(2 + 1) noncommutative gravity and conical spacetimes

P. Valtancoli

Dipartimento di Fisica, Polo Scientifico Universitá di Firenze
and INFN, Sezione di Firenze (Italy)
Via G. Sansone 1, 50019 Sesto Fiorentino, Italy

Abstract

We solve (2 + 1) noncommutative gravity coupled to point-like sources. We find continuity with Einstein gravity since we recover the classical gravitational field in the $\theta \to 0$ limit or at large distance from the source. It appears a limitation on the mass which is twice than expected. Since the distance is not gauge invariant, the measure of the deficit angle near the source is intrinsically ambiguous, with the gauge group playing the role of statistical ensemble. Einstein determinism can be recovered only at large distance from the source, compared with the scale of the noncommutative parameter $\sqrt{\theta}$. 
1 Introduction

General relativity has always been regarded as one of the most important achievements of mankind, despite some philosophers have criticized it for its strict determinism.

In the meantime the level of abstraction in mathematics has increased from Einstein’s age and it has permitted us conceiving spaces in which the locality principle is lost. Nowadays space-time can be substituted by a noncommutative algebra and the ordinary differential structures, i.e. derivative and integration, can be generalized to more abstract operations such as the commutator and the trace. Non-commutativity, connected to these structures, has certain analogy with quantum mechanics, so one might expect that, once inserted in the physical laws, it produces a new fundamental limit on the possibility of determining and measuring the physical quantities, which are object of experimental tests.

Following such general reasoning, we have decided to study in a concrete example how the inspiring principles of general relativity are extended in the noncommutative case [1]-[2]-[3]. We have chosen the case of (2 + 1) gravity [4]-[5] coupled to a point mass source [6]-[7], where meaningful results are simple to reach.

In classical Einstein gravity mass is the primary source for deforming space-time, and a test particle is gravitationally influenced by a mass source because it follows the geodesic of a deformed space-time.

In the literature ([6]) it is questioned that mass is the source for deforming space-time in noncommutative gravity. Our first objective was then to build an explicit solution of the noncommutative equations of motion confirming that mass is really the only cause for deforming space-time. To achieve such result we have taken advantage of the definition given in [6] of point-like source, an extended source which only in the $\theta \to 0$ limit reduces to a delta-function singularity. We have then rewritten the equations of motion in the form of a commutator between operators, following the matrix model formalism. With simple manipulations we can prove the existence of a non trivial solution for the spin connection and the vierbein.

By carrying out the correct classical limit, we can show that the noncommutative gravitational field is a smooth deformation of the classical one [8]-[9]. Moreover it is possible to extend such results to the massive and spinning case, i.e. in presence of a torsion source.

At this point it is natural to rise the following question. In what sense noncommutative gravity can be distinguished from general relativity? What is its characteristic signal?

To answer such question we have to verify the other inspiring principle of general relativity, i.e. that the influence on a test particle of a massive source is determined by the
geodesic motion on a deformed space-time.

In (2 + 1) dimensions the scattering angle of a test particle is strictly related to the deficit angle of the conical space-time. To reveal such deficit angle it is necessary integrating the distance on a circumference centered on the mass source. However here we encounter a serious problem, since the distance is not a gauge invariant concept in noncommutative gravity, differently from the classical case. Two options are open: i) finding a gauge-invariant quantity substituting the distance in the noncommutative case (hard task), ii) working with the distance, as it is defined in general relativity, keeping in mind that its meaning is intrinsically ambiguous since it is not gauge invariant.

In the final discussion of this paper we attempt to give a partial answer to this question, reducing such arbitrariness in the proximity of the source, by imposing that the internal gauge transformations reduce to a constant transformation at spatial infinity. Proceeding this way we can at least assure continuity with general relativity. At large distance from the source, compared with the scale $\sqrt{\theta}$ of the noncommutative parameter, both theories give coincident predictions, but at small distance from the source Einstein determinism is irremediably lost, and the reason of such indeterminism comes from the clash between the concept of distance and the noncommutative symmetries of the model.

2 Lagrangian of (2 + 1) noncommutative gravity

Noncommutative gravity is a modification of classical Einstein gravity compatible with a noncommutative algebra between the coordinates. In (2 + 1) dimensions a particularly convenient example for such an algebra is represented by a noncommutative plane:

$$[\hat{x}, \hat{y}] = i\theta \Rightarrow [\hat{z}, \hat{\tau}] = 2\theta \quad \hat{z} = \hat{x} + i\hat{y} \quad (2.1)$$

The noncommutative analogue of the Einstein action can be easily obtained in the first order formalism by gauging the internal $SO(2,1)$ Lorentz invariance, extended to $U(1,1)$ gauge group. The generators of $U(1,1)$ group are defined as

$$\tau_1 = i\sigma_1 \quad \tau_2 = i\sigma_2 \quad \tau_3 = \sigma_3 \quad (2.2)$$

They satisfy the following hermiticity conditions

$$\tau_1^\dagger = \tau_3\tau_1\tau_3 \quad (2.3)$$
and the composition property

\[ \tau_i \tau_j = \eta_{ij} - i \epsilon_{ijk} \eta^{km} \tau_m \]  

(2.4)

The $U(1,1)$ gauge group is necessary to take into account the signature of the flat metric tensor $\eta^{\mu \nu} = (- - +)$, typical of $(2 + 1)$ dimensions.

