Denseness of Ashtekar-Lewandowski states
and a generalized cut-off in loop quantum gravity

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Abstract

We show that the set of states of the Ashtekar-Isham-Lewandowski holonomy algebra defined by elements of the Ashtekar-Lewandowski Hilbert space is dense in the space of all states. We consider weak convergence properties of a modified version of the cut-off procedure currently in use in loop quantum gravity. This version is adapted to vector states rather than to general distributions.

1 Introduction and overview

The kinematical algebra on which loop quantum gravity (LQG) is based, namely the algebra of holonomies and fluxes, has a preferred, seemingly unique, representation defined by a state, or measure $\mu_0$. This state gives rise to the so-called kinematical Hilbert space, a non-separable space $\mathcal{H}_0 := L^2(\mathcal{A}, \mu_0)$ of square integrable functions on the compact space $\mathcal{A}$ of generalized connections (see [AL5, R, T1] for reviews of LQG and [F, LOST, OL1, OL2, ST] for recent uniqueness results concerning the $\mathcal{H}_0$ representation). Nevertheless, different states and corresponding representations of the holonomy $C^*$-algebra $\mathcal{C}(\mathcal{A})$ – the commutative sub-algebra of the full kinematical algebra corresponding to configuration variables – are still worth considering, in particular in relation to the active search for low energy or semiclassical states [AG, AL4, Bo, T1, T3, Va1, Va2, Va3, Ve1].

In the present work we address the question of completeness of the Hilbert space $\mathcal{H}_0$ with respect to the space of all states of the holonomy
algebra \( C(\overline{A}) \), in the following sense. Let us consider the set \( E(H_0) \) of states of \( C(\overline{A}) \) defined by normalized vectors \( \psi \in H_0 \), i.e. states of the form
\[
C(\overline{A}) \ni f \mapsto \langle \psi, f\psi \rangle_0,
\]
where \( \langle \cdot, \cdot \rangle_0 \) denotes the \( H_0 \) inner product. The first question we wish to address is whether an arbitrary state of \( C(\overline{A}) \) can be approximated, or obtained in some limit, from elements of \( E(H_0) \). A variant of this question has been considered in [T3], where it was argued that new representations can indeed be obtained from \( H_0 \) through standard limiting procedures. Partly motivated by that work, we aim at a rigorous formulation of completeness results of the above type. Specifically, we consider the natural weak-* topology on the space of states of \( C(\overline{A}) \) and present a proof of the denseness of the set \( E(H_0) \).

A second, more subtle issue is inspired by the so-called ”cut-off” procedure, which is a consistent way to deal with formal, non-normalizable linear combinations of \( H_0 \) vectors that arise naturally in the context of LQG semiclassical analysis [AL4, T3]. In broad terms, to be refined below, we want to consider how to map a given state of \( C(\overline{A}) \), and elements of the corresponding Hilbert space, to nets of \( H_0 \) vectors, preserving as much information as possible. It turns out that the projective structure built in the framework of LQG provides, as in the cut-off procedure, the key ingredients to address this question.

As discussed in more detail in section 2.2, the space \( \overline{A} \) is a projective limit of a family of finite dimensional spaces \( A_\gamma \), each one associated to a graph \( \gamma \). Moreover, a state of \( C(\overline{A}) \), or a measure \( \mu \) in \( \overline{A} \), is equivalent to a family of measures \( \mu_\gamma \), each defined in the corresponding space \( A_\gamma \). Typically for measures \( \mu \) of potential interest one expects that, while the Radon-Nikodym derivative \( d\mu/d\mu_0 \) fails to exist, the corresponding derivatives \( d\mu_\gamma/d\mu_{0,\gamma} \) remain well defined. It follows that, for every measure as above, one can naturally define a map \( \psi \mapsto \{\psi_\gamma\} \) from \( H = L^2(\overline{A}, \mu) \) to nets of \( H_0 \) vectors. As in the cut-off procedure, these nets are labeled by graphs and each \( \psi_\gamma \) is (the pull-back of) an element of \( L^2(A_\gamma, \mu_{0,\gamma}) \). We consider here convergence properties of this map. Besides convergence of expectation values of elements of \( C(\overline{A}) \), we will see that convergence of arbitrary matrix elements is also achieved, i.e. \( \langle \psi_\gamma', f\psi_\gamma \rangle_0 \) converges to \( \langle \psi', f\psi \rangle \), \( \forall \psi', \psi \in H \), \( \forall f \in C(\overline{A}) \). The map \( \psi \mapsto \{\psi_\gamma\} \) can be seen as a (weak) generalization of the would be transformation
\[
H \ni \psi \mapsto (d\mu/d\mu_0)^{1/2}\psi \in H_0,
\]
which, of course, depends on the existence of $d\mu/d\mu_0$.

The questions considered in the present work have well defined, and certainly well studied analogues in standard quantum field theory (QFT). The reasons to reconsider them in the LQG context are twofold. First, QFT typically deals with infinite dimensional linear spaces, and measures that give rise to familiar separable Hilbert spaces. It is thus not obvious what kind of results have straightforward analogues in LQG, where one finds an infinite dimensional compact space $\mathcal{A}$, and a fiducial Hilbert space $\mathcal{H}_0$ which is non-separable.

