On particle dynamics in $AdS_{N+1}$ space-time

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Abstract: We summarize part of a systematic study of particle dynamics on $AdS_{N+1}$ space-time based on Hamiltonian methods. New explicit UIR’s of $SO(2,N)$, defined on certain spaces of holomorphic functions, are constructed. The connection to some field theoretic results, including the construction of propagators, is discussed.

1 Introduction

The concept of quantized particles in Minkowski space is intimately connected with the unitary irreducible representations (UIR’s) of its isometry group, the Poincare group. In generic curved spacetime manifolds, due to the lack of enough isometries, the concept of quantized particle becomes dubiously. The description of matter is more appropriate in terms of quantized fields, although part of the difficulties then comes back via ambiguities in defining vacuum states.

In $AdS$, as a constant curvature background, one has an isometry group of the same dimensionality as in Minkowski space and particle quantization along the standard Hamiltonian techniques is possible. This has been done for lower dimensional cases [1, 2, 3, 4]. Our present contribution will summarize some aspects of our study of both the classical and quantum particle dynamics on $AdS_{N+1}$ for general $N$. With this study we will find new explicit UIR’s of $SO(2,N)$. Another motivation for our investigation comes from possible interrelations with the corresponding quantum field theories on $AdS$, which play a crucial role in the $AdS$/CFT correspondence, see e.g. [5, 6].

In this note we do not refer to the reach literature on $SO(2,N)$ representations in detail, leaving this account for an extended work in progress [7].

The $AdS_{N+1}$ space can be related to $\mathbb{H}^R_N$, the hyperboloid $X_0^2 + X_0^2 - X_nX_n = R^2$ embedded in $\mathbb{R}^2_N$ with metric tensor $G_{AB} = \text{diag}(+,+,−,\ldots,−)$. As global coordinates we use $x_0 = \theta$, $x_n$, ($n = 1,\ldots,N$)

$$X_0 = r \cos \theta\ , \quad X_0' = r \sin \theta\ , \quad X_n = x_n\ , \quad \text{with} \quad r = \sqrt{R^2 + x_n x_n} . \quad (1)$$

The cyclic coordinate $\theta \in S^1$ is identified with time. Obviously, $SO(2,N)$ is the isometry group of the obtained space-time. Unwrapping the time coordinate $\theta \in \mathbb{R}^1$, one gets the universal covering of $\mathbb{H}^R_N$, and $AdS_{N+1}$ is usually associated with it.
2 Classical dynamics

The dynamics of a particle with mass $m$ moving on $\mathbb{R}_N^R$ is described by the action

$$S = -\int d\tau \left( \frac{\dot{X}^A \dot{X}_A}{2e} + \frac{e m^2}{2} + \frac{\mu}{2} (X^A X_A - R^2) \right),$$

(2)

where $e$ and $\mu$ are Lagrange multipliers and $\tau$ is an evolution parameter. To fix the time direction on $\mathbb{R}_N^R$, we make the $SO(2, N)$ invariant choice [7] $\dot{\theta} > 0$, which via (1) is equivalent to

$$X_0 \dot{X}_0 - X_0' \dot{X}_0 > 0.$$  

(3)

We also assume $e > 0$, to have a positive kinetic term for the spacelike coordinates.

The $SO(2, N)$ symmetry of (2) provides the Noether dynamical integrals

$$J_{AB} = P_A X_B - P_B X_A,$$

(4)

where $P_A$ are the canonical momenta $P_A = -\dot{X}_A/e$. The generators $J_{0n}$ and $J_{0'n}$ are related to the non compact $SO(2, N)$ transformations, while $J_{0'0'}$ and $J_{mn}$ to the compact ones, which form the subgroup $SO(2) \times SO(N)$. We will use the notations $J_{0n} = K_n$, $J_{0'n} = L_n$ and $J_{00'} = E$. Since $\theta$ is the time coordinate, $E$ is identified with the particle energy, and due to (3) it is positive.