We prefer working in the operator formalism instead of the star product formalism, and in analogy with the vierbein and spin connection fields we define two basic operators:

\[
Y_{\mu} = e_{\mu}^a \tau_a + e_{\mu}^0 \\
X_{\mu} = \hat{p}_{\mu} + \frac{\omega_{\mu}^a}{2} \tau_a + \omega_{\mu}^0
\]

(2.5)

where all the components $(e_{\mu}^a, e_{\mu}^0)$ and $(\omega_{\mu}^a, \omega_{\mu}^0)$ are operators acting on the Hilbert space on which the basic commutation relations (2.1) are represented.

The classical property that the vierbein and spin connection are real fields can be taken into account in this scheme by requiring the following hermiticity conditions for the basic matrices $(X_{\mu}, Y_{\mu})$:

\[
Y_{\mu}^\dagger = \tau_3 Y_{\mu} \tau_3 \quad \Rightarrow \quad e_{\mu}^{a\dagger} = e_{\mu}^a \quad e_{\mu}^{0\dagger} = e_{\mu}^0 \\
X_{\mu}^\dagger = \tau_3 X_{\mu} \tau_3 \quad \Rightarrow \quad \omega_{\mu}^{a\dagger} = \omega_{\mu}^a \quad \omega_{\mu}^{0\dagger} = \omega_{\mu}^0
\]

(2.6)

All these component are by construction hermitian operators. The background of the matrix model $\hat{p}_{\mu}$ has to satisfy the commutation relations:

\[
[\hat{p}_{\mu}, \hat{p}_{\nu}] = i \theta_{\mu\nu}^{-1}
\]

(2.7)

where the tensor on the right hand side is defined for the noncommutative plane as

\[
\theta_{\mu\nu}^{-1} = \frac{1}{\hat{\theta}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

(2.8)

The classical pure Einstein action on a noncommutative plane is

\[
S_E = \beta \, e^{\mu \nu} \, Tr \left[ Y_{\mu} \left( [X_{\nu}, X_{\rho}] - i \theta_{\nu\rho}^{-1} \right) \right]
\]

(2.9)
To discuss the properties of this action, it is convenient to define the curvature and torsion tensors:

\[
R_{\mu\nu} = [X_\mu, X_\nu] - i\theta^{-1}_{\mu\nu}
\]
\[
T_{\mu\nu} = [X_\mu, Y_\nu] - [X_\nu, Y_\mu]
\] (2.10)

By using the results of ref. [4] we can state that the Einstein action \( S_E \) is endowed with a deformed diffeomorphism invariance defined as

\[
\delta_v X_\mu = \frac{1}{2} \{ v^\alpha, R_{\mu\alpha} \}
\]
\[
\delta_v Y_\mu = \frac{1}{2} \{ v^\alpha, T_{\mu\alpha} \}
\] (2.11)

By substituting these transformations into the action we find

\[
\delta_v S_E = \frac{1}{2} \epsilon^{\mu\nu\alpha} Tr \left( \delta_v Y_\mu R_{\nu\alpha} + \delta_v X_\mu T_{\nu\alpha} \right) = 0
\]
\[
= \frac{1}{4} Tr \left( v^\beta \epsilon^{\mu\nu\alpha} - v^\alpha \epsilon^{\mu\nu\beta} - v^\mu \epsilon^{\beta\nu\alpha} - v^\nu \epsilon^{\alpha\mu\beta} \right) T_{\mu\beta} R_{\nu\alpha}
\] (2.12)

but the tensor in parenthesis is trivially null. A similar trick applies in two dimensions [10]-[11].

These observations are generalizable to the more general action of \((2 + 1)\) gravity containing the cosmological constant term:

\[
S_T = S_+ - S_-
\]
\[
S_\pm = \epsilon^{\mu\nu\alpha} Tr \left[ X_\pm^\mu \left( \frac{1}{3} [X_\mu^\pm, X_\alpha^\pm] - i\theta^{-1}_{\nu\alpha} \right) \right]
\]
\[
X_\pm^\mu = X_\mu \pm Y_\mu
\] (2.13)

