \left( \begin{array}{c} g(r)\Omega_{\kappa m}(n) \\ i f(r)\Omega_{-\kappa m}(n) \end{array} \right), \]  

(2.3)

which are determined by solving the Dirac equation with the Fermi distribution of the nuclear charge (\( 4\pi \int dr r^2 \rho(r) = 1 \)):

\[ \rho(r) = \rho_0 \frac{1}{1 + \exp\left(\frac{r - c}{\alpha}\right)}. \]  

(2.4)

Here \( c \) is the half-density radius and \( \alpha \) is related to the skin thickness \( t \) by \( t = (4 \log 3) \alpha \), defined as the distance over which the charge density falls from 90% to 10% of its maximum value.

The individual contributions to \( \Delta E \) for muonic atoms of \(^{203}\text{Tl}, {^{205}\text{Tl}, \text{and} {^{209}\text{Bi}} \) are presented in Table 1. The \( \Delta E_{NS} \) values are given in the second column. In the third column we present the BW correction evaluated within the single-particle nuclear model according to the prescriptions given in [6, 7, 8] (see also the next section of the present paper). The wave function of the odd nucleon was determined by solving the Schrödinger equation with the Woods-Saxon potential. In muonic atoms, the QED correction is mainly determined by the vacuum polarization (VP) contribution, which consists of the electronic electric-loop and magnetic-loop parts. In the Uehling approximation, the electric-loop part is determined by the potential:

\[ U^\text{EL}_{\text{Ue}}(r) = -\alpha Z \frac{2\alpha}{3\pi} \int_0^\infty dr' 4\pi \rho(r') \int_1^\infty dt \left( 1 + \frac{1}{2t^2} \right) \frac{\sqrt{t^2 - 1}}{t^2} \times \exp(-2m|r - r'|t) - \exp(-2m(r + r')t)}{4mr'}, \]  

(2.5)

where \( m \) is the electron mass. The corresponding correction (\( \Delta E^\text{EL}_{\text{VP}} \)) is derived as the difference of equations (2.2) with the wave functions obtained by solving the Dirac equation with and without the Uehling potential (2.5). For the correction to the hyperfine splitting due to the magnetic loop one obtains:

\[ \Delta E^\text{ML}_{\text{VP}} = \langle A|U^\text{ML}_{\text{VP}}(r)|A \rangle, \]  

(2.6)

where \( |A \rangle \) is the state vector of the whole atomic system,

\[ U^\text{ML}_{\text{VP}}(r) = H_{\text{hfs}}(r) \frac{2\alpha}{3\pi} \int_1^\infty dt \left( 1 + \frac{1}{2t^2} \right) \frac{\sqrt{t^2 - 1}}{t^2} (1 + 2mr) \exp(-2mrt), \]  

(2.7)

\[ H_{\text{hfs}}(r) \] is the hyperfine interaction operator:

\[ H_{\text{hfs}}(r) = \frac{|e|}{4\pi} \frac{\alpha \cdot [\mu \times r]}{r^3}, \]  

(2.8)
Table 1: Individual contributions to the hyperfine splitting in muonic atoms, in keV.

<table>
<thead>
<tr>
<th>Atom</th>
<th>$\Delta E_{NS}$</th>
<th>$\Delta E_{BW}$</th>
<th>$\Delta E_{QED}$</th>
<th>$\Delta E_{\text{theor}}$</th>
<th>$\Delta E_{\mu}^{\text{exp}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{205}$Tl</td>
<td>4.70</td>
<td>-2.06(66)</td>
<td>0.06</td>
<td>2.70(66)</td>
<td>2.66(30)</td>
</tr>
<tr>
<td>$^{205}$Tl</td>
<td>4.73</td>
<td>-2.06(66)</td>
<td>0.06</td>
<td>2.73(66)</td>
<td>2.32(6)</td>
</tr>
<tr>
<td>$^{209}$Bi</td>
<td>6.69</td>
<td>-2.49(80)</td>
<td>0.09</td>
<td>4.29(80)</td>
<td>4.44(15)</td>
</tr>
</tbody>
</table>