The dynamical integrals (4) allow to represent the set of all trajectories geometrically without solving the dynamical equations. From (4) we find $N$ equations

$$E X_n = K_n X_0 - L_n X_0,$$

(5)

which define a 2-dimensional plane in the embedding space $\mathbb{R}_N^2$, containing the origin. Its intersection with the hyperboloid is a particle trajectory.

For further calculations it is convenient to introduce the complex variables

$$z_n = L_n - i K_n, \quad z_n^* = L_n + i K_n.$$

(6)

Then, from (1) and (5) we obtain

$$X_0 = r(\theta) \cos \theta, \quad X_0' = r(\theta) \sin \theta, \quad X_n = -\frac{r(\theta)}{2E} \left( z_n^* e^{i\theta} + z_n e^{-i\theta} \right),$$

(7)

where

$$r(\theta) = \frac{2ER}{\sqrt{4 E^2 - 2 \lambda^2 - 2 \rho^2 \cos(2\theta + \beta)}},$$

$$\lambda^2 = z_n^* z_n, \quad \rho^2 e^{-i\beta} = z_n^* z_n, \quad \rho^2 e^{i\beta} = z_n z_n^*.$$  

(8)

The action (2) is gauge invariant under the reparameterizations $\tau \rightarrow f(\tau)$ combined with $\mu \rightarrow \mu / \mu'$, $e \rightarrow e / e'$. This gauge symmetry, as usual, leads to dynamical constraints. Applying Dirac’s procedure, we find three constraints

$$X^A X_A - R^2 = 0, \quad P_A P^A - m^2 = 0, \quad P_A X^A = 0,$$

(9)

which fix the quadratic Casimir number of the symmetry group

$$C = \frac{1}{2} J_{AB} J^{AB} = m^2 R^2.$$  

(10)

This equation can be rewritten as

$$E^2 + J^2 = \lambda^2 + \alpha^2, \quad \text{with} \quad J^2 = \frac{1}{2} J_{mn} J_{mn} \quad \text{and} \quad \alpha = m R.$$  

(11)
Another set of quadratic relations follows from (14) as identities in the \((P,X)\) variables
\[
J_{AB} J_{A'B'} = J_{AA'} J_{BB'} - J_{AB'} J_{BA'} .
\] (12)
These equations are nontrivial in terms of the symmetry generators, if all indices are different. Taking \(A = 0, B = 0', A' = m\) and \(B' = n\) \((m \neq n)\) we obtain
\[
2iE J_{mn} = z^*_m z_n - z^*_n z_m ,
\] (13)
which provides
\[
4E^2 J^2 = \lambda^4 - \rho^4 .
\] (14)
By eqs (11) and (14) \(E^2\) and \(J^2\) are roots of the quadratic equation
\[
4x^2 - 4(\lambda^2 + \alpha^2) x + \lambda^4 - \rho^4 = 0 .
\] (15)
We choose
\[
E^2 = \frac{1}{2} \left( \lambda^2 + \alpha^2 + \sqrt{\alpha^4 + 2\alpha^2\lambda^2 + \rho^4} \right) ,
\] (16)
\[
J^2 = \frac{1}{2} \left( \lambda^2 + \alpha^2 - \sqrt{\alpha^4 + 2\alpha^2\lambda^2 + \rho^4} \right) .
\] (17)
The other choice would be unphysical, since with \(E^2\) chosen as the small root of (15), \(r(\theta)\) in (7) would become imaginary. According to (16) \(\alpha\) is the lowest value of energy. Using in addition (17), one can show that \(E \geq J + \alpha\).

We now study Hamiltonian reduction. The generators \(E\) and \(M_{mn}\) are functions of \((z_n, z^*_n)\) via (16) and (13). Therefore, \((z_n, z^*_n) \sim (K_n, L_n)\) are global coordinates on the space of dynamical integrals. There are no restrictions on these coordinates for \(\alpha > 0\) and they cover all \(\mathbb{R}^{2N}\), but for the massless case one has to remove the submanifold \(z_n z^*_n = 0\).