In this case the generator of extended general covariance is defined as

\[
\delta_v X_\pm^\mu = \frac{1}{2} \{ v^\alpha, R_{\pm\alpha}^\mu \}
\]
\[
R_{\pm \mu\nu} = [X_\pm^\mu, X_\pm^\nu] - i\theta^{-1}_{\pm \mu\nu}
\] (2.14)
The equations of motion of $(2 + 1)$ noncommutative Einstein gravity coincide with the vanishing of both curvature and torsion tensors:

\[
R_{\mu\nu} = [X_\mu, X_\nu] - i\theta_{\mu\nu}^{-1} = 0
\]
\[
T_{\mu\nu} = [X_\mu, Y_\nu] - [X_\nu, Y_\mu] = 0
\] (2.15)

These conditions are mapped into themselves by gauge invariance

\[
X_\mu \rightarrow U^{-1}X_\mu U
\]
\[
Y_\mu \rightarrow U^{-1}Y_\mu U
\] (2.16)

where $U$ is a $U(1,1)$ gauge transformation satisfying the unitarity condition

\[
U^\dagger \tau_3 U \tau_3 = 1
\] (2.17)

and by general covariance.

The classical limit ( $\theta \to 0$ limit ) is obtained by imposing that the background $\hat{p}_\mu$ has the following representation on the symbols of the operators ($e^a_\mu, \ldots, \omega^a_\mu, \ldots$):

\[
[\hat{p}_\mu, \ldots] \rightarrow -i\partial_\mu(\ldots)
\] (2.18)

While the classical limit is better understood in the star product formalism, we will prefer to solve the gravitational equations of motion directly in the operator formalism, where we will find a considerable simplification allowing us to generalize the typical conical space-times of Einstein gravity.

### 3 Conical solutions: spin connection

Our motivation is to study the deformation of space-time induced by a static mass source, in presence of a noncommutative plane. The first step is the definition of a mass source, which has been already solved in ref. [6], with the introduction of ”point-like” source ( although we are afraid that this coupling doesn’t respect any kind of general covariance for $\theta \neq 0$ ). This is a distributed source which only in the $\theta \to 0$ limit produces a singular delta-function.
source. In the Hilbert space there is in fact a natural candidate for such a source, i.e. a simple operator whose symbol is a well known representation of a delta-function:

\[ \text{Pointlike source} \equiv P_0 = |0><0| \]  

(3.1)

The symbol corresponding to \( P_0 \) is in fact

\[ P_0 = |0><0| \longrightarrow 2e^{-r^2/\theta} \]  

(3.2)

and the representation of a delta-function can be realized as:

\[ \lim_{\theta \to 0} \frac{1}{2\pi \theta} P_0 = \delta^2(x) \]  

(3.3)

Formula (3.2) is a particular example of the general transformation rule between operators and symbols on the noncommutative plane:

\[ |n >< n + l| \equiv 2(-1)^n \sqrt{\frac{n!}{(n+l)!}} \left( \frac{2r^2}{\theta} \right)^{\frac{l}{2}} L^l_n \left( \frac{2r^2}{\theta} \right)^{\frac{l}{2}} e^{-\frac{r^2}{\theta}} e^{il\phi} \]  

(3.4)

where \( L^l_n(z) \) are the generalized Laguerre polynomials. For example the coordinate \( z \) can be replaced by the following operator:

\[ \hat{z} = \sqrt{2\theta} \sum_{n=0}^{\infty} \sqrt{n+1} |n >< n + 1| \]

\[ \hat{\bar{z}} = \sqrt{2\theta} \sum_{n=0}^{\infty} \sqrt{n+1} |n + 1 >< n| \]  

(3.5)

and it automatically satisfies the commutation relations:

\[ [\hat{z}, \hat{\bar{z}}] = 2\theta \]  

(3.6)

In the classical solutions of (2 + 1) gravity \[8-9\] the inverses of \( z \) and \( \bar{z} \) appear. To match the noncommutative solution with the standard results it is useful to introduce the following operators:

\[ \frac{1}{\hat{z}} = \frac{1}{\sqrt{2\theta}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n + 1 >< n| = \frac{1}{r} (1 - e^{-r^2/\theta}) e^{-i\phi} \]  

(3.7)
\[
\frac{1}{z} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n > < n+1| = \frac{1}{r}(1 - e^{-r^2/\theta})e^{i\phi}
\]

\[
\frac{1}{z^\dagger} \hat{z} = \frac{\hat{z}}{z^\dagger} = 1 - |0 > < 0|
\]  

(3.7)

Therefore the classical formula

\[
\partial_z \frac{1}{z} = \pi \delta^2(x)
\]  

(3.8)

can be replaced in the noncommutative theory by:

\[
\frac{1}{2\theta} \left[ \hat{z}, \frac{1}{z} \right] = \frac{1}{2\theta} |0 > < 0| = \frac{1}{\theta} e^{-r^2/\theta} \theta \to 0 \pi \delta^2(x)
\]  

(3.9)