$\mu$ is the nuclear-magnetic-moment operator, and $\alpha$ is a vector incorporating the Dirac matrices. The total QED correction ($\Delta E_{\text{QED}} \approx \Delta E_{\text{ELVP}} + \Delta E_{\text{MLVP}}$) is given in the fourth column of Table 1. The total theoretical values obtained in this work (fifth column) are in a good agreement with the experimental ones (sixth column) and with the previous theoretical calculations [14], which do not account for the QED corrections. Since the theoretical uncertainty is mainly determined by the uncertainty of the BW effect, the experimental values of the hyperfine splitting in muonic atoms can be employed to determine the BW contribution and, therefore, the parameters of the nuclear magnetization distribution for a given nuclear model. Then, with these parameters, the BW correction to the hyperfine splitting in the corresponding electronic ions can be calculated. Such calculations are presented in the third and fourth sections for the single-particle and configuration-mixing nuclear models, respectively.

3 Nuclear magnetization in the single-particle model

In the single particle model, the nuclear magnetization is ascribed to the odd nucleon. The nuclear magnetic moment is given by

$$
\frac{\mu}{\mu_N} = \begin{cases} 
\frac{1}{2} [g_S + (2I - 1)g_L], & I = L + 1/2 \\
\frac{I}{2I+1} [-gs + (2I + 3)g_L], & I = L - 1/2,
\end{cases}
$$

(3.1)

where $I$ and $L$ are the total and orbital angular momenta of the odd nucleon. For proton $g_L = 1$ and for neutron $g_L = 0$. In calculations of the HFS the $g_S$ factor is usually chosen to yield the observed value of the nuclear magnetic moment. To calculate the BW effect within the single-particle model, one has to adopt the replacement $\mu \rightarrow \mu(r) = F(r)\mu$ in the HFS operator. The function $F(r)$ is given by [8]:

$$
F(r) = \frac{\mu_N}{\mu} \left\{ \frac{1}{2} g_S + \left( I - \frac{1}{2} \right) g_L \right\} \int_0^r dr' r'^2 u^2(r') \\
+ \left\{ -\frac{2I - 1}{8(I + 1)} g_S + \left( I - \frac{1}{2} \right) g_L \right\} \int_r^{\infty} dr' r'^2 u^2(r') \left( \frac{r}{r'} \right)^3 \right\}
$$

(3.2)

for $I = L + 1/2$ and

$$
F(r) = \frac{\mu_N}{\mu} \left\{ \left[ -\frac{I}{2(1 + I)} g_S + \frac{I(2I + 3)}{2(1 + I)} g_L \right] \right\} \int_0^r dr' r'^2 u^2(r') \\
+ \left\{ \frac{2I + 3}{8(I + 1)} g_S + \frac{I(2I + 3)}{2(1 + I)} g_L \right\} \int_r^{\infty} dr' r'^2 u^2(r') \left( \frac{r}{r'} \right)^3 \right\}
$$

(3.3)
Table 2: The Bohr-Weisskopf correction to the HFS in electronic hydrogenlike ions, derived from experiments on muonic atoms within the single-particle nuclear model. $\delta_{\varepsilon_{\text{s.p.}}}^{\text{mod}}$ denotes the uncertainty of $\varepsilon_{\text{s.p.}}$ caused by the different parameterizations used in the calculation (see equations (3.5), (3.6)).

