Algebraic holography in asymptotically simple, asymptotically AdS spacetimes

This work is dedicated to Professor Jacques Bros, on the occasion of his 70th birthday.

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Abstract

I’ll describe a general geometric setup allowing a generalization of Rehren duality to asymptotically anti-de Sitter spacetimes whose classical matter distribution is sufficiently well-behaved to prevent the occurrence of singularities in the sense of null geodesic incompleteness. I’ll also comment on the issues involved in the reconstruction of an additive and locally covariant bulk net of observables from a corresponding boundary net in this more general situation.

1 Introduction

The inception of quantum field theory in curved spacetime, about forty years ago, brought into evidence a host of new conceptual problems hitherto absent or left unnoticed due to the peculiarities of Minkowski spacetime, such as the very definitions of the notions of vacuum, particle, S-matrix, etc.. One hopes that the clarification of such issues may bring some new insights into the deeper problem of the quantization of gravity. An example of a possible interface between QFT in curved spacetime – based on the principle of locality – and a would-be quantum theory of gravity – where such principle is likely to be only macroscopically valid – is black hole thermodynamics. The idea that a stationary black hole is a “black” object in the quantum sense of the word – i.e., it produces a thermal

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bath with a certain universal temperature – suggests, together with the peculiar geometrical behaviour of its event horizon, some remarkable consequences, such as: 1.) the notion of (thermodynamical) entropy is no longer extensive as in usual thermodynamics, but leads to a quantity depending linearly on the area of the event horizon (Bekenstein-Hawking entropy) 2.) A black hole can “evaporate”, i.e., lose all its mass by thermally radiating it to infinity (Hawking radiation), in finite time, which leads to a complete decoherence of an initially pure global state through entanglement with the partial state inside the horizon of the vanishing black hole. To explain these phenomena without violating basic postulates of quantum mechanics, 't Hooft and Susskind have put forward the holographic principle – namely, that the horizon has already all physical degrees of freedom, in the sense that one can completely reconstruct the physical data contained in a (bulk) volume from the physical system living on the boundary of this volume, in pretty much the same way a tridimensional picture is rebuilt from a two-dimensional hologram.

A concrete implementation of this principle in string theory was conjectured by Maldacena and Witten – the famous AdS-CFT correspondence, which triggered an impressive amount of theoretical development afterwards. Surprisingly, although it was thought the holographic principle to be inconsistent with the principle of locality, it is possible to rigorously prove that the essentials of the AdS-CFT correspondence – more precisely, the peculiar geometry of the spacetimes involved – allow the reformulation of this correspondence in a manner consistent with the principle of locality, that is, within the context of QFT in curved spacetime. Such a result is proven in Rehren’s paper, which is formulated within the framework of Local Quantum Physics (Algebraic Quantum Field Theory). It states that “theories of local observables in Anti-de Sitter (AdS) spacetime that are covariant under global (rigid) isometries can be put in an one-to-one correspondence to theories of local observables in AdS’s boundary – that is conformal to Minkowski spacetime of one dimension less – that are covariant under global (rigid) conformal transformations”. This result did not call the attention it deserved outside the realm of Local Quantum Physics, being misinterpreted as a “fake proof” of Maldacena’s AdS-CFT correspondence, and therefore deemed useless by string theorists; this comes as a consequence of the fact that only rigid isometries are implemented, i.e., the quantum observables are completely decoupled from the gravitational degrees of freedom – there’s no clue to how the bulk quantum system transforms under arbitrary diffeomorphisms of spacetime, let alone how it reacts to arbitrary, but compactly localized, changes of the metric, and how these changes manifest themselves in the holographic dual theory.

Here we understand Rehren’s theorem – called heretofore algebraic holography or Rehren duality – as an independent result, that, at the same time, poses questions with a counterpart in the “stringy” AdS-CFT correspondence, and issues deeply related to the foundations of relativistic quantum theory itself. It’s from this perspective that the author’s work starts.
In Section 2 after recalling some basic definitions and results in Lorentzian geometry that will be needed in the sequel, we’ll extend Rehren’s geometrical setup to asymptotically simple, asymptotically AdS spacetimes of any dimension greater than 2, based in the simple, but crucial remark: wedges in AdS are simply diamonds with both tips belonging to the conformal infinity. This not only renders Rehren duality quite natural, but also shows that it depends essentially on the global causal structure of AdS’s conformal infinity, therefore begging for a generalization to spacetimes who share these properties. We’ll see, however, in Section 3 that there are some subtle, but important aspects in this more general setting. Namely, one need some global constraints on the classical matter distribution (which can be put into strictly geometrical terms) in order to algebraic holography to preserve causality when going from the bulk to the boundary. We’ll see that these conditions also open up the possibility of encoding bulk gravitational effects in a non-geometrical way at the boundary – namely, in the form of spontaneous symmetry breaking (breakdown of Haag duality for diamonds at the boundary), if the bulk theory is causal and Haag dual. This has remarkable consequences, due to previous results by Brunetti, Guido and Longo about modular covariance in conformal QFT[BrGL]. These same conditions raise, on the other hand, great difficulties when it comes to reconstruct the (compact) localization of the bulk observables using only boundary CFT data and the (bulk wedge ⇔ boundary diamond) correspondence. It can be shown, nevertheless, that for sufficiently small bulk diamonds this reconstruction can indeed be done. This is just enough for additive bulk theories, which can thus be “holographically rebuilt”. Section 4 closes with some remarks on open problems and further work to be done by the author.

The developments to be presented in what follows are, first and foremost, geometric. We’ll center in two essential aspects: causality and localization (in the sense of manifold topology – see Section 3). We’ll refrain from studying covariance aspects of our construction in detail, as they demand a separate paper of their own for a proper discussion – we’ll limit ourselves to some remarks at the end.

2 Doing away with coordinates in Rehren duality

2.1 Some tools in Lorentzian geometry

Let’s recapitulate some definitions. For details, see the monographs of Wald[Wald], Hawking and Ellis[HawE], O’Neill[ONei] and Beem, Ehrlich and
Easley[BeEE]. By a spacetime it will be understood a pair \((\hat{\mathcal{M}}, \hat{g})\), where \(\mathcal{M}\) is a paracompact, connected and orientable \(\mathcal{C}^\infty\) manifold, and \(\hat{g}\) is a time-orientable, Lorentzian \(\mathcal{C}^\infty\) metric, with Levi-Civita connection \(\nabla_a\).