We are ready to discuss the solution for the spin connection in presence of a massive source. Let us define the following complex combinations of $X_\mu$ matrices:

\[
X = X_z = \frac{1}{2}(X_1 - iX_2) = \hat{p}_z + \frac{\omega^a}{2} r_a + \omega^0
\]

\[
\overline{X} = \overline{X}_z = \overline{\hat{p}_z} + \frac{\omega^a}{2} r_a^\dagger + \omega^0
\]

\[
\hat{p}_z = \frac{1}{2}(\hat{p}_1 - i\hat{p}_2) \to [\hat{p}_z, \ldots] \theta \to 0 -i\partial_z(\ldots)
\]  

(3.10)

The second step is showing that the equations of motion with point-like source can be written in the form of a commutator:

\[
[ X, \overline{X} ] = -\frac{1}{2\theta} \left( 1 - \frac{m}{4\pi} P_0 \tau_3 \right)
\]

\[
\lim_{\theta \to 0} \frac{1}{\theta} P_0 = 2\pi \delta^2(x)
\]  

(3.11)

where $m$ is the mass of the point-like source.

It is enough to show that this equation has as classical limit the Einstein equation coupled to point sources (see ref. [8]-[9]):

\[
(\partial_{[\mu} \omega_{\nu]} + \omega_{[\mu}, \omega_{\nu]})^a_b = -\epsilon_{\mu\nu\lambda} e_b e^\lambda P^c \delta^a_a (x^\mu - \xi^\mu(t)) \quad (8\pi G = 1 \text{ units})
\]  

(3.12)

where the impulse $P^a$ and the velocity vector $v^a$ are defined as
\[ P^a = mv^a = m\gamma(\vec{v},1) \quad (3.13) \]

and \( \xi^\mu(t) \) is the position vector of the source.

In the simplified case we are discussing of a rest particle \( (\vec{v} = 0) \), and introducing the auxiliary spin connection

\[ \omega^a_\mu = -\frac{1}{2} \eta^{aa'} \epsilon_{a'bc} \omega^b_{\mu} \Leftrightarrow \omega_{\mu,ab} = -\epsilon_{abc} \omega^c_\mu \quad (3.14) \]

the equations of motion \((3.12)\) become

\[ \partial_\mu \omega^a_\nu + \frac{1}{2} \eta^{aa'} \epsilon_{a'bc} \omega^b_{[\mu} \omega^c_{\nu]} = \epsilon_{\mu\nu3} m \delta^a_3 \delta^2(x) \quad (3.15) \]

To adjust the classical limit of the commutator equation \((3.11)\) we must choose the background \( \hat{p}_\mu \) as

\[ \begin{align*}
\hat{p}_x &= \frac{1}{\theta} \hat{y} & \hat{p}_x, \ldots & \rightarrow -i \partial_x \\
\hat{p}_y &= -\frac{1}{\theta} \hat{x} & \hat{p}_y, \ldots & \rightarrow -i \partial_y \\
\hat{p}_z &= \frac{i}{2\theta} \hat{z}
\end{align*} \quad (3.16) \]

Then the commutator term can be developed in the \( \theta \rightarrow 0 \) limit:

\[ \begin{align*}
[X_x, X_y] & \xrightarrow{\theta \rightarrow 0} \frac{i}{\theta} \left[ (\partial_x \omega^a_y - \partial_y \omega^a_x) + \frac{1}{2} \eta^{aa'} \epsilon_{a'bc} \{ \omega^b_{\mu}, \omega^c_{\nu} \} \right] \tau_a \\
&= \frac{i}{\theta} - i \frac{m}{2} \tau_3 \delta^2(x) & \theta \rightarrow 0 & \frac{i}{\theta} \left[ 1 - \frac{m}{4\pi} P_0 \tau_3 \right]
\end{align*} \quad (3.17) \]

The second equality can be obtained using the classical equation of motion \((3.13)\) or as a classical limit of the source of the commutator equation \((3.11)\). We have then proved the equivalence of the two equations of motion \((3.11)\) and \((3.12)\).

It is interesting to note that the factor in front of the projector \( P_0 \) is proportional to the combination \( \mu = \frac{m}{2\pi} \) entering in the condition \( (\mu < 1) \) for the existence of conical solutions.