<table>
<thead>
<tr>
<th>Ion</th>
<th>$I^z$</th>
<th>$\Delta E_{\text{exp}}^\mu$ (keV)</th>
<th>$\varepsilon_{\text{s.p.}}$</th>
<th>$\delta_{\varepsilon_{\text{s.p.}}}^{\text{mod}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{203}$Tl$^{80+}$ $^{1}f$</td>
<td>$\frac{1}{2}+$</td>
<td>2.66(30) [12]</td>
<td>0.0155(35)</td>
<td>9%</td>
</tr>
<tr>
<td>$^{205}$Tl$^{80+}$ $^{1}f$</td>
<td>$\frac{1}{2}+$</td>
<td>2.32(6) [12]</td>
<td>0.0193(24)</td>
<td>11%</td>
</tr>
<tr>
<td>$^{209}$Bi$^{82+}$ $^{9}g$</td>
<td>$\frac{9}{2}$−</td>
<td>4.44(15) [13]</td>
<td>0.0123(14)</td>
<td>8%</td>
</tr>
</tbody>
</table>

for $I = L - 1/2$. Here $u(r)$ is the radial part of the wave function of the odd nucleon. The relative value of the BW correction, defined by $\varepsilon = -\Delta E_{\text{BW}}/\Delta E_{\text{NS}}$, can be written in the form:

$$\varepsilon_{\text{s.p.}} = 1 - \frac{\int_0^\infty dr F(r)g(r)f(r)}{\int_0^\infty dr g(r)f(r)}.$$  (3.4)

To reproduce the experimental values for the HFS in muonic atoms within the single-particle nuclear model, we have considered the following parameterization for $u(r)$:

$$\begin{cases} 
  u(r) = u_0 r^n, & n = 0, 1, 2, \quad r \leq R_M \\
  u(r) = 0, & r > R_M 
\end{cases}$$  (3.5)

and

$$\begin{cases} 
  u(r) = u_0 (R_M - r)^n, & n = 1, 2, \quad r \leq R_M \\
  u(r) = 0, & r > R_M. 
\end{cases}$$  (3.6)

The constant $u_0$ is determined by: $\int_0^\infty dr r^2 u^2(r) = 1$. The magnetic radius $R_M$ is derived from the equation:

$$\Delta E_{\text{exp}}^\mu - \Delta E_{\text{QED}} = (1 - \varepsilon_{\text{s.p.}}(R_M))\Delta E_{\text{NS}},$$  (3.7)

where $\Delta E_{\text{exp}}^\mu$ and $\Delta E_{\text{QED}}$ are the experimental value of the HFS for muonic atom and the QED correction, accordingly. Then we can calculate the BW correction for the corresponding electronic ions with $R_M$ derived from equation (3.7). We have found that the results for the BW correction ($\varepsilon_{\text{s.p.}}$) in electronic H-like ions are stable enough within the parameterizations (3.5), (3.6). For this reason, the uncertainties of $\varepsilon_{\text{s.p.}}$ presented in Table 2 are mainly determined by the uncertainty of the experimental HFS values in muonic atoms.

### 4 Nuclear magnetization in the configuration-mixing model

In the configuration-mixing model the nuclear magnetism is determined by the last odd nucleon and the particle-hole excited states. So, the HFS (without the QED correction) can be represented as:

$$\Delta E_{\text{c.m.}} = \Delta E_{\text{s.p.}} + \delta E(\Delta \mu),$$  (4.1)

where $\Delta \mu$ is the correction to the nuclear magnetic moment, due to mixing particle-hole states. The nuclear magnetic moment can be written in the form:

$$\mu_{\text{exp}} = \mu_{\text{s.p.}} + \Delta \mu.$$  (4.2)
The formulas for $\Delta E_{s.p.}$ are well known (see, e.g., [14]):

$$\Delta E_{s.p.} = -\frac{4}{3} \frac{1}{m_p} \frac{2I+1}{2I} \left\{ \left[ \frac{1}{2}g_s + \left( I - \frac{1}{2} \right)g_L \right] K_a + \left[ -\frac{2I-1}{8(I+1)}g_s + \left( I - \frac{1}{2} \right)g_L \right] K_b \right\}, \quad I = L + \frac{1}{2},$$