Second, the status of the questions we consider is likely to be different in QFT and in LQG. To begin with, and still deeply related to the first point, in QFT one does not have a true analogue of the $\mu_0$ measure, as this would have to be the non-existing “Lebesgue measure on an infinite dimensional linear space”. Gaussian measures take its place as fiducial measures, but these are not as distinguished, or singled out in the way $\mu_0$ is. Moreover, there are specific reasons why maps between inequivalent representation Hilbert spaces, such as the map $\psi \mapsto \{\psi_{\gamma}\}$ above, may be useful in LQG, while they can be regarded as constructions of disputable interest in QFT. Let us point out that the convergence $\langle \psi'_{\gamma}, \cdot \psi_{\gamma}\rangle_0 \rightarrow \langle \psi', \cdot \psi \rangle$ one obtains is usually far from being uniform. Thus, typically no fixed $\psi_{\gamma}$ approximates well the properties of $\psi$ with respect to a large set of operators. This diminishes the interest of such maps when, as in QFT, the operators in question are physical observables. In LQG, however, variables like $f \in C(\mathcal{A})$ are not observables (as they do not commute with the constraints of general relativity). There is thus hope that images $\psi_{\gamma}$ can approximate the properties of corresponding $\psi$ with respect to a more meaningful restricted set of operators, constructed e.g. from elements of $C(\mathcal{A})$. That this is a real possibility, regarding e.g. a sufficient set of spatially diffeomorphism invariant operators, is one of the conclusions of [T3].

This work is organized as follows. In section 2 we review selected mathematical aspects of the LQG approach. In section 3 the denseness of the set of states $E(\mathcal{H}_0)$ is proved. In section 4 the above mentioned convergence properties of the nets $\{\psi_{\gamma}\}$ are proved.

2 Preliminaries

In this section we review very briefly the necessary mathematical aspects of LQG (see e.g. [AL5, T1, T2, Ve2] for a more thorough review). We also
include a summary of notions and results used throughout the present work, mainly topics from the theory of commutative $C^*$-algebras (see e.g. [BR] and [RS] for an extensive coverage).

2.1 Summary of commutative $C^*$-algebra results

Let $\mathcal{C}$ be a commutative $C^*$-algebra with identity. There exists a uniquely defined compact Hausdorff space $X$ – called the spectrum of $\mathcal{C}$ – such that $\mathcal{C}$ is isomorphic to the algebra $C(X)$ of complex continuous functions on $X$. One can therefore identify $\mathcal{C}$ with $C(X)$.

A positive linear functional $\omega : C(X) \to \mathbb{C}$ is called a state (of the algebra $C(X)$) if it is normalized, i.e. if $\|\omega\| = \omega(1) = 1$. It follows that the set of all states, or state space $E$, is a convex subset of the topological dual of $C(X)$.

Cyclic representations of $C(X)$ are in 1-1 correspondence with states. By the Riesz-Markov theorem, states are in turn in 1-1 correspondence with regular normalized Borel measures on $X$. Given such a measure $\mu$, one has a representation of $C(X)$ by multiplication operators in the Hilbert space $L^2(X, \mu)$:

$$(f\psi)(x) = f(x)\psi(x), \quad \psi \in L^2(X, \mu), \quad f \in C(X), \quad (1)$$

with corresponding state $\omega$ defined by:

$$\omega(f) = \langle 1, f1 \rangle = \int_X f \, d\mu. \quad (2)$$

Conversely, every cyclic representation of $C(X)$ is (unitarily equivalent to) a representation of the above type (1). Given a representation Hilbert space $\mathcal{H}$, we will say that a state of the form $f \mapsto \langle \psi, f\psi \rangle$, with a normalized $\psi \in \mathcal{H}$, is a $\mathcal{H}$ vector state.

A representation of a $C^*$-algebra is said to be faithful if it is injective, and a (regular Borel) measure $\mu$ on a compact space $X$ is called faithful if $\mu(B) \neq 0$ for all non-empty open sets $B \subset X$. It follows that a (cyclic) representation of $C(X)$ is faithful if and only if the corresponding measure $\mu$ is faithful.

Let us consider the state space $E$ equipped with the weak-$\ast$ topology, i.e. the weakest topology such that all complex maps $E \ni \omega \mapsto \omega(f)$ are continuous, $f \in C(X)$. The state space $E$ then becomes a compact space, with convergence given by: a net $\{\omega_\lambda\}_{\lambda \in I}$ converges to $\omega$ if and only if every complex net $\{\omega_\lambda(f)\}_{\lambda \in I}$ converges to $\omega(f)$, $\forall f$. (Nets are a generalization of
sequences: a net is a family indexed by a directed set $I$.\footnote{A directed set $I$ is a set equipped with a partial order relation "$\geq$" with the property that for any $\lambda, \lambda' \in I$ there exists $\lambda'' \in I$ such that $\lambda'' \geq \lambda, \lambda'$.} Only in spaces that are first countable can one construct closures using sequences; in a general topological space a point $x$ is in the closure of a set $S$ if and only if there is a net in $S$ converging to $x$).

The next result provides a dense set in $E$. Let us consider the subset of Dirac states, i.e. states $\omega_x, x \in X$, defined by $\omega_x(f) = f(x)$. It follows that the convex hull of Dirac states is (weak-$*$) dense in $E$. Thus, given any state $\omega \in E$ there is a net of states of the type

$$f \mapsto \sum_{i=1}^{n} t_i f(x_i),$$

with $n \in \mathbb{N}$, $x_i \in X$, $t_i > 0$, $\sum^{n} t_i = 1$, that converges to $\omega$.