Let \(\mathcal{U} \subset \mathcal{M}\) be an open set, and \(p \in \mathcal{U}\). The chronological (resp. causal) future of \(p\) with respect to \(\mathcal{U}\), denoted by \(I^+(p, \mathcal{U})\) (resp. \(J^+(p, \mathcal{U})\)) is given by the following sets:

\[
I^+(p, \mathcal{U}) \doteq \{ x \in \mathcal{U} : \exists \gamma : [0, a] \xrightarrow{\mathcal{C}^\infty} \mathcal{U} \text{ timelike and future such that } \gamma(0) = p, \gamma(a) = x \}; \tag{1}
\]

\[
J^+(p, \mathcal{U}) \doteq \{ x \in \mathcal{U} : x = p \text{ on } \exists \gamma : [0, a] \xrightarrow{\mathcal{C}^\infty} \mathcal{U} \text{ causal and future such that } \gamma(0) = p, \gamma(a) = x \}. \tag{2}
\]

Exchanging future with past, one can define in a dual fashion the chronological (resp. causal) past \(I^-(p, \mathcal{U})\) (resp. \(J^-(p, \mathcal{U})\)) of \(p\) with respect to \(\mathcal{U}\). It follows from these definitions that \(I^\pm(p, \mathcal{U})\) are open and \(\text{int}(J^\pm(p, \mathcal{U})) = I^\pm(p, \mathcal{U})\). Using such sets we can define chronology and causality relations between two points. Let \(p, q \in \mathcal{U} \subset \mathcal{M}\). We say that \(p\) chronologically (resp. causally) precedes \(q\) with respect to \(\mathcal{U}\) if \(p \in I^-(q, \mathcal{U})\) (resp. \(p \in J^-(q, \mathcal{U})\)). We denote this relation by \(p \ll_{\mathcal{U}} q\) (resp. \(p \leq_{\mathcal{U}} q\)). If \(p \leq_{\mathcal{U}} q\) and \(p \neq q\), we write \(p <_{\mathcal{U}} q\). If \(p \ll_{\mathcal{U}} q\) and \(p \gg_{\mathcal{U}} q\) (resp. \(p \not\ll_{\mathcal{U}} q\) and \(p \not\gg_{\mathcal{U}} q\)), we say that \(p\) and \(q\) are chronologically (resp. causally) disjoint — in this case, we write \(p \lor_{\mathcal{U}} q\) (resp. \(p \land_{\mathcal{U}} q\)). All chronology and causality relations defined above, as well as the notions of chronological/causal future/past, are defined for arbitrary nonvoid sets in an obvious way. If \(p, q \in \mathcal{U} \subset \mathcal{M}\) implies that \(p \lor_{\mathcal{U}} q\) (resp. \(p \land_{\mathcal{U}} q\)), we say that \(\mathcal{U}\) is achronal (resp. acausal) with respect to \(\mathcal{M}\).

For instance, given \(\mathcal{O} \subset \mathcal{M}\), the set \(\partial I^+/-(\mathcal{O}, \mathcal{U}) = \partial J^+/-(\mathcal{O}, \mathcal{U})\) is said to be the future/past achronal boundary of \(\mathcal{O}\) with respect to \(\mathcal{U}\), and constitutes an achronal, topological submanifold of \(\mathcal{M}\) such that every \(p \in \partial I^+/-(\mathcal{O}, \mathcal{U})\) belongs to a (necessarily unique) null geodesic segment, achronal with respect to \(\mathcal{U}\) and contained in \(\partial I^+/-(\mathcal{O}, \mathcal{U})\), which is either past/future inextendible or possesses a past/future endpoint in \(\mathcal{O}\). Such geodesics are the null generators of \(\partial I^+/-(\mathcal{O}, \mathcal{U})\).

Let now \(\mathcal{I} \subset \mathcal{M}\) be closed and achronal. The future (resp. past) domain of dependence of \(\mathcal{I}\), denoted by \(D^+(\mathcal{I})\) (resp. \(D^-\mathcal{I})\)) is given by:

\[
D^+/-\mathcal{I} \doteq \{ p \in \mathcal{V} : \forall \gamma : [0, a] \rightarrow \mathcal{V} \text{ past/future inextendible, causal such that } \gamma(0) = p, \exists b < a \text{ such that } \gamma(b) \in \mathcal{I} \}. \tag{3}
\]

\(D(\mathcal{I}) \doteq D^+(\mathcal{I}) \cup D^-\mathcal{I}\) is said to be the domain of dependence or Cauchy development of \(\mathcal{I}\). The edge of \(\mathcal{I}\) (notation: \(\partial \mathcal{I}\)) is given by the points \(p \in \mathcal{I}\)
such that any open neighborhood of \( p \) possesses points \( q \in I^- (p) \), \( r \in I^+ (p) \) and a timelike curve \( \gamma \) linking \( q \) to \( r \), and with empty intersection with \( \mathcal{I} \). If \( S \) is not only achronal but also acausal, then the set \( \text{int}(D(\mathcal{I})) \) is globally hyperbolic (in such a case, we say that \( S \) is a Cauchy surface for \( \text{int}(D(\mathcal{I})) \)). The closed, achronal set \( H^+(\mathcal{I}) \equiv \overline{D^+(\mathcal{I}) \setminus I^-(D^+(\mathcal{I}))} \), denoted future Cauchy horizon of \( \mathcal{I} \), possess the following property: any \( p \in H^+(\mathcal{I}) \) is contained in an achronal, null geodesic segment contained in \( H^+(\mathcal{I}) \), which is either past inextendible or has a past endpoint in \( \mathcal{I} \). An analogous property holds for \( H^-(\mathcal{I}) \), the past Cauchy horizon of \( \mathcal{I} \); the (full) Cauchy horizon \( H(\mathcal{I}) \equiv H^+(\mathcal{I}) \cup H^-(\mathcal{I}) \) equals \( \partial D(\mathcal{I}) \).

Now, for the notion of conformal infinity:

**Definition 2.1** The conformal infinity or conformal boundary of a \( n \)-dimensional spacetime \((\mathcal{M}, \hat{g})\) is a \( n - 1 \)-dimensional semi-Riemannian manifold \((I, b)\) such that there exists a \( n \)-dimensional, Lorentzian manifold-with-boundary \((\mathcal{M}, g)\) (the conformal closure or conformal completion of \((\mathcal{M}, \hat{g})\)) satisfying:

- \( I \equiv \partial \mathcal{M} \); there is a diffeomorphism \( \Phi \) of \( \mathcal{M} \) onto \( \Phi(\mathcal{M}) = \mathcal{M} \setminus \partial \mathcal{M} \);
- \( b \) is the (possibly degenerate) semi-Riemannian metric induced by \( g \) in \( I \);
- There exists a conformal (Weyl) factor (that is, a real-valued, positive \( C^\infty \) function \( \Omega \) in \( \mathcal{M} \), that admits a \( C^\infty \) extension to \( \mathcal{M} \) such that \( \Omega \mid_{\mathcal{I}} = 0 \) and \( d\Omega \mid_{\mathcal{I}} \neq 0 \) em \( \mathcal{I} \) satisfying \( g_{ab} = \Omega^2 \hat{g}_{ab} \).

Note that, if \( n_a := \nabla_a \Omega \neq 0 \) in \( \mathcal{I} \) (\( n_a \) is the normal (co)vector to \( \mathcal{I} \); this condition can be made to hold if the Einstein equations are satisfied in a neighborhood of \((\mathcal{I}, b)\) and the classical matter fields satisfy certain decay conditions in this neighborhood), then \( \Omega \) can be chosen in such a way that the extrinsic curvature (second fundamental form) \( K_{ab} := \nabla_a n_b \) vanishes in \( \mathcal{I} \). It is enough to multiply \( \Omega \) by a real-valued, positive, nowhere vanishing \( C^\infty \) function \( \omega \) in \( \mathcal{M} \) – the new factor \( \Omega \) still satisfies all the conditions in Definition 2.1. Nonetheless, this choice by no means constrains the values \( \omega \) can take in \( I \) [AsMa]. Therefore, \((\mathcal{I}, b)\) can be taken totally geodesic, regardless of the choice of representative of the conformal structure of \( b \).