$$\Delta E_{s.p.} = -\frac{4}{3} \frac{1}{m_p} \frac{2I+1}{2I} \left\{ \left[ -\frac{I}{2(I+1)}g_s + \frac{I(2I+3)}{2(I+1)}g_L \right] K_a + \left[ \frac{2I+3}{8(I+1)}g_s + \frac{I(2I+3)}{2(I+1)}g_L \right] K_b \right\}, \quad I = L - \frac{1}{2},$$

where

$$K_a = \int_0^\infty dR R^2 u_{n,L}(R) \int_0^\infty dr g(r)f(r),$$

$$K_b = \int_0^\infty dR R^{-1} u_{n,L}(R) \int_0^R dr g(r)f(r).$$

The correction for the $\Delta L = 0$ mixing terms is given by [14]:

$$\delta E(\Delta \mu) = -\frac{4}{3} \frac{1}{m_p} \frac{2I+1}{2I} \int_0^\infty dr f(r)g(r) \sum_{L'} \zeta_{L'} \Delta \mu$$

$$\times \left\{ (1-K_s)_{L'} + \frac{1}{S} \left[ g_s - g_L \right] \left\{ K_S - K_L \right\}_{L'} \right\}, \quad (4.7)$$

$$\sum_{L'} \zeta_{L'} = 1. \quad (4.8)$$

The quantities $\zeta_{L'}$ determine the re-distribution $\Delta \mu$ between the configurations. We use the following designations:

$$\{ A \}_{L'} = \int_0^\infty dR R^2 u_{n,L',j}(R)u_{n,L',j'}(R)A(R), \quad (4.9)$$

$$K_S(R) = \frac{\int_0^R dr g(r)f(r)}{\int_0^\infty dr g(r)f(r)}, \quad (4.10)$$

$$K_L(R) = \frac{\int_0^R dr (1 - \frac{s^3}{2})g(r)f(r)}{\int_0^\infty dr g(r)f(r)}. \quad (4.11)$$

Here $u_{n,L}$ and $u_{n,L,j}$ are the radial parts of the wave functions of the odd nucleon. They are obtained by solving the Schrödinger equation with the Woods-Saxon potential. We consider $\zeta_{L'} = \frac{1}{M}$, where $M$ is the number of the configurations. The quantities $\Delta \mu$ and $g_S$ are derived from the equations:

$$\frac{\mu_{exp}}{\mu_N} = \begin{cases} \frac{1}{2}g_s + \left( I - \frac{1}{2} \right)g_L + \frac{\Delta \mu}{\mu_N} \sum_{L'} \zeta_{L'} \{1\}_{L'}', & I = L + \frac{1}{2} \\ \frac{I}{2(I+1)}g_s + \frac{I(2I+3)}{2(I+1)}g_L + \frac{\Delta \mu}{\mu_N} \sum_{L'} \zeta_{L'} \{1\}_{L'}', & I = L - \frac{1}{2}, \end{cases} \quad (4.12)$$
Table 3: The Bohr-Weisskopf correction to the HFS in electronic hydrogen-like ions, derived from experiments on muonic atoms within the configuration-mixing model.

<table>
<thead>
<tr>
<th>Ion</th>
<th>( I^\pi )</th>
<th>( \Delta E_{\text{exp}}^{\mu} ) (keV)</th>
<th>( \varepsilon_{\text{c.m.}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{203}\text{Ti}^{80+})</td>
<td>( \frac{1}{2}^+ )</td>
<td>2.66(30) ([12])</td>
<td>0.0179(36)</td>
</tr>
<tr>
<td>(^{205}\text{Ti}^{80+})</td>
<td>( \frac{1}{2}^+ )</td>
<td>2.32(6) ([12])</td>
<td>0.0214(6)</td>
</tr>
<tr>
<td>(^{209}\text{Bi}^{82+})</td>
<td>( \frac{5}{2}^- )</td>
<td>4.44(15) ([13])</td>
<td>0.0119(11)</td>
</tr>
</tbody>
</table>

\[
\Delta E_{\text{exp}}^{\mu} - \Delta E_{\text{QED}} = \Delta E_{s.p.}(g_S) + \delta E(\Delta \mu).
\] (4.13)