We can anticipate that we will be able to obtain a noncommutative limit on mass, which is however twice than expected, by imposing that the solution of the commutator equation respects the hermicity condition:
\[ X^\dagger_\mu = \tau_3 X\mu \tau_3 \]

\[
[X_x, X_y] = \frac{i}{\theta} \left[ 1 - \frac{\mu}{2} P_0 \tau_3 \right] \quad \Leftrightarrow \quad \frac{\mu}{2} < 1 \quad (3.18)
\]

Fortunately the solution we are going to discuss is contained in the following simple ansatz:

\[
X = i \sum_{n=0}^{\infty} (f(n) - \tau_3 g(n)) |n + 1><n| \]

\[
X = -i \sum_{n=0}^{\infty} (f(n) - \tau_3 g(n)) |n><n+1| \quad (3.19)
\]

The hermicity condition (2.6) is satisfied by the reality condition on the unknown functions \( f(n) \) and \( g(n) \):

\[
f^\dagger(n) = f(n) \quad g^\dagger(n) = g(n) \quad (3.20)
\]

In the massless limit \( \mu \to 0 \), the commutator equation is solved by the background, i.e. by the choice

\[
f(n) = \sqrt{\frac{n+1}{2\theta}} \quad g(n) = 0 \quad (3.21)
\]

The commutator equation (3.11), together with the ansatz (3.19), produces the following recursive relations:

\[
f^2(n) + g^2(n) = f^2(n-1) + g^2(n-1) + \frac{1}{2\theta} = \frac{n+1}{2\theta}
\]

\[
f(n)g(n) = f(n-1)g(n-1) = \frac{\mu}{8\theta} \quad (3.22)
\]

which are solved by:

\[
f(n) = \frac{1}{2\sqrt{2\theta}} \left( \sqrt{n+1 + \frac{\mu}{2}} + \sqrt{n+1 - \frac{\mu}{2}} \right)
\]

\[
g(n) = \frac{1}{2\sqrt{2\theta}} \left( \sqrt{n+1 + \frac{\mu}{2}} - \sqrt{n+1 - \frac{\mu}{2}} \right) \quad (3.23)
\]
In the $\mu \to 0$ limit we recover the background solution (3.21). It is now clear that the hermicity condition is respected if and only if

$$f^\dagger(n) = f(n) \quad g^\dagger(n) = g(n) \quad \iff \quad n + 1 - \frac{\mu}{2} > 0 \quad \forall n \quad \iff \quad \frac{\mu}{2} < 1 \quad (3.24)$$

We have found that the operator formalism of noncommutative gravity has some similarity with the results of classical Einstein gravity. A better confirmation comes from the discussion of the classical limit (see section 5).

## 4 Conical solutions: vierbein

A gravity theory is based on the concept of metric. Therefore to complete the solution for point-like sources we need to extract the vierbein from the null torsion condition. In the massless case the natural choice is the flat vierbein

$$Y_\mu = \delta^a_\mu \tau_a \quad (4.1)$$

which can be recast in a more convenient form:

$$Y_\mu = i[X_\mu, \Lambda] \quad \Lambda = \hat{x}\tau_1 + \hat{y}\tau_2 + t\tau_3 \quad (4.2)$$

The null torsion condition

$$T_{\mu\nu} = [X_\mu, Y_\nu] - [X_\nu, Y_\mu] = 0 \quad (4.3)$$

is automatically satisfied by the ansatz (4.2), since due to the Jacobi identity

$$T_{\mu\nu} = i[[X_\mu, X_\nu], \Lambda] = -\theta^{-1}_{\mu\nu}[1, \Lambda] = 0 \quad (4.4)$$

the equation of motion reduces to a commutator with a c-number, which is trivially null, for every choice of $\Lambda$. However requiring that the metric is flat in absence of sources fixes for $\Lambda$ the form given in eq. (4.2).

Let us study the case $\mu \neq 0$, with $X_\mu$ given by eqs. (3.19) and (3.23). The ansatz (4.2) is again useful:
\[ Y_\mu = i[X_\mu, \Lambda] \quad \Lambda^\dagger = \tau_3 \Lambda \tau_3 \]  
\hspace{1cm} (4.5) 

with \( \Lambda \) an unknown operator.

By using the Jacobi identity, since the commutator of two spin connections is proportional to the projector operator \( P_0 \), the null torsion condition is solved by the following condition on \( \Lambda \):

\[ [\Lambda, P_0 \tau_3] = 0 \quad \Lambda \xrightarrow{\theta \to 0} \hat{x}_a \tau_a \]  
\hspace{1cm} (4.6) 

A natural choice is dressing the flat solution with quasi-unitary operators:

\[ \Lambda = U^\dagger \hat{x}_a U \tau_a \]  
\[ U = \sum_{n=0}^\infty |n \rangle \langle n + 1| \]  
\hspace{1cm} (4.7) 

due to the properties

\[ U P_0 = P_0 U^\dagger = 0 \]  
\hspace{1cm} (4.8) 

Let us define some new coordinate operators:

\[ \hat{z}' = \sqrt{2\theta} \sum_{n=0}^\infty \sqrt{n} \ |n \rangle \langle n + 1| = U^\dagger \hat{z} U \]  
\[ \hat{z}' = \sqrt{2\theta} \sum_{n=0}^\infty \sqrt{n} \ |n + 1 \rangle \langle n| = U^\dagger \hat{z} U \]  
\hspace{1cm} (4.9) 

The coordinate operators \( \hat{z}' \) and \( \hat{z}' \) share the same classical limit with \( \hat{z} \) and \( \hat{z}' \) and differ only for terms of the order \( \sqrt{\theta} \) at large distance from the source. However near the source we have \( \hat{z}' P_0 = P_0 \hat{z}' = 0 \) while \( [\hat{z}, P_0] \neq 0 \).