**Definition 2.2** Let \((\mathcal{M}, \hat{g})\) be a \( n \)-dimensional spacetime with conformal infinity \((\mathcal{I}, b)\). We say that \((\mathcal{M}, \hat{g})\) is asymptotically simple if any inextendible null geodesic \((\mathcal{M}, \hat{g})\) has an unique extension to \((\mathcal{M}, g)\) such that \( \mathcal{I} \) contains precisely its both endpoints.

Obviously, this is only possible if \((\mathcal{M}, \hat{g})\) is null geodesically complete. Actually, when \((\mathcal{I}, b)\) is timelike (and therefore a spacetime in its own right), one can say more, justifying the name “asymptotically simple”:
Theorem 2.1 If $(\tilde{\mathcal{M}}, \tilde{g})$ is asymptotically simple and has a timelike conformal infinity, then it is causally simple, that is, $J^\pm(p, \tilde{\mathcal{M}})$ is closed in $\tilde{\mathcal{M}}$ (and therefore equal to $I^\pm(p, \tilde{\mathcal{M}})$) for all $p \in \tilde{\mathcal{M}}$.

Proof. First, notice that if $p, q \in \tilde{\mathcal{M}}$ are such that $p \prec \tilde{\mathcal{M}} q$ and likewise exchanging future with past, for a timelike curve in $\mathcal{M}$ linking $p$ to $q$ can always be slightly deformed so as to give a timelike curve contained in $\tilde{\mathcal{M}}$ and linking $p$ to $q$. Now, suppose that $p \in \partial I^-(q, \tilde{\mathcal{M}})$ and $p /\in J^-(q, \tilde{\mathcal{M}})$. By the reasoning above, we have $p \in \partial I^-(q, \tilde{\mathcal{M}})$. Moreover, by hypothesis, a null generator $\gamma$ of $\partial I^-(q, \tilde{\mathcal{M}})$ must reach its future endpoint at infinity without crossing $q$ before this. Let $r$ be such an endpoint. Then, $r \in \partial I^-(q, \tilde{\mathcal{M}})$ since this set is closed. Since the infinity is totally geodesic, $\gamma$ must hit it transversally and thus any future causal extension of $\gamma$ must be broken. Therefore, if one extends $\gamma$ slightly to the future by a null generator segment $\gamma'$ of $\partial I^-(q, \tilde{\mathcal{M}})$ crossing $r$ (say, by setting the affine parameter $t$ of $\gamma'$ equal to zero in $r$ and extending up to $t = \epsilon > 0$), then there is a timelike curve in $\mathcal{M}$ linking $p$ to $\gamma'(\epsilon)$, which violates the achronality of $\partial I^-(q, \tilde{\mathcal{M}})$. Repeat the argument exchanging future with past.

An asymptotically simple spacetime, however, need not be globally hyperbolic – a prime example is AdS spacetime, which will be studied in the next Subsection.

2.2 Asymptotically Anti-de Sitter (AAdS) spacetimes and Rehren duality

We’ll recapitulate some definitions given in [Ribe]. Recall that $n$-dimensional AdS spacetime (notation: $AdS_n$, $n \geq 3$) is given by the hyperquadric in $\mathbb{R}^{n+1}$ ($X = (X^1, X^2, \ldots, X^{n-2})$

$$-X^0X^0 + X \cdot X + X^{n-1}X^{n-1} - X^nX^n = A^2, \quad A > 0,$$

(4)

where the $X^0 - X^n$ plane determines the time orientation. $AdS_n$ is an homogeneous space for the isometry group $SO(2, n - 1)$. Consider now the region $AdS_n^+ = \{ X \in AdS_n : X^{n-1} + X^n > 0 \}$. One can build a chart for this region (denoted horocyclic or Poincaré parametrization) with the parameters $(x, z)$, where $x \in \mathbb{R}^{1,n-2}$ and $z \in \mathbb{R}_+$:

$$\begin{align*}
X^n &= 4x^\mu (\mu = 0, \ldots, n-2) \\
\frac{1}{A}X^{n-1} &= \frac{1}{2}z^2 + \frac{1}{2}x^\mu x^\mu \\
\frac{1}{A}X^n &= \frac{1+z^2}{2z} - \frac{1}{2z}x^\mu x^\mu
\end{align*}

(5)

One can see from the formulae (5) that each timelike hypersurface given by $z = const.$ is conformal to $\mathbb{R}^{1,n-2}$ by a factor

$$(X^{n-1} + X^n)^2 = \frac{A^2}{z^2}.$$

(6)
In this chart, the $AdS_n$ metric is written as
\[
ds^2 = \frac{A^2}{z^2} (dx_\mu dx^\mu + dz^2),
\]
that is, $AdS^+_n$ is a semi-Riemannian “warped product” of $\mathbb{R}^{1,n-2}$ with $\mathbb{R}_+^*$. The universal covering of $AdS_n$ (notation: $\tilde{AdS}_n$) is asymptotically simple, and possesses the Einstein static universe (ESU) $\mathcal{I} = \mathbb{R} \times S^{n-2}$ as conformal infinity. Specializing to the Poincaré chart, we see that in $AdS^+_n$ the conformal factor is given by (6), that is, $z = 0$ corresponds to the conformal embedding of Minkowski spacetime into the Einstein static universe. $\tilde{AdS}_n$ satisfies the empty space Einstein equations with (negative) cosmological constant $\Lambda = -\frac{(n-1)(n-2)}{2A^2}$.

Now, let us define, for $p, q \in \mathcal{I}$, $p \ll \mathcal{I} q$:
\[
W_{p,q} = (\mathcal{I}^- (p, AdS_n) \cap \mathcal{I}^+(q, AdS_n)) \cap \tilde{AdS}_n \text{ ((bulk) wedge); (8)}
\]
\[
D_{p,q} = (\mathcal{I}^- (p, AdS_n) \cap \mathcal{I}^+(q, AdS_n)) \cap \mathcal{I} = \mathcal{I}^- (p, \mathcal{I}) \cap \mathcal{I}^+(q, \mathcal{I}) \text{ ((boundary) diamond). (9)}
\]

Let $p \in \mathcal{I}$. All future null geodesics emanating from $p$ will focus at a single point of $\mathcal{I}$, which is the future endpoint of all null generators of $\mathcal{I}^+ (p, \mathcal{I})$. This point is denoted antipodal of $p$ (notation: $\tilde{p}$). The antipodal of $p$ has the following properties:
\[
\partial I^+(p, \mathcal{I}) = \partial I^- (\tilde{p}, \mathcal{I});
\]
\[
\partial I^+(p, AdS_n) = \partial I^- (\tilde{p}, AdS_n).
\]

Let $\tilde{\bar{p}} = \overline{(p)}$, and define $\mathcal{M}in(p) = D_{p,\tilde{\bar{p}}}$, the Minkowski domain to the future of $p \in \mathcal{I}$. This region corresponds to the conformal embedding of $\mathbb{R}^{1,n-2}$ into $\mathcal{I}$ such that $p$ corresponds to the past timelike infinity of $\mathbb{R}^{1,n-2}$. $\mathcal{P}oi(p) \equiv W_{p,\tilde{\bar{p}}}$ corresponds to the domain of a Poincaré chart in $AdS_n$, therefore being denominated Poincaré domain to the future of $p$. Given the objects defined above, we can define the geometrical setup for Rehren duality as follows:

1. The isometry group of $\tilde{AdS}_n$ acts transitively on the collections $\mathfrak{W} \equiv \{W_{p,q} : p, q \in \mathcal{I}\}$ of bulk wedges and $\mathfrak{D} \equiv \{D_{p,q} : p, q \in \mathcal{I}\}$ of boundary diamonds.
2. $\mathfrak{W}_{p,q}$ and $\mathfrak{D}_{p,q}$ share the same isotropy subgroup.
3. From (10) and (11), it follows respectively that $\mathfrak{W}$ and $\mathfrak{D}$ are closed under causal complements. More precisely (by $O_{\mathcal{I}}'$ we mean the causal complement of $O$ with respect to $\mathfrak{W} \supset O$), we have, for all $p, q \in \mathcal{I}in(r)$, $r \in \mathcal{I}$, $p \ll \mathcal{I} q$,
\[
\mathfrak{W}_{q,\tilde{\bar{p}}} \cap \tilde{AdS}_n = (\mathfrak{W}_{p,q} \bigcap \tilde{AdS}_n)
\]
\[ D_{\bar{q}, \bar{p}} = (D_{p, q})' \]

4. As a consequence of the above statements, the Rehren bijection

\[ \rho : \mathcal{W} \rightarrow \mathcal{D} \]

\[ \mathcal{W}_{p, q} \mapsto \alpha(\mathcal{W}_{p, q}) = D_{p, q} \]

is one-to-one and onto, preserves inclusions and causal complements, and intertwines the action of \( AdS_n \)'s isometry group, which is also the conformal group of \( (\mathcal{I}, b) \) and the universal covering of the conformal group of Minkowski spacetime. **Rehren duality = algebraic holography** is simply the transplantation (change of index set) of theories of local observables under the map \( \rho^2 \).

Now, what happens if we “perturb” AdS spacetime in such a way that we still have the ESU \( (\mathcal{I}, b) \) as conformal infinity? This correspond to the class of **asymptotically AdS** spacetimes. More precisely, by employing a definition similar to the ones given in \[\text{AsMa}\] and \[\text{AshD}\], one can write:

**Definition 2.3** A \( n \)-dimensional spacetime \( (n \geq 3) \) \( (\hat{\mathcal{M}}, \hat{g}) \) with conformal infinity \( (\mathcal{I}, b) \) is said to be **asymptotically anti-de Sitter** (notation: AAdS) if:

1. It satisfies Einstein’s equations \( \hat{R}_{ab} - \frac{1}{2} \hat{R} \hat{g}_{ab} - \Lambda \hat{g}_{ab} = 8\pi G_{(n)} \hat{T}_{ab} \), where \( G_{(n)} \) is the \( n \)-dimensional Newton’s constant, and the cosmological constant \( \Lambda < 0 \) (one can attribute an “AdS radius” to such spacetimes, by setting \( A = \sqrt{-\frac{(n-1)(n-2)}{\Lambda}} \));

2. \( (\mathcal{I}, b) \) is globally conformally diffeomorphic to the \( (n-1) \)-dimensional Einstein static universe;

3. The (classical) energy-momentum tensor \( \hat{T}_{ab} \) of \( (\hat{\mathcal{M}}, \hat{g}) \) decays fast enough close to \( \mathcal{I} \) for \( \Omega^{2-n} \hat{T}^a_b \) to possess a \( \mathcal{C}^\infty \) extension to the conformal closure \( (\mathcal{M}, g) \).

The condition on the decay of \( \hat{T}_{ab} \) is motivated by considering the behaviour of classical fields emanating from compactly localized sources in \( AdS_n \), specially massless fields (electromagnetic, Yang-Mills). The global condition on the conformal infinity makes sense in general because solutions of Einstein’s equations with negative cosmological constant possess a timelike conformal infinity.

In what follows, we shall make two additional demands on the class of AAdS spacetimes we’ll deal with:

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\[2\text{This coordinate-free form of the Rehren bijection, which solely makes use of causal relationships in the conformal closure, is based in a suggestion from prof. K.-H. Rehren [Reh2], and was employed in this form by Bousso and Randall [BoRa] for studying qualitative aspects of the AdS-CFT correspondence.}\]
Asymptotic simplicity. This is indispensable for rebuilding bulk localization from wedges. The existence of a large class of asymptotically simple AAdS spacetimes was proven by Friedrich [Fri1, Fri3].

Global focusing of null geodesics. More precisely, it’s demanded that all inextendible null geodesics shall possess a pair of conjugate points (recall that a pair of points p, q in a null geodesic γ are said to be conjugate if there’s a Jacobi field on γ – i.e., a vector field that satisfies the geodesic deviation equation on each point of γ – nowhere vanishing on the open segment of γ linking p to q but vanishing at both p and q). It’s well known [BeEE, Wald, HawE] that, in this case, any point of γ to the future (resp. past) of q (resp. p) can be linked to p (resp. q) by a timelike curve. It’s precisely this condition that guarantees the Rehren bijection will preserve causality, and it also ends up playing an important role in the reconstruction of bulk localization. Even if one does not require asymptotic simplicity, one can still show that any chronological spacetime which satisfies this focusing condition is strongly causal [BeEE], and therefore its topology is generated by diamonds. AdS does not satisfy this condition, but it follows from energy conditions on the energy-momentum tensor as weak as the NEC, ANEC and the Borde energy condition [Bord], and does look natural from the viewpoint of certain rigidity theorems for asymptotically simple spacetimes: for asymptotically flat and de Sitter spacetimes which satisfy, say, NEC, it was proven by Galloway [Gall, Gal2], by employing the stability results of Friedrich [Fri1, Fri2], that if such spacetimes possess a so-called null line (a complete, achronal null geodesic), then they are globally isometric to Minkowski spacetime (resp. de Sitter spacetime, with radius determined by the value of the cosmological constant appearing in the Einstein equations). From the viewpoint of stability of the (conformal) [Fri1] mixed Cauchy/boundary problem, one can see that the occurrence of null lines is an unstable feature of such spacetimes, i.e., any arbitrarily small perturbation of Cauchy data that preserves boundary conditions at the conformal infinity destroys all null lines, i.e., all complete null geodesics acquire a pair of conjugate points. There is still no similar result for asymptotically simple AAdS spacetimes in the sense of Definition 2.3, yet the the global structure of conformal infinity suggests that this may still be true. If so, our analysis complements and extends Rehren’s.

We shall now study how this framework behaves in the more general situation of AAdS spacetimes complying with the conditions above.