2.2 LQG basics: holonomy algebra, projective structure and uniform measure

Loop quantum gravity starts from the $SU(2)$ version of Ashtekar’s canonical formulation of general relativity as a special kind of gauge theory [A, Bar]. For generality and to simplify notation we consider in what follows a general connected and compact gauge Lie group $G$. The classical configuration space of the theory is thus the space $A$ of $G$-connections $A$ on a principle bundle $P(\Sigma, G)$ over a spatial manifold $\Sigma$. For general base manifold $\Sigma$ and gauge group $G$, the bundle $P(\Sigma, G)$ may be non-trivial (such a situation is considered in [AL3, Ba] and more recently in [LOST, OL2]). However, for the purposes of the present work it is irrelevant whether the bundle is trivializable or not, as our results do not depend on this. Furthermore, in the case of greatest interest, namely gravity in $3+1$ dimensions formulated as a $G = SU(2)$ gauge theory, the bundle is actually trivial [LOST, T1]. Thus, for simplicity, we assume in what follows that the bundle $P(\Sigma, G)$ is trivial. This allows us to identify connections $A$ with globally defined one-forms on the base manifold $\Sigma$, with values on the Lie algebra of $G$. Moreover, one can then look at holonomies of connections as taking values on the group $G$.

A key ingredient in LQG is the particular choice of configuration functions, collected in a commutative $C^*$-algebra with identity called the holonomy algebra. The basic ($G$-valued) variables are the holonomies or parallel
transports

\[ A \mapsto A(e) := \mathcal{P} \exp(- \int_e A) \]

along (analytically embedded oriented compact) curves \( e \) in \( \Sigma \), called edges.

Basic complex functions – called cylindrical functions – are of the form

\[ f(A) = F(A(e_1), \ldots, A(e_N)), \tag{3} \]

for arbitrary integer \( N \) and \( F \in C(G^N) \). The set \( \text{Cyl} \) of all cylindrical functions is a commutative \( \ast \)-algebra with identity. The \( C^\ast \)-completion of \( \text{Cyl} \) in the supremum norm is then the holonomy \( C^\ast \)-algebra \([AI]\). The spectrum of the algebra is the so-called space of generalized connections \( \mathcal{A} \), a compact extension of \( \mathcal{A} \) that plays the role of quantum configuration space. Following section 2.1, we identify the holonomy algebra with \( C(\mathcal{A}) \). It turns out that \( \mathcal{A} \) coincides with a subclass of maps from the set of edges of \( \Sigma \) to the group \( G \). More precisely, each element \( \bar{A} \) of \( \mathcal{A} \) is a map \( e \mapsto \bar{A}(e) \in G \), preserving the natural composition of edges \([AL1, AL3, Ba]\). Conversely, every edge \( e \) defines a \( G \)-valued function on \( \mathcal{A} \), \( \bar{A} \mapsto \bar{A}(e) \).

A quantization of configuration variables is, by construction, a representation of \( C(\mathcal{A}) \). Given a measure \( \mu \) on \( \mathcal{A} \) one thus has a quantization of configuration variables by multiplication operators in \( L^2(\mathcal{A}, \mu) \). In particular, cylindrical functions \( f(A) \) are quantized by the functions \( f(\bar{A}) \).

Measure theory in \( \mathcal{A} \) is well understood, due to the projective nature of \( \mathcal{A} \) \([AL1, AL2, AL3, Ba, MM]\). One may start with the set \( \Gamma \) of (appropriate) finite collections of edges, called graphs in \( \Sigma \). The set of graphs \( \Gamma \) is naturally directed, a graph \( \gamma' \) being ”greater than” \( \gamma \) \((\gamma' \geq \gamma)\) if \( \gamma \) is a subgraph of \( \gamma' \). To each graph \( \gamma \) corresponds a finite dimensional configuration space \( \mathcal{A}_\gamma \), which captures the (finite number of) degrees of freedom associated to parallel transports along the edges of \( \gamma \). Each \( \mathcal{A}_\gamma \) is a compact space diffeomorphic to \( G^{N_\gamma} \), where the integer \( N_\gamma \) is the number of (independent) edges in \( \gamma \). There is thus an identification between \( C(\mathcal{A}_\gamma) \) and \( C(G^{N_\gamma}) \), essentially given by (3) above. The family of spaces \( \{\mathcal{A}_\gamma\}_{\gamma \in \Gamma} \) forms a projective family, i.e. for every pair \( \gamma, \gamma' \in \Gamma \) such that \( \gamma' \geq \gamma \), there is a continuous surjective projection \( p_{\gamma'\gamma} : \mathcal{A}_{\gamma'} \to \mathcal{A}_\gamma \) satisfying the consistency conditions

\[ p_{\gamma''\gamma} = p_{\gamma'\gamma} p_{\gamma''\gamma'}, \quad \forall \gamma'' \geq \gamma' \geq \gamma. \tag{4} \]

The space \( \mathcal{A} \) is a limit – the so-called projective limit – of the family \( \{\mathcal{A}_\gamma\}_{\gamma \in \Gamma} \). In particular, this means that there are continuous surjective projections
\[ p_\gamma : \overline{\mathcal{A}} \to \mathcal{A}_\gamma \] satisfying the conditions
\[ p_\gamma = p_{\gamma'} p_{\gamma'_{\gamma}}, \ \forall \gamma' \geq \gamma. \tag{5} \]

Notice that the \(*\)-algebra of functions on \( \overline{\mathcal{A}} \) of the form \( p_\gamma^* f, \ f \in C(\mathcal{A}_\gamma), \ \gamma \in \Gamma \), where \( p_\gamma^* \) denotes pull-back, is naturally isomorphic to the algebra \( \text{Cyl} \) of cylindrical functions. As usual, we will not distinguish between the two algebras.