To obtain the final result we have only to develop equation (4.5):

\[ Y = \frac{1}{2} (Y_1 - iY_2) = i[X, \Lambda] = i \left[ X, \hat{z}' \left( \frac{\tau_1 - i\tau_2}{2} \right) + \hat{z}' \left( \frac{\tau_1 + i\tau_2}{2} \right) \right] \]  
\hspace{1cm} (4.10)
Using the Pauli matrices algebra

\[ \tau_3 (\tau_1 \pm i \tau_2) = -(\tau_1 \pm i \tau_2) \tau_3 = \pm (\tau_1 \pm i \tau_2) \]  

(4.11)

we can simplify the result as

\[
Y = - \left( \frac{\tau_1 - i \tau_2}{2} \right) \sum_{n=0}^{\infty} \left[ \sqrt{n} \left( n + 1 + \frac{\mu}{2} \right) |n+1><n+1| - \sqrt{n} \left( n + 1 - \frac{\mu}{2} \right) |n><n| - \left( \frac{\tau_1 + i \tau_2}{2} \right) \sum_{n=0}^{\infty} \left[ \sqrt{n} \left( n + 2 - \frac{\mu}{2} \right) |n+2><n+2| - \sqrt{(n+1) (n+1 + \frac{\mu}{2})} |n+2><n| \right] \]

(4.12)

5 Classical limit

To check the classical limit of the noncommutative solution (eqs. (3.23) and (4.12)), we must recall the classical results of ref. [8]-[9] in presence of a point source:

\[
e^a_\mu = \delta^a_\mu + \mu n^a \mu,
\]
\[
n_\mu = \left( \epsilon_{ij} x^j r, 0 \right) = \left( \frac{y}{r}, -\frac{x}{r}, 0 \right)
\]
\[
\omega_{\mu,ab} = \epsilon_{ab3} \frac{\mu n_\mu}{r},
\]
\[
\omega^a_i = -\mu \delta^a_3 \epsilon_{ij} x^j r^2
\]

(5.1)

Working out the components we obtain

\[
\omega^3_z = \frac{1}{2} (\omega^3_x - i \omega^3_y) = \frac{i \mu}{2z}
\]
\[
e^z_x = 1 - \frac{\mu}{2},
\]
\[
e^z_\mu = \frac{\mu}{2 z}
\]

(5.2)

The conical singularity becomes evident in the metric tensor built from this form of the vierbein:

\[
g_{zz} = e^z_\mu e^\mu_\mu = \frac{\mu}{2} \left( 1 - \frac{\mu}{2} \frac{\tau}{z} \right)
\]
\( g_{z\bar{z}} = \frac{1}{2} (e^z \bar{e}^z + e^\bar{z} \bar{e}^\bar{z}) = \frac{1}{2} \left( 1 - \mu + \frac{\mu^2}{2} \right) \)

\[ ds^2 = \bar{d}z d\bar{z} + \frac{\mu}{2} \left( 1 - \frac{\mu}{2} \right) r^2 \left( \bar{z} d\bar{z} - z d\bar{z} \right)^2 = dr^2 + (1 - \mu)^2 r^2 d\phi^2 \]  

\( (5.3) \)

At a noncommutative level it is simpler to compare the spin connection given by eq. (3.19):

\[ X = \hat{p}_z + \frac{\omega^a}{2} \tau_a + \omega^0 = i \sum_{n=0}^{\infty} (f(n) - \tau_3 g(n)) \left| n + 1 \right\rangle < n \right| \]  

\( (5.4) \)

Identifying the components we finally find:

\[ \omega^3_z = -2i \sum_{n=0}^{\infty} g(n) \left| n + 1 \right\rangle < n \right| = -i \frac{\mu}{\sqrt{2}} \sum_{n=0}^{\infty} \left( \sqrt{n + 1 + \frac{\mu}{2}} - \sqrt{n + 1 - \frac{\mu}{2}} \right) \left| n + 1 \right\rangle < n \right| \]

\[ \omega^0_z = \frac{i}{2\sqrt{2}} \sum_{n=0}^{\infty} \left( \sqrt{n + 1 + \frac{\mu}{2}} + \sqrt{n + 1 - \frac{\mu}{2}} - 2\sqrt{n + 1} \right) \left| n + 1 \right\rangle < n \right| \]  

\( (5.5) \)

A first look shows that \( \omega^3_z \sim O(\mu) \) while \( \omega^0_z \sim O(\mu^2) \). However since the development parameter is really \( \frac{\mu}{n+1} \), every power in \( \mu \) corresponds to a large distance behavior of the order \( \frac{\mu}{r} \). We therefore conclude that \( \omega^3_z \) has a nontrivial classical limit, while \( \omega^0_z \to 0 \), as it should be.