4If one wants to extend our considerations to semiclassical AAdS spacetimes, i.e., with quantum backreaction, we remark that quantum energy inequalities seem to be capable of guaranteeing that a “Planck-scale coarse-grained (i.e., transversally smeared)” ANEC holds for the renormalized quantum energy-momentum tensor [Perd], but whether this implies, say, the Borde energy condition, and thus gives rise to the needed focusing theorems [Bord], or not, it’s an open question so far.
3 Properties of the Rehren bijection in AAdS spacetimes

3.1 Causality (bulk-to-boundary)

In principle, gravitational effects deep inside the bulk may produce causal shortcuts through the bulk linking causally disjoint point at the boundary, i.e., it may happen that $I^+(p, \mathcal{I}) \cap I^-(q, \mathcal{I}) \subset (I^+(p, \mathcal{M}) \cap I^-(q, \mathcal{M})) \cap \mathcal{I}$, rendering the second identity in (9) false. Such a thing would be ruinous to the Rehren bijection to keep preserving causality in AAdS spacetimes. We shall show now that, luckily, (9) still holds under our set of hypotheses. Here, we’ll make use of the notion of gravitational time delay [GaWa, PeSW, Wool] of complete null geodesics in AAdS spacetimes. The Einstein static universe $(\mathcal{I}, b)$ is globally hyperbolic; let it be given a foliation of it in Cauchy surfaces such that the orbits of the global time function $t$ (supposed to be oriented in the same way as the time orientation of $(\mathcal{I}, b)$) generating the foliation are complete timelike geodesics, and the values of the global time function correspond to a common affine parametrization of this family of geodesics. Now, let $\gamma$ be a complete null geodesic in $\mathcal{M}$ traversing $\mathcal{M}$, with past endpoint $p$ and future endpoint $q$ belonging to the orbits $T_p$ (resp. $T_q$), and $\gamma'$ a null geodesic segment in $\mathcal{I}$ starting at $p$ and ending at, say, $q' \in T_q$. The gravitational time delay of $\gamma$ with respect to $\mathcal{I}$ is given by the difference $\Delta t = t(q) - t(q')$ (notice that, due to the properties of null geodesics in ESU, it follows that any other null geodesic segment in $\mathcal{I}$ starting at $p$ that crosses $T_q$ afterwards will necessarily do it at $q'$). Although this value depends on the choice of foliation, the sign of $\Delta t$ ($< 0, = 0, > 0$) does not. Under our set of hypotheses, the gravitational time delay in AAdS spacetimes is always positive:

**Theorem 3.1** Let $(\mathcal{M}, \hat{g})$ be an asymptotically simple AAdS spacetime, such that every inextendible null geodesic has a pair of conjugate points, and $p \in \mathcal{I}$. Then, every null geodesic segment emanating from $p$ which doesn’t belong to $\mathcal{I}$ has its future endpoint in $I^+(p, \mathcal{I})$.

**Proof.** Let $\gamma$ be a null geodesic segment emanating from $p$ and traversing the bulk, and let $p'$ be the future endpoint of $\gamma$ in $\mathcal{I}$. Since we’ve assumed that $(\mathcal{M}, \hat{g})$ is causal, one can see that $p' \notin I^-(p, \mathcal{M})$. Now, we prove two Lemmata:

**Lemma 3.2** (Absence of causal shortcuts) Let $p, p' \in \mathcal{I}$. If $p \perp_{\mathcal{I}} p'$, then there is no causal curve in $(\mathcal{M}, g)$ linking $p$ to $p'$.

4The method of proof was communicated to me by Sumati Surya [Sury]. It’s analogous to the proof of a positive mass theorem for asymptotically flat spacetimes due to Penrose, Sorkin and Woolgar [PeSW] and for AAdS spacetimes due to Woolgar [Wool]. Another proof of this, using a somewhat different strategy, was provided by Gao and Wald [GaWa].
**Proof.** Suppose that $p' >_{\mathcal{M}} p$ (the opposite case is treated analogously). We’ll prove that the gravitational time delay implied by the presence of a pair of conjugate points contradicts the causal disjointness of $p$ and $p'$ with respect to $\mathcal{I}$. Denote by $T(p')$ the timelike generator of $(\mathcal{I}, b)$ containing $p'$.

Note that $\partial I^+(p, \mathcal{I}) \cong \Sigma$ is a closed, achronal surface that cuts $(\mathcal{I}, b)$, in two disjoint subsets $I^+(p, \mathcal{I}) \cong A$ and $\mathcal{I} \setminus T^+(p, \mathcal{I}) \cong B$ and intersects each timelike generator of $\mathcal{I}$ in precisely one point, as every timelike generator has points in $I^+(p, \mathcal{I})$ and $I^-(\bar{p}, \mathcal{I})$. By hypothesis, $p' \in B$. Moreover, $T(p')$ must cross $\Sigma$ at some instant of time. Therefore, there exists $p'' \in T(p')$ such that $p'' \not\in \mathcal{I}$ and $p'' \in \Sigma$. Let $\gamma$ be a null generator of $\Sigma$ that contains $p''$. As the segment of $\gamma$ that links $p$ to $p''$ is null and achronal, $\gamma$ is necessarily the fastest curve in $(\mathcal{I}, b)$ linking $p$ to $T(p')$.

Now, consider the achronal boundary $\partial I^+(p, \mathcal{M}) = \partial I^+(p, \mathcal{M}) \cong \Sigma$, $\Sigma \cap \mathcal{I}$ is closed, achronal and intersects each timelike generator of $\mathcal{I}$ in precisely one point, as every timelike generator has points in $I^+(p, \mathcal{I})$ and $I^-(\bar{p}, \mathcal{I})$, and $\Sigma$ separates $\mathcal{I}$ in two disjoint open sets (from the viewpoint of a manifold-with-boundary, of course) $I^+(p, \mathcal{I}) \cong A$ and $\mathcal{I} \setminus T^+(p, \mathcal{I}) \cong B$. Thus, $T(p')$ must cross $\Sigma$ in, say, $p'''$. Since $p <_{\mathcal{M}} p'$, we must have $p''' \not\in \mathcal{I}$ or $p''' = p'$. In both cases, we have $p''' \not\in \mathcal{I}$, which implies that any null generator $\gamma$ of $\Sigma$ containing $p'''$ is strictly faster than $\gamma$. As $\gamma$ was the fastest curve in $(\mathcal{I}, q\mathcal{M})$ linking $p$ to $T(p')$, $\gamma$ necessarily traverses $\Sigma$. Hence, we’ve built a complete and achronal null geodesic in $(\mathcal{M}, \bar{g})$. However, such a geodesic cannot exist since it must have a pair of conjugate points and therefore cannot be achronal. 

**Lemma 3.3** Let $p, p' \in \mathcal{I}$. If $p' \in \partial I^+(p, \mathcal{I})$ and $p' \not\in \bar{p}$, then there is no null geodesic segment in $(\mathcal{M}, g)$ that doesn’t belong to $\mathcal{I}$ and links $p$ to $p'$.