It follows from the above structure [AL2] that (normalized regular Borel) measures on \( \overline{\mathcal{A}} \) are in 1-1 correspondence with families \( \{\mu_\gamma\}_{\gamma \in \Gamma} \) of measures \( \mu_\gamma \) on the spaces \( \mathcal{A}_\gamma \), satisfying the consistency conditions:
\[ \int_{\mathcal{A}_{\gamma'}} p_{\gamma'\gamma}^* f \, d\mu_{\gamma'} = \int_{\mathcal{A}_\gamma} f \, d\mu_\gamma, \ \forall \gamma' \geq \gamma. \tag{6} \]

The correspondence between a measure \( \mu \) on \( \overline{\mathcal{A}} \) and the associated family \( \{\mu_\gamma\}_{\gamma \in \Gamma} \) is given by
\[ \int_{\overline{\mathcal{A}}} p_\gamma^* f \, d\mu = \int_{\mathcal{A}_\gamma} f \, d\mu_\gamma, \ \forall f \in C(\mathcal{A}_\gamma). \tag{7} \]

As a counterpart of the projective structure of \( \overline{\mathcal{A}} \), every Hilbert space \( \mathcal{H} = L^2(\overline{\mathcal{A}}, \mu) \) acquires a so-called inductive structure, as follows (see [T1] for further details). Let \( \mathcal{H}_\gamma \) denote the Hilbert space \( L^2(\mathcal{A}_\gamma, \mu_\gamma) \). Due to (6), the pull-back's \( p_{\gamma'\gamma}^* \), \( \gamma' \geq \gamma \), define injective isometries \( p_{\gamma'\gamma}^* : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma'} \) satisfying consistency conditions following from (4). Likewise, it follows from (7) that the pull-back's \( p_\gamma^* \) define transformations \( p_\gamma^* : \mathcal{H}_\gamma \to \mathcal{H} \), which are unitary when considered as maps
\[ p_\gamma^* : \mathcal{H}_\gamma \to p_\gamma^* \mathcal{H}_\gamma \tag{8} \]
onto their images, the closed subspaces \( p_\gamma^* \mathcal{H}_\gamma \). The linear maps \( p_\gamma^* \) and \( p_{\gamma'\gamma}^* \) are related by consistency conditions corresponding to (5), namely:
\[ p_\gamma^* = p_{\gamma'\gamma}^* p_{\gamma'_{\gamma}}^*, \ \forall \gamma' \geq \gamma. \tag{9} \]

Notice that the subspace \( \text{Cyl} = \bigcup_{\gamma \in \Gamma} p_\gamma^* C(\mathcal{A}_\gamma) \), and therefore the reunion \( \bigcup_{\gamma \in \Gamma} p_\gamma^* \mathcal{H}_\gamma \), is dense in \( \mathcal{H} \). This fact will be used repeatedly in section 4.

The representation of \( C(\overline{\mathcal{A}}) \) upon which LQG is actually developed is based on the Ashtekar-Lewandowski, or uniform measure \( \mu_0 \) [AL1]. The measure \( \mu_0 \) is defined by a family \( \{\mu_{0,\gamma}\}_{\gamma \in \Gamma} \) such that, for each \( \gamma \), \( \mu_{0,\gamma} \) is
the image of the Haar measure on $G^N\gamma$, under the natural identification $A_{\gamma} \equiv G^N\gamma$. The corresponding Hilbert space is the so-called kinematical Hilbert space $\mathcal{H}_0 := L^2(\mathcal{A}, \mu_0)$. The measure $\mu_0$ has remarkable and unique properties, allowing a quantization of the full LQG kinematical algebra together with implementations of the constraints of general relativity (see [F, LOST, OL1, OL2, ST] for uniqueness results concerning the quantization of flux variables together with the implementation of the diffeomorphism constraint, [T1, T4] for a discussion on the implementation of the hamiltonian constraint, and [AL5, R, T1] for general treatments of LQG).

In what concerns us here, only one property of $\mu_0$ will be used, namely its faithfulness [AL1, AL2], which, as seen before, is the same as faithfulness of the corresponding representation of $C(\mathcal{A})$.

3 Denseness of $\mathcal{H}_0$ vector states

It is argued in [T2, T3] that the $\mu_0$ representation is in some sense a ”fundamental representation” of the algebra $C(\mathcal{A})$, meaning that new representations can be obtained from the $\mathcal{H}_0$ inner product. Using results from section 2.1 and well known separation properties of compact Hausdorff spaces, we will now show that every $C(\mathcal{A})$ state can indeed be obtained, as a weak-star limit, from $\mathcal{H}_0$ vector states. This is in fact a very general result, relying on faithfulness only.