Let us work out the first order contribution in \( \mu \):

\[ \omega^3_z \approx -i \frac{\mu}{2\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n + 1}} \left| n + 1 \right\rangle < n \right| = -i \frac{\mu}{2\sqrt{2}} \]  

\( (5.6) \)

and \( \omega^3_z \) coincides with the operator corresponding to the inverse of \( z \), confirming that we have reached the right classical limit, while \( \omega^0_z \) contains only terms \( O(\frac{\mu}{\sqrt{r}}) \) which vanish in the \( \theta \to 0 \) limit.

The vierbein operator can be simplified by replacing

\[ f(n) \to 0 \quad \sqrt{\frac{n + 1}{2}} \quad g(n) \to 0 \quad \frac{\mu}{4\sqrt{2}\sqrt{n + 1}} \]  

\( (5.7) \)

and by approximating the factors \( \sqrt{n} \approx \sqrt{n + 1} \). In this way we obtain the correct classical limit
\[ Y \approx \left(1 - \frac{\mu}{2}\right) \left(\frac{\tau_1 - i\tau_2}{2}\right) + \frac{\mu}{4} \left(\frac{\tau_1 + i\tau_2}{2}\right) \sum_{n=0}^{\infty} \left(\sqrt{\frac{n+2}{n+1}} + \sqrt{\frac{n+1}{n+2}}\right) |n + 2 < n| \]

\[ + O\left(\frac{\sqrt{\theta}}{r}\right) = \left(1 - \frac{\mu}{2}\right) \left(\frac{\tau_1 - i\tau_2}{2}\right) + \frac{\mu}{4} \left(\frac{\tau_1 + i\tau_2}{2}\right) \left(\frac{1}{z\bar{z}} + \frac{1}{z^*\bar{z}}\right) + O\left(\frac{\sqrt{\theta}}{r}\right) \] (5.8)

in perfect agreement with

\[ e_z^z = 1 - \frac{\mu}{2} \quad e_{\bar{z}}^{\bar{z}} = \frac{\mu}{2} \left(\frac{\bar{z}}{z}\right) \] (5.9)

6 The spinning case

It is natural to generalize all these results to the case of a massive and spinning source (the \( \mu \to 0 \) is already solved in ref. [7]). This is easily obtained by adding to the massive solution an extra particular solution for the vierbein corresponding to the torsion source.

Again the torsion source is defined through a representation of a delta-function and the equations of motion to solve are

\[ [X_\mu, X_\nu] = i \frac{\theta}{\epsilon_{\mu\nu3}} \left[1 - \frac{m}{4\pi P_0\tau_3}\right] \]

\[ T_{\mu\nu} = [X_\mu, Y_\nu] - [X_\nu, Y_\mu] = -i \frac{s}{2\pi\theta} \epsilon_{\mu\nu3} P_0\tau_3 \quad (8\pi G = 1 \text{ units}) \] (6.1)

As in the classical case, the solution is obtained by adding to the vierbein operator (4.12) an extra term \( Y_\mu^S \):

\[ X_\mu = X_\mu (\mu \neq 0, s = 0) \quad Y_\mu = Y_\mu (\mu \neq 0, s = 0) + Y_\mu^S \] (6.2)

In complex coordinates the torsion equation reads:

\[ [X, Y^S] - [\bar{X}, Y^S] = \frac{s}{4\pi\theta} P_0\tau_3 \] (6.3)

By introducing for the unknown operator \( Y^S \) the same ansatz of \( X_\mu \):
\[ Y^S = i \sum_{n=0}^{\infty} (h(n) - \tau_3 k(n)) |n + 1 >> n| \]  

(6.4)

The torsion equation produces the recursive relations for the coefficients \( h(n), k(n) \):

\[
\begin{align*}
  f(n)h(n) + g(n)k(n) &= 0 \\
  f(n)k(n) + g(n)h(n) &= \frac{s}{8\pi \theta}
\end{align*}
\]

(6.5)

which are solved by

\[
\begin{align*}
  h(n) &= \frac{s}{8\pi \sqrt{2}} \left( \frac{1}{\sqrt{n+1+\frac{\mu}{2}}} - \frac{1}{\sqrt{n+1-\frac{\mu}{2}}} \right) \\
  k(n) &= \frac{s}{8\pi \sqrt{2}} \left( \frac{1}{\sqrt{n+1+\frac{\mu}{2}}} + \frac{1}{\sqrt{n+1-\frac{\mu}{2}}} \right)
\end{align*}
\]

(6.6)