Eq. (7) can be considered as a parameterization of \(X_A\) by the dynamical integrals and the time coordinate \(\theta\). The canonical momenta \(P_A\) are parameterized in a similar way. These parameterizations together define the gauge orbits on the \(2N+1\) dimensional surface constrained by (9), and \(\theta\) is the parameter along the orbits. The physical phase space \(\Gamma_{ph}\) is the set of these orbits, it is \(2N\)-dimensional and is given by the gauge invariant variables \(z_n, z^*_n\). Thus, \(\Gamma_{ph}\) is identified with a \(2N\)-dimensional space of dynamical integrals.

Our aim is to describe the Poisson bracket structure on \(\Gamma_{ph}\) and to calculate the corresponding reduced symplectic form \(\omega\). Starting with the canonical \(\{P_A, X^B\} = \delta_A^B\) the Poisson bracket algebra of the generators (11) turns out to be \(so(2,N)\)
\[
\{J_{AB}, J_{A'B'}\} = G_{AA'} J_{BB'} + G_{BB'} J_{AA'} - G_{AB'} J_{BA'} - G_{BA'} J_{AB'} .
\] (18)
Since the generators are gauge invariant, their Poisson bracket algebra is the same after the reduction to \(\Gamma_{ph}\) and we have
\[
\{z^*_m, z_n\} = 2J_{mn} - 2i\delta_{mn} E ,
\] (19)
\[
\{E, z_n\} = -i z_n , \quad \{J_{lm}, z_n\} = z_l \delta_{mn} - z_m \delta_{ln} ,
\] (20)
\[
\{z_m, z_n\} = 0 = \{z^*_m, z^*_n\} , \quad \{E, J_{mn}\} = 0 .
\] (21)
These Poisson brackets are essentially non-linear in terms of the independent variables \(z_n, z^*_n\) and their quantum realization becomes problematic. Below we apply the method of geometric quantization, which is based on the symplectic structure of the classical system.
The symplectic form $\omega$ can be obtained as a reduction of the canonical form $dP_A \wedge dX^A$. To simplify this procedure, we use the relation valid on the constraint surface \(9\)

\[
dP_A \wedge dX^A = \frac{1}{2\alpha^2} J_{AB} dJ^{AC} \wedge dJ^{B}_C .
\]

Then, parameterizing all generators by $z_n$ and $z^*_n$, a straightforward calculation gives

\[
\omega = \omega_{mn} \frac{dz_m \wedge dz^*_n}{2iE} ,
\]

where

\[
\omega_{mn} = \delta_{mn} - \frac{z^{*2}_{m}}{4E^2(E^2 - J^2)} z_{m} z_{n} - \frac{ z^{2}_{m}}{4E^2(E^2 - J^2)} z^{*}_{m} z^{*}_{n} + \frac{ z^{*}_{m} z + 2E^2}{4E^2(E^2 - J^2)} z_{m}^{*} z_{n} + \frac{ z^{*}_{m} z - 2E^2}{4E^2(E^2 - J^2)} z_{m}^{*} z_{n} ,
\]

and use has been made of \(16\) and \(17\).

Since the phase space for $\alpha > 0$ has the structure of $\mathbb{R}^{2N}$ the symplectic form \(23\) is exact $\omega = d\Omega$. The integration of the 2-form \(23\) yields

\[
\Omega = \frac{1}{4iE} \left( z + \frac{z^{2}}{F^2} z^{*}_{n} \right) d z^{*}_{n} - \frac{1}{4iE} \left( z^{*} + \frac{z^{*2}}{F^2} z_{n} \right) d z_{n} ,
\]

with

\[
F = \sqrt{(E + \alpha)^2 - J^2} .
\]