Let us start by constructing a directed set needed in what follows. Let $\{\bar{A}_1, \ldots, \bar{A}_n\}$ be a finite set of distinct points of $\mathcal{A}$ and consider a set of disjoint open sets $\{U_1, \ldots, U_n\}$, with $\bar{A}_i \in U_i$. Then the set of ordered $n$-tuples of the form $(B_1, \ldots, B_n)$ with $B_i \subset U_i$, $\bar{A}_i \in B_i$, $B_i$ open, is clearly a directed set with respect to the following partial order relation:

$$(B_1, \ldots, B_n) \geq (D_1, \ldots, D_n) \text{ whenever } B_i \subset D_i \forall i.$$  

Let then $E(\mathcal{H}_0)$ be the set of $\mathcal{H}_0$ vector states, i.e. the set of states of the algebra $C(\mathcal{A})$ defined by normalized vectors $\psi \in \mathcal{H}_0$ as follows:

$$C(\mathcal{A}) \ni f \mapsto \langle \psi, f\psi \rangle_0,$$

where $\langle \cdot, \cdot \rangle_0$ denotes the $\mathcal{H}_0$ inner product. We will now show that the convex hull of Dirac states lies in the weak-star closure of $E(\mathcal{H}_0)$. To prove it, let us fix a state in the convex hull of Dirac states:

$$f \mapsto \sum_{i=1}^n t_i f(\bar{A}_i), \quad n \in \mathbb{N}, \bar{A}_i \in \mathcal{A}, \quad t_i > 0, \quad \sum_{i=1}^n t_i = 1.$$
Consider then disjoint open sets \( \{U_1, \ldots, U_n\} \) in \( \overline{A} \), with \( \bar{A}_i \in U_i \), and let \( I \) be the directed set \( \{(B_1, \ldots, B_n)\} \) defined as above, with \( \bar{A}_i \in B_i \subset U_i \). Let \( \chi_{B_i} \) be the characteristic function of the open set \( B_i \). Clearly \( \|\chi_{B_i}\|_0^2 = \mu_0(B_i) \neq 0 \), since \( \mu_0 \) is faithful. Let

\[
\psi_{B_i} := \frac{\chi_{B_i}}{\|\chi_{B_i}\|_0} \in \mathcal{H}_0
\]

and consider still

\[
\psi_{B_1 \ldots B_n} := \sum_{i=1}^{n} t_i^{1/2} \psi_{B_i}.
\]

We also have

\[
\|\psi_{B_1 \ldots B_n}\|_0^2 = 1,
\]

since the sets \( B_i \) are disjoint and \( \sum t_i = 1 \). Consider finally the net of states indexed by \( I \):

\[
f \mapsto \langle \psi_{B_1 \ldots B_n}, f \psi_{B_1 \ldots B_n} \rangle_0 = \sum_{i=1}^{n} t_i \int_{B_i} f \, d\mu_0 \ , \ (B_1, \ldots, B_n) \in I.
\]

One can now show that \( \langle \psi_{B_1 \ldots B_n}, f \psi_{B_1 \ldots B_n} \rangle_0 \) converges, \( \forall f \), to the given value \( \sum t_i f(\bar{A}_i) \). In order to do this, let us fix \( f \) and consider any \( \epsilon > 0 \). Let \( \epsilon' > 0 \) be such that \( n\epsilon' < \epsilon \). Since \( f \) is continuous, for each \( \bar{A}_i \) there is an open set \( V_i \ni \bar{A}_i \) such that

\[
|f(\bar{A}) - f(\bar{A}_i)| < \epsilon' \quad \forall \bar{A} \in V_i.
\]

In particular, there are open sets \( B_i^0 \),

\[
B_i^0 := V_i \cap U_i \subset U_i \ , \ \bar{A}_i \in B_i^0,
\]

such that

\[
|f(\bar{A}) - f(\bar{A}_i)| < \epsilon' \quad \forall \bar{A} \in B_i^0.
\]

Let us take

\[
I \ni (B_1, \ldots, B_n) \supseteq (B_1^0, \ldots, B_n^0).
\]

Then

\[
|\langle \psi_{B_1 \ldots B_n}, f \psi_{B_1 \ldots B_n} \rangle_0 - \sum_{i=1}^{n} t_i f(\bar{A}_i)\rangle_0| = \left| \sum_{i=1}^{n} t_i \int_{B_i} (f - f(\bar{A}_i)) \, d\mu_0 \right| \leq \\
\leq \sum_{i=1}^{n} \sup_{B_i} |f(\bar{A}) - f(\bar{A}_i)| \leq n\epsilon' < \epsilon \ , \forall (B_1 \ldots B_n) \supseteq (B_1^0 \ldots B_n^0),
\]
showing that the convex hull of Dirac states is in the closure of $E(\mathcal{H}_0)$.

Combining the above result with the denseness of the convex hull of Dirac states (section 2.1) one then concludes that the set $E(\mathcal{H}_0)$ is weak-$^\star$ dense in the space $E$ of all states of $C(\mathcal{A})$. Thus, given any state $\omega$ of $C(\mathcal{A})$, there is a net of vectors $\psi_\lambda \in \mathcal{H}_0$, where $\lambda$ belongs to some directed set $I$, such that the net $\langle \psi_\lambda, f \psi_\lambda \rangle_0$ converges to $\omega(f)$, $\forall f \in C(\mathcal{A})$. In particular, given any (regular Borel) measure $\mu$ and $\psi \in L^2(\mathcal{A}, \mu)$, there is a $\mathcal{H}_0$ net $\{\psi_\lambda\}$ such that $\langle \psi_\lambda, f \psi_\lambda \rangle_0$ converges to $\langle \psi, f \psi \rangle := \int f \psi |^2 d\mu$, $\forall f$.

4 Mapping vector states to $\mathcal{H}_0$ cylindrical nets

Despite major progresses in the LQG programme, the identification of semiclassical or low energy states has proved very hard, and this task is not yet completed. Moreover, naturally constructed candidate semiclassical states are typically not elements of $\mathcal{H}_0$, but of some extension thereof. As extensions of $\mathcal{H}_0$ are already required for different reasons, e.g. in order to solve the diffeomorphism constraint [ALMMT, MTV], it is perhaps not surprising that semiclassical analysis leads us to consider extensions of $\mathcal{H}_0$ as well. It does, however, create difficulties concerning the interpretation of candidate states, since well defined quantum operators that ultimately can give meaning to those states are defined in $\mathcal{H}_0$, and not, a priori, in the required extensions.