The Bohr-Weisskopf correction is determined by:

\[
1 - \varepsilon_{\text{c.m.}} = \frac{\Delta E_{s.p.}(g_S) + \delta E(\Delta \mu)}{\Delta E_{\text{NS}}}. \tag{4.14}
\]

With \( g_S \) and \( \Delta \mu \) derived from equations (4.12), (4.13), we can calculate the BW correction for the electronic ions. The results of these calculations are presented in Table 3. The values of \( \varepsilon_{\text{c.m.}} \) given in Table 3 are in a good agreement with \( \varepsilon_{s.p.} \) presented in Table 2.

5 Results and discussion

In Table 3 we compare the BW correction to the HFS in electronic H-like ions, derived from the experiments on muonic atoms. \( \varepsilon_{s.p.} \) and \( \varepsilon_{\text{c.m.}} \) are obtained employing the single-particle and configuration-mixing models, respectively. \( \varepsilon \) is a value, which corresponds to the parameterization (3.5) with \( n = 0 \) and whose uncertainty covers all the \( \varepsilon_{s.p.} \) and \( \varepsilon_{\text{c.m.}} \) values. For comparison, the values of the BW correction obtained previously by direct calculations within the single-particle and many-particle models are presented as well. We conclude that our results for \( \varepsilon \) are stable enough with respect to a change of the nuclear model. They also have a better accuracy than the previous single-particle results \([6, 7, 8, 9]\).

In Table 4 we present our final theoretical results for the HFS in electronic hydrogen-like ions. These results, which include the BW correction derived in the present work and the QED correction taken from \([7, 15, 16, 17]\), are compared with previous calculations and with experiment. As one can see from the table, our results are closer to the experimental ones, compared to the results based on the direct calculations within the single-particle nuclear model \([8]\). However, due to a higher accuracy of the present results, a small discrepancy between the theory and experiment occurs for \(^{209}\text{Bi}^{82+}\). The reason for this discrepancy is unclear to us.

Acknowledgments

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Table 4: The Bohr-Weisskopf correction to the HFS in electronic H-like ions, derived from experiments on muonic atoms.

<table>
<thead>
<tr>
<th></th>
<th>209Bi82+</th>
<th>205Tl80+</th>
<th>203Tl80+</th>
</tr>
</thead>
<tbody>
<tr>
<td>ε_{s.p.}</td>
<td>0.0123(14)</td>
<td>0.0193(24)</td>
<td>0.0155(35)</td>
</tr>
<tr>
<td>ε_{c.m.}</td>
<td>0.0119(11)</td>
<td>0.0214(6)</td>
<td>0.0179(36)</td>
</tr>
<tr>
<td>ε</td>
<td>0.0123(15)</td>
<td>0.0193(27)</td>
<td>0.0155(40)</td>
</tr>
</tbody>
</table>

Shabaev et al. [8] ε 0.0118 0.0179 0.0179
Labzowsky et al. [9] ε 0.0131 0.0210 0.0210
Tomaselli et al. [10] ε 0.0210 0.0210 0.0210
Sen’kov and Dmitriev [11] ε 0.0095(−7−3) 0.0095(−7−3) 0.0095(−7−3)

Table 5: The total theoretical results for the hyperfine splitting in electronic H-like ions, in eV.

<table>
<thead>
<tr>
<th></th>
<th>Theory</th>
<th>Theory</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[this work]</td>
<td>[8]</td>
<td>[5]</td>
</tr>
<tr>
<td>209Bi82+</td>
<td>5.098(7)</td>
<td>5.101(27)</td>
<td>5.0840(8)</td>
</tr>
<tr>
<td>205Tl80+</td>
<td>3.238(9)</td>
<td>3.261(18)</td>
<td>3.24410(29)</td>
</tr>
<tr>
<td>203Tl80+</td>
<td>3.220(20)</td>
<td>3.229(17)</td>
<td>3.21351(25)</td>
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</tbody>
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References