3 Quantization

The quantum theory of particle dynamics on AdS space can be constructed quantizing the classical system given on $\Gamma_{ph}$. A consistent quantization should provide an unitary irreducible representation of the symmetry group in some Hilbert space. The cases $N = 1$ and $N = 2$ have been discussed in several papers \[1\]-\[4\]. $N = 2$ splits in two $N = 1$ cases. For $N = 1$ one usually uses the (Bargman) Hilbert space \[8\] formed by the holomorphic functions $\psi(\zeta)$ inside the unit disk $|\zeta| < 1$ with the scalar product

\[
\langle \psi_1 | \psi_2 \rangle = \int_{|\zeta| < 1} d\zeta d\zeta^* \left( 1 - |\zeta|^2 \right)^{2\alpha - 2} \psi_1^*(\zeta) \psi_2(\zeta) .
\]

The dynamical integrals \[1\] are transformed by the co-adjoint representation of $SO(2, N)$ group $J_{AB} \rightarrow \Lambda_A \Lambda^*_B J_{A'B'}$, and $\Gamma_{ph}$ is the orbit of the point $z_n = 0, J_{mn} = 0$ and $E = \alpha$. Therefore, it is natural to apply the method of geometric quantization \[9\].

At the first stage of this approach we construct pre-quantization operators. They are given by the map $f \mapsto \hat{O}_f$ from observables $f$, which are functions on $\Gamma_{ph}$, to the operators $\hat{O}_f$ acting on the Hilbert space $L^2(\Gamma_{ph})$

\[
\hat{O}_f = f - \Omega(V_f) - iV_f .
\]

Here $V_f$ is the Hamiltonian vector field

\[
V_f = \{ f, z_n \} \partial_{z_n} + \{ f, z^*_n \} \partial_{z^*_n} ,
\]
and $\Omega(V_f)$ is the value of the 1-form (26) on $V_f$. $\Omega(V_f)$ is calculated in a standard way $\Omega(V_f) = \Omega_{z_n}\{f, z_n\} + \Omega_{z_n^*}\{f, z_n^*\}$, where $\Omega_{z_n}$ and $\Omega_{z_n^*}$ are the coefficients in (25) of the differentials $dz_n$ and $dz_n^*$, respectively. The operators (28) act on wave functions $\Psi(z, z^*)$ and they are hermitian with respect to the scalar product based on the Liouville measure on $\Gamma_{ph}$. The Hamiltonian vector fields of the symmetry generators are obtained from (28) and after calculation of the values of $\Omega$ on these fields we find the pre-quantization operators

\[
\hat{O}_E = \alpha - z_n \partial_{z_n} + z_n^* \partial_{z_n^*},
\]

\[
\hat{O}_{Jmn} = i \left( z_n^* \partial_{z_m^*} - z_m^* \partial_{z_n^*} \right) + i \left( z_n \partial_{z_m} - z_m \partial_{z_n} \right),
\]

\[
\hat{O}_{z_n} = \frac{\alpha}{2E} \left( z_n + \frac{z^2}{F^2} z_n^* \right) + 2(\delta_{nm} E - iJ_{nm}) \partial_{z_m^*},
\]

\[
\hat{O}_{z_n^*} = \frac{\alpha}{2E} \left( z_n^* + \frac{z^2}{F^2} z_n \right) - 2(\delta_{nm} E + iJ_{nm}) \partial_{z_m}.
\]

The operators (30)-(33) give an unitary representation of the $so(2, N)$ algebra, but it is not a representation we are looking for, since it is reducible and the spectrum of $\hat{O}_E$ is not positive. The representation becomes irreducible on a subspace of $\mathcal{L}^2(\Gamma_{ph})$ which is defined by a choice of polarization [9]. This subspace we will construct in a more physical way, rather than via a choice of polarization.