Two types of "generalized states" occur naturally in relation to LQG semiclassical analysis, namely complex (not necessarily continuous) linear functionals over the space Cyl of cylindrical functions, and states of $C(\mathcal{A})$ that are not realizable as $\mathcal{H}_0$ vector states [AG, AL4, Bo, T1, T3, Va1, Va2, Va4, Ve1]. Although there is a very interesting interplay between these two types of objects [AL4, T3, Va2], the relation between them is not fully clear in general.

The way in which linear functionals $\Psi$ — also called distributions in this context — are dealt with in LQG is to trade them for corresponding graph-labelled nets $\{p^*_\gamma \Psi_\gamma\}_{\gamma \in \Gamma}$ of $\mathcal{H}_0$ vectors [AL4, T3]. The mapping from distributions to $\mathcal{H}_0$ nets is achieved through the so-called cut-off procedure, as follows. The cut-off, up to a graph $\gamma$, of a linear functional $\Psi$ over Cyl is defined (when it exists) as the unique $\Psi_\gamma \in \mathcal{H}_{0\gamma}$ such that $\langle \Psi_\gamma, f \rangle_{0\gamma} = \Psi(p^*_\gamma f)$ is satisfied $\forall f \in C(\mathcal{A}_\gamma)$, where $\mathcal{H}_{0\gamma} := L^2(\mathcal{A}_\gamma, \mu_{0\gamma})$ and $\langle , \rangle_{0\gamma}$ denotes the corresponding inner product. The nets $\{p^*_\gamma \Psi_\gamma\}_{\gamma \in \Gamma}$ typically do not converge in the $\mathcal{H}_0$ norm. Nevertheless, important properties of the original
distribution $\Psi$ are captured in a weaker sense [AL4, T3].

The purpose of the present section is to point out that methods similar to the cut-off procedure can also be applied to an important class of states of $\tilde{C}(\tilde{A})$, and to study convergence properties of such a generalized cut-off construction.

In particular, we consider vector states corresponding to some Hilbert space $\mathcal{H} = L^2(\mathcal{A}, \mu)$, with the requirement that the measure $\mu$ is such that each measure $\mu_\gamma$ of the associated family is absolutely continuous with respect to $\mu_{0_\gamma}$. While this is not the general case, it covers the most potentially interesting situations. The point is simply that $\mu_{0_\gamma}$ is (essentially) the uniform Haar measure on $G^{N_\gamma}$, and therefore more general measures would give rise to representations that look pathological already at the finite dimensional level, since they assign non-zero measure values to subsets of $G^{N_\gamma}$ of zero Haar measure.\(^2\)

Let us then consider a measure $\mu$ on $\tilde{A}$, defined by a family $\{\mu_\gamma\}_{\gamma \in \Gamma}$ such that there is a family of positive functions $R_\gamma \in L^1(\tilde{A}_\gamma, \mu_{0_\gamma})$ satisfying $d\mu_\gamma = R_\gamma d\mu_{0_\gamma}, \forall \gamma$. Let $\mathcal{H}$ be the corresponding Hilbert space $L^2(\mathcal{A}, \mu)$. The non-trivial situation occurs when the Radon-Nikodym derivative $d\mu/d\mu_{0_\gamma}$ fails to exist, so that the natural inner product preserving transformation

$$\mathcal{H} \ni \psi \mapsto \left(\frac{d\mu}{d\mu_0}\right)^{1/2} \psi \in \mathcal{H}_0$$  \hspace{1cm} (10)

is not available. In fact, when $d\mu/d\mu_0$ exists the transformation (10) maps $\mathcal{H}$ to a subspace $\mathcal{H}_{0_\gamma} \subset \mathcal{H}_0$ of functions supported on the support of $\mu$, and transforms the $\mu$ representation of $\tilde{C}(\tilde{A})$ into the restriction to $\mathcal{H}_{0_\gamma}$ of the $\mu_0$ representation. If, moreover, $d\mu_0/d\mu$ is also defined, then (10) establishes a unitary equivalence between representations.

In our case, due to the existence of $d\mu_\gamma/d\mu_{0_\gamma}$, $\forall \gamma$, one has a family of inner product preserving maps from $\mathcal{H}_{\gamma} = L^2(\tilde{A}_\gamma, \mu_\gamma)$ to $\mathcal{H}_{0_\gamma}$. Combining these maps with orthogonal projections on the $\mathcal{H}$ side and pull-back’s on the $\mathcal{H}_0$ side, one can then define a weak version of the transformation (10), consisting of a map from $\mathcal{H}$ to graph-labelled nets in $\mathcal{H}_0$.