The reality conditions on \( h(n) \) and \( k(n) \) do not induce extra constraints on \( \mu \) other than the limit \( \mu < 2 \). In the massless limit we find agreement with the results of ref. [7]:

\[
\begin{align*}
  h(n) \to 0 \quad k(n) \to \frac{s}{4\pi \sqrt{2}} \frac{1}{\sqrt{n+1}}
\end{align*}
\]

(6.7)

7 Measuring the deficit angle

Since we have at disposition an explicit solution of noncommutative gravity we can discuss its properties. For example, we know that the conical metric can be revealed with an integral of the distance around the source. The deficit angle is obtained by choosing as a path a circle with the source in its center:

\[
\alpha = \frac{1}{r} \oint_C ds = \frac{1}{r} \int_0^{2\pi} (1-\mu) r d\phi = 2\pi (1-\mu)
\]

(7.1)

Moreover this measure is not dependent on the distance from the source.

What can we say at a noncommutative level? Firstly to compute the distance we need to build the metric given by the star product of two vierbeins, calculated in eq. (4.12). Our solution is expressed only as an infinite series on the Laguerre polynomials. In any case suppose that we have at disposition a formula in terms of known functions for \( ds \):
\[ ds^2 = g^{NC}_{\mu\nu} dx^\mu dx^\nu \]
\[ g^{NC}_{\mu\nu} = \frac{1}{2} \{ (e^a_\mu \tau_a + e^0_\mu) \ast, (e^b_\nu \tau_b + e^0_\nu) \} ||_1 \]  
(7.2)

where the symbol \( ||_1 \) means that we need to extract the identity part of this product, once it is expressed in the basis of the \( U(1, 1) \) group.

Our particular formula (4.12) for the vierbein generates a correction \( O(\sqrt{\theta} r) \) to the deficit angle:

\[ \alpha_{NC} = \frac{1}{r} \oint ds^{NC} = 2\pi (1 - \mu) + O\left(\frac{\sqrt{\theta}}{r}\right) \]  
(7.3)

The problem we want to discuss is that it doesn’t make sense to look for a deterministic correction to the deficit angle in order to characterize noncommutative gravity with respect to Einstein gravity.

In a previous article [3], we have discussed how the metric (and therefore the distance) is not invariant under \( U(1, 1) \) gauge group but it transform covariantly as

\[ G_{\mu\nu} = \frac{1}{2} \{ Y_\mu, Y_\nu \} \]
\[ G_{\mu\nu} \rightarrow U^{-1} G_{\mu\nu} U \]  
(7.4)

However we can still partially recover a continuity with classical Einstein gravity by restricting the gauge group in order that at spatial infinity the gauge transformations reduce to the identity. The infinitesimal correction to \( g^{NC}_{\mu\nu} \), due to the application of this restricted gauge group, is

\[ \delta g^{NC}_{\mu\nu} \approx O\left(\frac{\sqrt{\theta}}{r}\right) \]  
(7.5)

i.e. the same order of the correction to the deficit angle extrapolated by the explicit formula for the vierbein as in eq. (4.12).

Therefore the gauge group induces a statistical fluctuation of the value of the deficit angle comparable with \( O(\sqrt{\theta} r) \).

Only at large distance from the source it is possible to recover the typical determinism of general relativity. The same observation applies to the geodesics around the source, whose
definition is dependent on the distance. Such an ambiguity of the deficit angle automatically produces an ambiguity for the scattering angle of a test particle in presence of a point-like source, which can be avoided only for large impact parameter, compared with the scale of the noncommutative parameter. This ambiguity problem is an intriguing property of noncommutative gravity which deserves a better understanding and it is probably the distinguishing feature of such a theory.

Such considerations do not apply to the flat metric, which is unaffected by the $U(1,1)$ gauge group. Therefore such a chaotic effect here discussed requires the combination of a nontrivial massive source with the noncommutativity of space-time.

8 Conclusions

Noncommutative gravity is a relatively new subject and in literature rather few applications exist. The aim of this article was to build a concrete and exactly solvable example, that helps us in clarifying its physical meaning.

We have in fact succeeded in solving the noncommutative gravitational field produced by a mass in $(2 + 1)$ dimensions by reformulating this problem in the operator formalism and making use of the concept of a point-like source, an extended source which only in $\theta \to 0$ limit produces a delta-function singularity. We show that it is possible to solve the equations of motion with mass as unique source, responding to ref. [6]. The noncommutative gravitational field is then a smooth deformation of the classical one, which can be recovered in the $\theta \to 0$ limit. Since the source is extended, the noncommutative field is regular around the source, while the classical one is singular.

Another question puzzles us and it could be the distinguishing feature of such a theory, i.e. the gauge ambiguity of the distance and consequently the geodesic field. Restricting gauge transformations to be constant at spatial infinity we can recover the classical determinism at large distance from the source.

Briefly speaking, noncommutative gravity contains two behaviors, one which is deterministic at macroscopic level, giving coinciding predictions with general relativity, and another one chaotic at microscopic level.
References