Let us define the vacuum as a lowest energy state, which is invariant under $SO(N) \times SO(2)$, i.e.

\[
\hat{O}_{Jmn} \Psi_0 = 0 , \quad \hat{O}_E \Psi_0 = \alpha \Psi_0 , \quad \hat{O}_{z_n} \Psi_0 = 0 .
\]

By the first two equations $\Psi_0(z, z^*)$ depends only on the scalar quantities $\lambda^2$ and $\rho^2$ from (3), and then, due to the third equation we find

\[
\Psi_0 = c_N F^{-\alpha} ,
\]

where $c_N$ is a normalization constant and the function $F$ is given by (20). Our representation subspace we will form by linear combinations of the states $\Psi = (\hat{O}_{\zeta_1})^{n_1}(\hat{O}_{\zeta_2})^{n_2}... \Psi_0$. The action of $\hat{O}_{z_n}$ on the vacuum state is given by

\[
\hat{O}_{z_n} \Psi_0 = 2\alpha \zeta_n \Psi_0 ,
\]

where $\zeta_n$ are complex variables

\[
\zeta_n = \frac{1}{2E} \left( z_n^* + \frac{z^2}{F^2} z_n \right) .
\]

On the level of $SO(N)$ scalars the last definition implies

\[
\zeta^2 = \frac{z^2}{F^2} , \quad \zeta^* \zeta = \frac{E^2 - J^2 - \alpha^2}{(E + \alpha)^2 - J^2} , \quad \zeta^* \zeta = \frac{E - \alpha)^2 - J^2}{(E + \alpha)^2 - J^2} ,
\]

with the notations: $\zeta^2 = \zeta_n \zeta_n , \quad \zeta^2 = \zeta_n^* \zeta_n^* , \quad \zeta^* \zeta = \zeta_n^* \zeta_n , \quad z^2 = z_n^* z_n^*$. This can be used to express $E$ and $F^2$ in terms of the $\zeta$'s

\[
F^2 = \frac{4\alpha^2}{1 - 2\zeta^2 \zeta + \zeta^* \zeta^2 \zeta^2} , \quad E = \alpha \frac{1 - \zeta^2 \zeta^2}{1 - 2\zeta^* \zeta + \zeta^* \zeta^2 \zeta^2} .
\]
and to finally arrive at the inversion of (37)

\[ z_n^* = \frac{2\alpha (\zeta_n - \zeta^2 \zeta^*_n)}{1 - 2\zeta^* \zeta + \zeta^*^2 \zeta^2} . \]  

(40)

Thus, \( \zeta, \zeta^* \) are global coordinates on \( \Gamma_{ph} \), and due to (38) and (39) they are constrained to the domain

\[ \zeta^* \zeta < 1 , \quad 1 - 2\zeta^* \zeta + \zeta^*^2 \zeta^2 > 0 . \]  

(41)

The \( \zeta_n \) variables have the remarkable property that the action of the Hamiltonian vector fields of the symmetry generators (29) on \( \zeta_n \)'s is expressed in terms of the \( \zeta_n \)'s alone

\[ V_E(\zeta_n) = i\zeta_n , \quad V_{jm}(\zeta_n) = \delta_{mn} \zeta_l - \delta_{ln} \zeta_m , \]

\[ V_{zn}(\zeta_n) = i\delta_{mn} , \quad V_{zn}(\zeta_n) = 2i\zeta_m \zeta_n - i\delta_{mn} \zeta^2 . \]

(42)

These equations provide the following structure of the states

\[ \Psi = \psi(\zeta_1, \ldots, \zeta_N) \Psi_0 , \]

(43)

which is invariant under the action of the symmetry generators (30)-(33). This yields an irreducible representation of the \( SO(2, N) \) group on the space \( \mathcal{H}_{ph} \) of holomorphic functions \( \psi(\zeta_1, \ldots, \zeta_N) \). Recalculating the operators (30)-(33) on \( \mathcal{H}_{ph} \) we find the following representation of the \( so(2, N) \) algebra

\[ \hat{E} = \alpha + \zeta_n \partial_{\zeta_n} , \quad \hat{J}_{mn} = i(\zeta_n \partial_{\zeta_m} - \zeta_m \partial_{\zeta_n}) , \]

\[ \hat{z}_n^* = 2\alpha \zeta_n + (2\zeta_n \zeta_m - \zeta^2 \delta_{mn}) \partial_{\zeta_m} , \quad \hat{z}_n = \partial_{\zeta_n} . \]

(44)

The energy spectrum on \( \mathcal{H}_{ph} \) is positive \( \langle \hat{E} \rangle \geq \alpha \).