Let then

$$P_\gamma : \mathcal{H} \rightarrow \mu_{\gamma}^* \mathcal{H}_{\gamma}$$  \hspace{1cm} (11)

\(^2\)This is reminiscent of the generic situation in quantum field theory, where interaction measures are typically equivalent to the Lebesgue measure when restricted to finite dimensions. Moreover, one could probably safely assume that $\mu_\gamma$ is in fact equivalent to $\mu_{0_\gamma}, \forall \gamma$, i.e. that $\mu_{0_\gamma}$ is in turn continuous with respect to $\mu_\gamma$. We work with the weaker condition for generality, as it poses no further difficulties.
denote the orthogonal projection onto the closed subspace \( p_{\gamma}^* \mathcal{H}_\gamma \). Notice that consistency condition (9) leads to \( p_{\gamma}^* \mathcal{H}_\gamma \subset p_{\gamma'}^* \mathcal{H}_{\gamma'} \), \( \forall \gamma' \geq \gamma \), and therefore \( P_{\gamma'} \) is the identity operator on every subspace \( p_{\gamma}^* \mathcal{H}_\gamma \) such that \( \gamma' \geq \gamma \). Let us also introduce the composition \( \pi_{\gamma} := (p_{\gamma}^*)^{-1} P_{\gamma} \), \( \pi_{\gamma} : \mathcal{H} \to \mathcal{H}_\gamma \). It is easily seen that \( \pi_{\gamma} \) satisfies

\[
\langle \pi_{\gamma} \psi, f \rangle_{\gamma} = \langle \psi, p_{\gamma}^* f \rangle, \quad \forall f \in \mathcal{H}_\gamma, \tag{12}
\]

where \( \langle \cdot, \cdot \rangle_{\gamma} \) and \( \langle \cdot, \cdot \rangle \) denote inner products in \( \mathcal{H}_{\gamma} \) and \( \mathcal{H} \), respectively. Notice also that nothing is lost at this stage when trading a given \( \psi \in \mathcal{H} \) by the family \( \{ \pi_{\gamma} \psi \}_{\gamma \in \Gamma} \), since the net of \( \mathcal{H} \) vectors \( \{ p_{\gamma}^* \pi_{\gamma} \psi \}_{\gamma \in \Gamma} = \{ P_{\gamma} \psi \}_{\gamma \in \Gamma} \) clearly converges to \( \psi \).

One can now bring in the multiplication operators \( R_{\gamma}^{1/2} \), obtaining a map from \( \mathcal{H} \) to \( \mathcal{H}_0 \) nets as follows. Each \( \psi \in \mathcal{H} \) is mapped to the net \( \{ \psi_{\gamma} \}_{\gamma \in \Gamma} \), where \( \psi_{\gamma} \in \mathcal{H}_0 \) is defined by

\[
\psi_{\gamma} = p_{\gamma}^* (R_{\gamma}^{1/2} \pi_{\gamma} \psi). \tag{13}
\]

To see in what sense the map (13) generalizes (10), let us first show that the image \( (d\mu/d\mu_0)^{1/2} \psi \) is recovered from the net \( \{ \psi_{\gamma} \}_{\gamma \in \Gamma} \) in the trivial case in which \( d\mu/d\mu_0 \) actually exists. More precisely, we will show that if there exists \( R \in L^1(\mathcal{A}, \mu_0) \) such that \( d\mu = Rd\mu_0 \), then, \( \forall \psi \in \mathcal{H} \), the net \( \{ \psi_{\gamma} \}_{\gamma \in \Gamma} \) converges in the \( \mathcal{H}_0 \) norm to \( R^{1/2} \psi \).

To prove the above we use the following facts: i) the set \( \text{Cyl} \) of cylindrical functions is dense in \( \mathcal{H} \), and ii) the net \( \{ p_{\gamma}^* R_{\gamma}^{1/2} \}_{\gamma \in \Gamma} \) converges, in the \( \mathcal{H}_0 \) norm, to \( R^{1/2} \) [Ya, B.2.7]. Let us then fix arbitrary \( \psi \in \mathcal{H} \) and \( \epsilon > 0 \). Let us choose \( \gamma_0 \) and \( f \in C(\mathcal{A}_{\gamma_0}) \) such that \( \| p_{\gamma_0}^* f - \psi \| < \epsilon/6 \), where \( \| \cdot \| \) denotes the \( \mathcal{H} \) norm. Also, let us choose \( \gamma_1 \) and \( \gamma_2 \) such that \( \| P_{\gamma_1} \psi - \psi \| < \epsilon/9 \), \( \forall \gamma \geq \gamma_1 \), and \( \| p_{\gamma_0}^* R_{\gamma}^{1/2} - R^{1/2} \|_0 < \epsilon/(3\| p_{\gamma_0}^* f \|_{C^*}) \), \( \forall \gamma \geq \gamma_2 \), where \( \| \cdot \|_{C^*} \) denotes the \( C(\mathcal{A}) C^* \)-algebra norm. Moreover, let \( \gamma_3 \) be such that \( \gamma_3 \geq \gamma_0, \gamma_1, \gamma_2 \). We now have

\[
\| \psi_{\gamma} - R^{1/2} \psi \|_0 = \| (p_{\gamma}^* R_{\gamma}^{1/2} - R^{1/2})P_{\gamma} \psi + R^{1/2}(P_{\gamma} \psi - \psi) \|_0 \leq \| (p_{\gamma}^* R_{\gamma}^{1/2} - R^{1/2})P_{\gamma} \psi \|_0 + \| P_{\gamma} \psi - \psi \|, \tag{14}
\]