**Proof.** Let $\gamma$ be the (necessarily unique) null generator of $\partial I^+(p, \mathcal{I})$ linking $p$ to $p'$. Suppose that there is another null geodesic, traversing $\mathcal{M}$ and linking $p$ to $p'$. Then, if one picks any point $p''$ in $\gamma$ after $p'$, there is a broken null geodesic segment linking $p$ to $p''$, which in turn implies that there is a timelike curve in $(\mathcal{M}, g)$ linking $p$ to $p''$. Let $T(p'')$ be the timelike generator of $(\mathcal{I}, b)$ containing $p''$. Now, consider $\Sigma$ as in the preceding Lemma. Again, $T(p'')$ must cross $\Sigma$, say, in $p'''$. Thus, necessarily $p''' \not\in \mathcal{I}$ or $p''' = p'$. But this implies that, since $(\mathcal{M}, g)$ is causal, $p \perp \mathcal{I}$ $p'$, which contradicts the preceding Lemma.
Both Lemmata above imply that, if \( p' \) isn’t dragged inside \( I^+(p, \mathcal{I}) \) by gravitational time delay, then \( p' \) coincides with \( \overline{p} \), just like in AdS. But, even in such a case, the presence of conjugate points in \( \gamma \) implies that there is a timelike curve traversing the bulk and linking \( p \) to \( \overline{p} \). Repeating the argument in Lemma 3.3, the result follows. □

All arguments above can be repeated exchanging future with past, and \( p \) with its antipodal \( \overline{p} \), yielding a similar result in the opposite time orientation.

**Remark 3.1** Our set of hypotheses, however, entails that, although the Rehren bijection preserves causality, it no longer does so in a maximal way – the collection of wedges is no longer closed under causal complements. More precisely:

**Proposition 3.4** Let \( (\hat{\mathcal{M}}, \hat{g}) \) be an AAdS spacetime satisfying the hypotheses of Theorem 3.1. Define \( \Xi^+_p = \partial I^+(p, \mathcal{M}) \setminus \{p, \overline{p}\} \) and \( \Xi^-_p = \partial I^-(\overline{p}, \mathcal{M}) \setminus \{p, \overline{p}\} \). Then:

(i) \( \Xi^+_p \cap I^- (\overline{p}, \mathcal{M}) = \Xi^-_p \cap I^+(p, \mathcal{M}) = \emptyset \).

(ii) \( \Xi^+_p \cap \Xi^-_p \cap \hat{\mathcal{M}} = \emptyset \).

**Proof.**

(i) We know that \( \Xi^+_p \cap \mathcal{I} = \Xi^-_p \cap \mathcal{I} \), so let’s concentrate only at the bulk. Namely, suppose that there is \( q \in \hat{\mathcal{M}} \) such that \( q \in \Xi^-_p \) and \( q \notin \Xi^+_p \). If \( q \gg p \), that contradicts the fact that there is no timelike curve linking \( p \) to \( \overline{p} \). Repeat the argument exchanging past with future, and the roles of \( p \) and \( \overline{p} \). (ii) If \( \Xi^+_p \) and \( \Xi^-_p \) coincide in some point \( q \in \hat{\mathcal{M}} \), then there is an at least broken null geodesic segment linking \( p \) to \( \overline{p} \), which implies that there is a timelike curve through the bulk doing the same. The result easily follows by repeating the reasoning at the end of the proof of Theorem 3.1. □

Note that the argument above is symmetric with respect to time orientation.

An important consequence of Proposition 3.4 is that our global focusing condition entails a nontrivial “shrinking” of the AAdS wedges towards the boundary, as a side effect of the gravitational time delay. As a consequence of the latter, it follows that, although by Theorem 3.1 \( \overline{p} \) still satisfies (10), it will certainly violate (11). Moreover, \( \mathcal{W}_{q, \overline{p}} \cap \hat{\mathcal{M}} \subseteq (\mathcal{W}_{p, \overline{q}})' \) and \( \mathcal{P}oi(r) \subseteq \{r\}' \cap \hat{\mathcal{M}} \). Hence, property 3 for wedges no longer remains valid, and therefore the collection of bulk wedges cannot be closed by causal complements in the sense of (12). The strict inclusions above suggest that Haag duality for boundary diamonds cannot be satisfied without violating bulk causality for local observables, as the local (von Neumann, for concreteness) algebra \( \mathfrak{A}(\mathcal{W}(p, \overline{q})') \) may be strictly larger than \( \mathfrak{A}(\mathcal{W}(q, \overline{p})) = \mathfrak{A}(\mathcal{D}(q, \overline{p})) \) (here the Borchers timelike tube theorem doesn’t
remove the strictness of the inclusion, as here one would need to extend the localization beyond the set of points causally between \( p \) and \( \bar{q} \). Hence, in such a case, if \( \mathfrak{A}(\mathcal{D}(p, \bar{q}))' = \mathfrak{A}(\mathcal{D}(q, \bar{p})) = \mathfrak{A}(\mathcal{D}(p, \bar{q}))' \), then the algebra \( \mathfrak{A}(\mathcal{W}(p, \bar{q}))' \) necessarily has elements that do not commute with \( \mathfrak{A}(\mathcal{W}(p, \bar{q})) = \mathfrak{A}(\mathcal{D}(p, \bar{q})) \), therefore violating local causality. That means that the local algebras also “shrink” from the viewpoint of the boundary – in this way, the boundary net can “feel” bulk gravitational effects as a spontaneous symmetry breaking that necessarily follows from the breakdown of Haag duality [Haag] – more on this at the end of this contribution.

3.2 Reconstruction (boundary-to-bulk)

Knowing the wedge localization of bulk observables may not be enough for the full reconstruction of the bulk quantum theory using only boundary data and the Rehren bijection \( \rho \). We need to be able to specify the localization of the local procedures in arbitrarily small open regions, or, which amounts to the same thing, the localization with respect to a basis of the bulk topology. This can, in principle, be performed by taking intersections of wedges, but it’s by no means clear whether these give a basis for the bulk topology or not. This must be done in a more precise way. In AdS, we’re fortunate, because any relatively compact AdS diamond (for the purpose of generating the manifold topology, these suffice) can be enveloped by AdS wedges: given

\[
\mathcal{O}_{p,q} = I^+(p, \text{AdS}_n) \cap I^-(q, \text{AdS}_n), \quad p \ll \text{AdS}_n, q,
\]

we can write

\[
\mathcal{O}_{p,q} = \bigcap_{r \in \partial I^-(p, \text{AdS}_n) \cap \mathcal{I}, s \in \partial I^+(q, \text{AdS}_n), r,s \in \mathcal{M}_{\text{in}(u)}} \mathcal{W}_{r,s}.
\]

We’ll see shortly that the achronality of the inextendible null geodesics in AdS is crucial for the precise enveloping. In an AAdS spacetime as in Theorem 3.1 the issue is much more delicate, because of the following

**Proposition 3.5**

Let \((\hat{\mathcal{M}}, \hat{g})\) be an AAdS spacetime satisfying the hypotheses of Theorem 3.1 and \( p \in \hat{\mathcal{M}} \). Then, \( \partial I^+(p, \hat{\mathcal{M}}) \) intersect each timelike generator if \((\mathcal{I}, b)\) precisely once.

**Proof.** By the achronality of \( \partial I^+(p, \hat{\mathcal{M}}) \), it intersects every timelike generator of \((\mathcal{I}, b)\) at most once. Suppose that the thesis is false. Then, given a timelike generator \( T \), we have the following possibilities:

(i) \( T \subset I^+(p, \hat{\mathcal{M}}) \) – Consider a complete null geodesic \( \gamma \) crossing \( p \), and let \( q \) be the past endpoint of \( \gamma \). Then, there exists a value \( t \) of the affine parameter of \( T \) such that \( T(t) \ll \mathcal{I}, q \). Therefore, \( T(t) \ll \mathcal{M}, p \), which is absurd since \((\mathcal{M}, g)\) is chronological.
Consider a complete null geodesic $\gamma$ crossing $p$, and let $r$ be the future endpoint of $\gamma$. Then, there exists a value $t$ of the affine parameter of $T$ such that $T(t) \gg r$. Therefore, $T(t) \gg p$, contradicting the hypothesis.