Recalculating the Casimir number of this representation one finds a quantum deformation relative to the classical \( C = \alpha^2 \) :

\[ C = \hat{E}^2 + \frac{1}{2} \hat{J}_{mn} \hat{J}_{mn} - \frac{1}{2} (\hat{z}_n^* \hat{z}_n + \hat{z}_n \hat{z}_n^*) = \alpha(\alpha - N) . \]

(45)

Starting with the Liouville measure and using (43) and (45) one finds for the scalar product on \( \mathcal{H}_{ph} \)

\[ \langle \psi_1 | \psi_2 \rangle = \int d^N \zeta d^N \zeta^* (1 - 2\zeta^* \zeta + \zeta^*^2 \zeta^2)^{\alpha-N} \psi_1^*(\zeta) \psi_2(\zeta) . \]

(46)

For \( N = 1 \) this result coincides with (27). The integration measure in the scalar product for \( N = 1 \) is regular for \( \alpha > 1/2 \), however, the regularity of the integration measure for \( N \geq 2 \) requires \( \alpha > N - 1 \). Thus, for \( N \geq 2 \) and \( \alpha > N - 1 \) eqs. (14) and (16) define UIR’s of the \( SO(2, N) \) group with the Casimir number \( C = \alpha(\alpha - N) \).

In this deformed relation \( C \) has to be interpreted as the squared mass of the quantum particle (multiplied by \( R^2 \)). The parameter \( \alpha \), as shown above, is the lowest value of the energy within the representation. The well-known unitarity bound for UIR’s of \( SO(2, N) \), see e.g. [5], [6], given by \( \alpha \geq N/2 - 1 \), is of course respected by our representations. However, it remains to be clarified why these representations are not valid in the whole range down to the bound. As long as we are on the hyperboloid, related to the original \( SO(2, N) \), due to time periodicity \( \alpha \) has to be integer. On the universal cover, i.e. on \( AdS_{N+1} \) \( \alpha \) is a continuous parameter.
4 Propagators

We now are interested in the quantum mechanical propagation kernel \( \langle X'|X \rangle \), where \( |X\rangle \) is understood as a state associated to the particle localized around the point \( X \) on the hyperboloid \( \mathbb{H}^N_{\mathbb{R}} \). The states \( |X\rangle \) have to transform under a \( SO(2, N) \) transformation \( X \rightarrow \Lambda X \), with unitary representative \( \hat{U}_\Lambda \), as \( |X\rangle \rightarrow \hat{U}_\Lambda |X\rangle = |\Lambda X\rangle \). For the transition from the \( \zeta \)-representation \( (\psi(\zeta) = \langle \zeta|\psi \rangle) \) of the previous section we need \( \langle \zeta|X \rangle \). Analyzing with \( \mathbf{10} \) the equations arising from

\[
\langle \zeta|\hat{U}_\Lambda|X \rangle = \langle \hat{U}_{\Lambda^{-1}}\zeta|X \rangle ,
\]

one gets up to a normalization factor

\[
\langle \zeta|X \rangle = [X_0 - iX_{0'} + 2\zeta_n X_n + \zeta^2 (X_0 + iX_{0'})]^{-\alpha} .
\]

One can check that \( \mathbf{16} \) solves the Klein-Gordon equation with mass squared equal to \( R^{-2}\alpha(\alpha - N) \), which is in consistence with the group theoretical interpretation given at the end of the last section.