and

\[
\| (p_{\gamma}^* R_{\gamma}^{1/2} - R^{1/2})P_{\gamma} \psi \|_0 = \| (p_{\gamma}^* R_{\gamma}^{1/2} - R^{1/2})(P_{\gamma} \psi - p_{\gamma_0} f) + (p_{\gamma}^* R_{\gamma}^{1/2} - R^{1/2})p_{\gamma_0} f \|_0 \leq \| (p_{\gamma}^* R_{\gamma}^{1/2} - R^{1/2})(P_{\gamma} \psi - p_{\gamma_0} f) \|_0 + \| p_{\gamma_0} f \|_{C^*} \| p_{\gamma}^* R_{\gamma}^{1/2} - R^{1/2} \|_0. \tag{15}
\]
Regarding the first term in the last line of (15), notice that \( P_γ \psi - P_{γ₀}^* f \) is an element of \( P_γ \mathcal{H}_γ \), \( ∀γ ≥ γ₀ \), and that both \( P_γ R_{γ/2} \) and \( R_{γ/2} \) are bounded operators of unit norm, from \( P_γ \mathcal{H}_γ \) to \( \mathcal{H}_0 \). Thus
\[
\| (P_γ R_{γ/2} - R_{γ/2}) (P_γ \psi - P_{γ₀}^* f) \|_0 ≤ 2\| P_γ \psi - P_{γ₀}^* f \| ≤ 2\| P_γ \psi - \psi \| + 2\| \psi - P_{γ₀}^* f \|. \tag{16}
\]

Putting (14), (15) and (16) together we get, \( ∀γ ≥ γ₀ \):
\[
\| \psi_γ - R_{γ/2} \psi \|_0 ≤ 3\| P_γ \psi - \psi \| + 2\| \psi - P_{γ₀}^* f \| + \| P_{γ₀}^* f \|_{C^*} \| P_γ R_{γ/2} - R_{γ/2} \|_0 < \epsilon ,
\]
thus concluding the proof.

Returning to the general case, we now establish the exact sense in which (13) generalizes the transformation (10). Specifically we show that, \( ∀ϕ', ψ ∈ \mathcal{H} \) and \( ∀F ∈ C(\mathcal{A}) \), the complex net \( \{ (ϕ'_γ, F ψγ) \}_γ \gamma ∈ Γ \) converges to \( (ϕ', F ψ) \).

We will use the fact that the set of cylindrical functions is dense in \( C(\mathcal{A}) \). We start precisely by proving the result for cylindrical \( F \), using (9) and unitarity of \( p_γ^* \) (8). Let us then fix \( ϕ', ψ ∈ \mathcal{H} \), \( γ₀ ∈ Γ \), \( f ∈ C(\mathcal{A}_{γ₀}) \) and take \( F = P_γ^* f \). For every \( γ ≥ γ₀ \) we have
\[
\langle ϕ'_γ, (p_{γ₀}^* f)ψγ \rangle_0 = \langle R_{γ/2}^\frac{1}{2} π_γ ϕ', (p_{γ₀}^* f) R_{γ/2}^\frac{1}{2} π_γ ψ \rangle_0, \gamma = \langle p_{γ₀}^* f \rangle π_γ ψγ = \langle P_γ ϕ', (p_{γ₀}^* f) P_γ ψ \rangle = \langle ϕ', (p_{γ₀}^* f) P_γ ψ \rangle, \tag{17}
\]
where in the last line we have used the fact that \( (p_{γ₀}^* f) P_γ ψ \) is actually an element of \( p_γ \mathcal{H}_γ \). Thus
\[
\| (ϕ'_γ, (p_{γ₀}^* f)ψγ) - (ϕ', (p_{γ₀}^* f)ψ) \| = \| (ϕ', (p_{γ₀}^* f) (P_γ ψ - ψ)) \| ≤ \| ϕ' \| p_{γ₀}^* \tilde{f} \| \| P_γ ψ - ψ \|.
\]
Convergence now follows, since \( ϕ' p_{γ₀}^* \tilde{f} \) is fixed. We turn next to general \( F \), using the denseness of cylindrical functions and the continuity of the representations of \( C(\mathcal{A}) \). Let us then fix \( ϕ', ψ ∈ \mathcal{H} \), \( F ∈ C(\mathcal{A}) \), \( ε > 0 \) and choose \( γ₀ \) and \( f ∈ C(\mathcal{A}_{γ₀}) \) such that
\[
\| p_{γ₀}^* f - F \|_{C^*} < ε/(3\| ϕ' \| \| ψ \|).
\]
Let \( γ₁ ≥ γ₀ \) be such that
\[
\| (ϕ'_γ, (p_{γ₀}^* f)ψγ) - (ϕ', (p_{γ₀}^* f)ψ) \| < ε/3, ∀γ ≥ γ₁.
\]
The proof can now be completed, since $\forall \gamma \geq \gamma_1$ we find

$$|\langle \psi'_\gamma, F\psi_\gamma \rangle_0 - \langle \psi', F\psi \rangle| =$$

$$|\langle \psi'_\gamma, (F - p_{\gamma_0}^* f)\psi_\gamma \rangle_0 + \langle \psi', (p_{\gamma_0}^* f - F)\psi \rangle + \langle \psi'_\gamma, (p_{\gamma_0}^* f)\psi_\gamma \rangle_0 - \langle \psi', (p_{\gamma_0}^* f)\psi \rangle|$$

$$< \|p_{\gamma_0}^* f - F\|_{C^*} (\|\psi'_\gamma\|_0 \|\psi_\gamma\|_0 + \|\psi'\| \|\psi\|) + \epsilon/3$$

$$= \|p_{\gamma_0}^* f - F\|_{C^*} (\|P_{\gamma} \psi'\| \|P_{\gamma} \psi\| + \|\psi'\| \|\psi\|) + \epsilon/3 < \epsilon.$$

The above results support the viewpoint of considering the map (13) as a natural weak substitute for transformation (10) (for the measures under consideration), thus generalizing the cut-off procedure in what concerns mapping elements of $\mathcal{H}$ into graph-labelled nets in $\mathcal{H}_0$.

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