□

**Proposition 3.6** Let $(\tilde{M}, \tilde{g})$ be an AAdS spacetime which satisfies the hypotheses of Theorem 3.1, $q, r \in \tilde{M}$ such that $r \in \partial I^-(q, \tilde{M})$, and $\gamma$ a null generator of $\partial I^-(q, \tilde{M})$ to which $r$ belongs. Let $s_1(r), s_2(r), s_3(r) \in I$ be defined as:

- $s_1(r)$ is the future endpoint of $\gamma$;
- $s_2(r)$ is the point where $\partial I^+(q, \tilde{M})$ intersects the timelike generator $T(s_1(r))$ to which $s_1(r)$ belongs;
- $s_3(r)$ is the point where $\partial I^+(r, \tilde{M})$ intersects $T(s_1(r))$.

Then:

1. $s_3(r) \leq s_2(r) \leq s_1(r)$.
2. $s_3(r) = s_2(r) = s_1(r)$ if and only if the segment of $\gamma$ linking $r$ to $s_1(r)$ is achronal.

**Proof.** (i) Immediate, as is (ii) $\Rightarrow$. It remains to prove (ii) $\Leftarrow$. Namely, suppose that $s_3(r)$ equals $s_2(r)$. Then, the null geodesic segment linking $q$ to $s_2(r)$ must belong to $\gamma$, for otherwise there would be a broken geodesic segment linking $r$ and $s_3(r)$, contradicting the definition of the latter (this, in particular, proves that $s_1(r) = s_3(r)$ even if we just assume $s_2(r) = s_3(r)$). If $\gamma$ is not achronal, once more we have a contradiction with the definition of $s_3(r)$.

□

Similar results are valid if we exchange future with past. Now, in an AAdS spacetime satisfying the hypotheses of Theorem 3.1 let $O_{p,q}$ be a relatively compact diamond with a contractible Cauchy surface – any sufficiently small diamond satisfies both conditions. Let’s now consider the region

\[ Q_{p,q} = \bigcap_{r \in \partial I^-(p, \tilde{M}) \cap I, s \in \partial I^+(q, \tilde{M}) \cap T(r)} \mathcal{W}_{r,s} \]

It follows naturally from the definition that $Q_{p,q} \supset O_{p,q}$, it is causally complete, as it is the intersection of causally complete regions, and

\[ Q_{p,q} \cap J^+(q, \tilde{M}) = Q_{p,q} \cap J^-(p, \tilde{M}) = \emptyset. \]
In AdS, $\mathcal{O}_{p,q} = \mathcal{O}_{p,q}^{\text{Reh1}}$. For AAdS spacetimes as in Theorem 3.1, however, it may happen that $\mathcal{O}_{p,q} \supsetneq \mathcal{O}_{p,q}$. Likewise, defining $E_{p,q} = \partial I^+ (p, \mathcal{M}) \cap \partial I^- (q, \mathcal{M})$, let’s start from

$$\tilde{\mathcal{D}}_{p,q} = \bigcap_{r \in E_{p,q}} W_{s_2(r), s_3(r)}, \tag{19}$$

where $s_2'(r)$ corresponds to $s_3(r)$ if we exchange future with past in the statement of Proposition 3.6. Here, $\tilde{\mathcal{D}}_{p,q} \subset \mathcal{D}_{p,q}$ is again causally complete, if nonvoid. However, if the spacetime metric deep inside the bulk is sufficiently “distorted”, and causing a sufficient number of null generators of, say, $\partial I^- (q, \mathcal{M})$ to acquire a pair of conjugate points between $r \in E_{p,q}$ and $\mathcal{I}$, for all we know (Proposition 3.6) $\tilde{\mathcal{D}}_{p,q}$ could very well be empty (the intersection of the corresponding algebras may even be nonvoid, but then we won’t be able to attribute any localization whatsoever to this algebra). This is suggested by the following remarks:

1. In a causally simple spacetime, any relatively compact diamond $\mathcal{O}_{p,q}$ is a globally hyperbolic region, for which any Cauchy surface has a boundary equal to $\mathcal{E}_{p,q}$;
2. Any causally complete region $\mathcal{U}$ has the following property: if $\mathcal{I} \subset \mathcal{U}$ is a closed, achronal set with respect to $\mathcal{U}$, then $D(\mathcal{I}) \subset \mathcal{U}$.

Both remarks together show that, if $r \in \mathcal{E}_{p,q}$ is such that a null generator of, say, $\partial I^- (q, \mathcal{M})$ crossing $r$ acquires a pair of conjugate points between $r$ and $s_1(r)$, then, by causal simplicity, it follows that there’s a neighbourhood of $q$ that is causally disjoint from $s_3(r)$, and, therefore, $\mathcal{I}^- (s_3(r), \mathcal{M})$ cannot contain a Cauchy surface for $\mathcal{O}_{p,q}$. Since, on the other hand, this doesn’t exclude the possibility that $\tilde{\mathcal{D}}_{p,q}$ may contain points outside $\mathcal{E}_{p,q}$ either, it’s by no means clear whether the collections of $\mathcal{D}_{p,q}$ and $\tilde{\mathcal{D}}_{p,q}$ give bases for the topology of $\mathcal{M}$ or not.

One way to circumvent these problems could be to restrict ourselves to sufficiently small diamonds, such that none of the null generators of $\partial I^+ (q, \mathcal{M})$ can travel far enough beyond $q$ in order to develop a pair of conjugate points. But there is a situation such that, no matter how small the extension, it will always cease to be achronal: it’s when $q$ itself is conjugate to $s_1(r)$. In this limiting case, $s_1(r) = s_2(r)$ but $s_2(r) \neq s_3(r)$.

We’ll show now that the key out of these problems is to try to build a region similar to $\tilde{\mathcal{D}}_{p,q}$, but employing, instead of the points $s_3(r)$, $s_2'(r)$ for $r \in \mathcal{E}_{p,q}$, the points for which the problem, entailed by Proposition 3.6 and mentioned above, is, in a certain sense, “minimized”. To perform this task, we’ll start from a different viewpoint, which will also eventually show that the critical situation in the preceding paragraph is ruled out by null geodesic completeness. First, notice that, using an argument similar to the one used in [Haw1] and
to prove the existence of a (Lipschitz) topological manifold structure for achronal boundaries, one can show that \(E_{p,q}\) is locally the graph of a \(\mathbb{R}\)-valued, locally Lipschitz function of \(n - 2\) real arguments, and therefore it’s a compact, acausal, embedded (Lipschitz) topological submanifold of \(\mathcal{M}\), with codimension 2. Notice as well that one can smoothly parametrize the family of timelike generators of \((\mathcal{I}, b)\) by a latter’s Cauchy surface \(\mathcal{S}\) which is homeomorphic to \(S^{n-2}\) and thus also compact. Let \(t\) be the common affine parametrization of the timelike generators of \((\mathcal{I}, b)\) mentioned in the previous Subsection. Define the function

\[
\tau : \mathcal{E}_{p,q} \times \mathcal{I} \ni (r, \theta) \mapsto \tau(r, \theta) \in \mathbb{R},
\]

where

\[
\partial I^+(R, \mathcal{M}) \cap T(\theta) = \{ T(\theta)(\tau(r, \theta)) \}. \tag{21}
\]