Then the propagation kernel \( \langle X'|X \rangle \) using \( \mathbf{16} \) is represented by

\[
\langle X'|X \rangle = \int d^N\zeta d^N\zeta^* \frac{(1 - 2\zeta^*\zeta + \zeta^*\zeta^2)^{N-\alpha}}{(Z^2 + 2\zeta_n X_n + Z^*\zeta^2)^\alpha} ,
\]

with \( Z = X_0 + iX_{0'} \). The integral is divergent for coinciding \( (X' = X) \) or antipodal \( (X' = -X) \) points. As a natural regularization, respecting the symmetry under \( (Z, X_n) \leftrightarrow (Z^*, X_n) \), we choose \( \theta \rightarrow \theta + i\epsilon, \ \theta' \rightarrow \theta' - i\epsilon \).

From the representation \( \mathbf{49} \) one finds the following interesting recursion in \( \alpha \)

\[
\langle X'|X \rangle_{\alpha+1} = \frac{1}{2\alpha^2} (2\partial_{Z'Z^*} - \partial_{X_n X_n'}^2 + 2\partial_{ZZ^*}^2) \langle X'|X \rangle_\alpha .
\]

The evaluation of the integral in \( \mathbf{49} \) becomes most simply for \( N = 1 \) and \( \alpha = 1 \)

\[
\langle X'|X \rangle_1 = \frac{1}{4R^2} \log \frac{(X - X')^2 - i\epsilon \sin(\theta' - \theta) - 4R^2}{(X - X')^2 - i\epsilon \sin(\theta' - \theta)} .
\]

The field theoretical Feynman propagator \( G_F(X, X') \) is given as usual by

\[
G_F(X', X) = \Theta(T_{X'X}) \langle X'|X \rangle + \Theta(T_{XX'}) \langle X|X' \rangle .
\]

Here the argument of the step function \( \Theta \) is \( T_{X'X} = \sin(\theta' - \theta) \) for \( \mathbb{H}^N_{\mathbb{R}} (SO(2, N) \) invariant time ordering), or \( T_{X'X} = (x' - x) \) for its universal cover, i.e. \( AdS_{N+1} \). The resulting propagator indeed coincides with the well-known field theoretical expression, see e.g. \( \mathbf{6} \).

The precise handling of the distributional properties of \( \mathbf{51} \) and \( \mathbf{52} \) generated by our regularization of the integral in \( \mathbf{49} \) agrees with \( \mathbf{10} \).

5 Conclusions

Based on the dynamics of massive spinless particles in \( AdS \) space-time we constructed explicitly UIR’s of \( SO(2, N) \) for generic \( N \) acting on spaces of holomorphic functions of \( N \) variables. The construction followed the lines of geometric quantization. In agreement with general \( SO(2, N) \) representation theory these representations are labelled by the
lowest energy value $\alpha$. They are valid for $\alpha > N - 1$ for $N \geq 2$ or $\alpha > 1/2$ for $N = 1$. These bounds are above the well-known unitarity bound $\alpha > N/2 - 1$. It remains to find a physical interpretation why these representations do not exhaust the whole $\alpha$ range allowed by unitarity.

Meanwhile we succeeded in constructing representations realized in terms of canonical variables. They are well defined for all $\alpha$ allowed by unitarity. A corresponding publication \cite{7}, in addition, will contain a treatment of the special situation of enhanced symmetry at $\alpha = N/2 \pm 1/2$, which gives the quantized particle a mass just enabling Weyl invariance in the associated field theoretic action.

Part of our motivation was to explore the potential of particle quantization techniques for the construction of quantum field propagators. Although their explicit form for scalar fields on $AdS$ is well-known for all masses and dimensions, for $AdS \times S$ only in special cases explicit closed expressions are available \cite{11}. There remains no doubt that propagators can be constructed this way. However, our study in section 4 apparently did not show up this method as more efficient than the direct investigation of the field theoretical differential equations. The potential of the recursion formula still has to be explored.

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