Proposition \ref{prop:futureFermatPotential} shows that the definition of \(\tau\) is not empty. Moreover:

**Proposition 3.7** \(\tau\) is upper semicontinuous in \(r\) for fixed \(\theta\).

**Proof.** Let \(\epsilon > 0\). \(r\) lies in the chronological past of the point \(T(\theta)(\tau(r, \theta) + \epsilon)\), and thus there’s an open neighborhood \(U\) of \(r\) in \(\mathcal{E}_{p,q}\) which lies in the chronological past of \(T(\theta)(\tau(r, \theta) + \epsilon)\). Therefore, for all \(r' \in U\), we must have \(\tau(r', \theta) < \tau(r, \theta) + \epsilon\). \(\Box\)

One can actually prove that \(\tau\) is Lipschitz continuous in \(\theta\) for fixed \(r\), but this won’t be used in the sequel. The function \(\tau(., \theta)\) will be called *future Fermat potential* with respect to \(\theta\). The name is remnant of the Huygens-Fermat principle of geometrical optics (see, for instance, pages 249-250 of [Arno]). Now, extend the definition of \(\tau(., \theta)\) to the closure \(\mathcal{F}_{p,q}\) of some Cauchy surface \(\mathcal{F}_{p,q}\) for \(\mathcal{E}_{p,q}\), denoting it by the same symbol, since no confusion arises here. By the same argument employed in Proposition \ref{prop:futureFermatPotential}, \(\tau(., \theta)\) is upper semicontinuous in \(\mathcal{F}_{p,q}\). Since both \(\mathcal{E}_{p,q} = \partial \mathcal{F}_{p,q}\) and \(\mathcal{F}_{p,q}\) are closed subsets of the compact set \(\mathcal{E}_{p,q}\), they are compact themselves. By a standard result of analysis (see, for instance, pages 110-111 of [KoFo]), \(\tau(., \theta)\) has a maximum value both in \(\mathcal{F}_{p,q}\) and \(\mathcal{E}_{p,q}\). The next theorem shows that \(\tau(., \theta)\) has indeed a distinguishing property of potentials:

**Theorem 3.8** (Maximum principle for the future Fermat potential)

The maximum value of \(\tau(., \theta)\) in \(\mathcal{F}_{p,q}\) is achieved at \(\mathcal{E}_{p,q}\).

**Proof.** Let \(r\) be a point of \(\mathcal{E}_{p,q}\) where \(\tau(., \theta)\) achieves its maximum in \(\mathcal{E}_{p,q}\), and let \(r'\) be a point of \(\mathcal{F}_{p,q}\) such that \(\tau(r', \theta) \geq \tau(r, \theta)\). In such a case, it’s obvious that \(\mathcal{E}_{p,q}\) lies in the causal past of \(T(\theta)(\tau(r', \theta))\). Pick a curve segment in \(\mathcal{F}_{p,q}\) starting at \(r'\), initially pointing outside \(J^-(T(\theta)(\tau(r', \theta)), \mathcal{M})\) and ending in some point of \(\mathcal{E}_{p,q}\). Then, any such a curve segment must cross \(\partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M})\) at least once more after \(r'\), and before or at \(\mathcal{E}_{p,q}\). This shows that \(\partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M}) \cap \mathcal{F}_{p,q}\)
encloses an open subset $X$ of $\mathcal{F}_{p,q}$ outside the causal past of $T(\theta)(\tau(r', \theta))$.

The remaining of the proof is analogous to the proof of Penrose’s singularity theorem [HawE, Wald]: namely, we’ll show that the properties of $\partial X$ imply that there must exist an incomplete null geodesic in $(\widetilde{\mathcal{M}}, \widetilde{g}_{\alpha\beta})$. First, we’ll show that the closed, acausal set $\partial X = \partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M}) \cap \mathcal{F}_{p,q}$ is past trapped, i.e., $\partial I^-(X, \mathcal{M})$ is compact. The past “ingoing” null geodesics of $\partial X$ constitute the past Cauchy horizon of $X$, which is thus contained in $\mathcal{F}_{p,q}$ and therefore compact, as it’s closed [HawE, Wald]. The past “outgoing” null geodesics are precisely the null generators of $\partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M})$ that cross $\partial X$. Given a common affine parametrization to the null generators of $\partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M})$ such that the zero value of the affine parameter corresponds to $\partial X$. Then, let $t_0$ the largest value of affine parameter for which a past endpoint of $\partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M})$ is achieved. It must be finite, for each inextendible null geodesic must acquire a pair of conjugate points before reaching infinity, although the value of the affine parameter at a past endpoint of the null generator segment starting at, say, $r'' \in \partial X$ can be zero if $r''$ happens to be itself a past endpoint. Anyway, the portion of $\partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M})$ in the causal past of $\partial X$, being closed, has a closed inverse image in the compact set $[0, t_0] \times \partial X$ under the chosen parametrization of the null generators, and is therefore compact. Hence, the set $\partial I^-(\partial X, \mathcal{M}) = H^-(X) \cup \partial X \cup (\partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M}) \cap J^-(\partial X, \mathcal{M}))$ is a compact, achronal subset of $\mathcal{M}$, as asserted.

However, any causally simple spacetime is stably causal [BEE]. That is, one can smoothly foliate $\mathcal{M}$ by “constant-time”, spacelike surfaces of codimension 1. By the structure of the conformal infinity, such surfaces (leaves) cannot be compact. Moreover, each timelike orbit of the foliation crosses an achronal set at most once. By following these orbits, one can continuously map $\partial I^-(\partial X, \mathcal{M})$ into a spacelike leaf of this foliation. As the image of this map is compact, it must have a nonvoid boundary. But it’s known that a set of the form $\partial I^-(Y, \mathcal{M}), Y \subset \mathcal{M}$ is a topological submanifold without boundary of $\mathcal{M}$, and, as such, it cannot have a boundary. This shows that some null generator of $\partial I^-(T(\theta)(\tau(r', \theta)), \mathcal{M})$ must terminate at a singularity before reaching its past endpoint. But this conflicts with the null geodesic completeness of $\mathcal{M}$, entailed by asymptotic simplicity. Hence, no point in $\mathcal{F}_{p,q}$ can achieve a maximum for $\tau(., \theta)$ in $\mathcal{F}_{p,q} -$ this maximum always takes place at $\mathcal{E}_{p,q}$. 

Propositions $\S3.7$ and Theorem $\S3.8$ together show that, for each $\theta$, there will always be a $r \in \mathcal{E}_{p,q}$ such that, given any Cauchy surface $\mathcal{F}_{p,q}$ for $\mathcal{O}_{p,q}$, the set $\mathcal{F}_{p,q}$ will always lie in the causal past of $T(\theta)(\tau(r, \theta))$. By Proposition $\S3.6$ and the remarks above, this can only happen if the achronal null geodesic segment $\gamma(r, \theta)$ linking $r$ to $T(\theta)(\tau(r, \theta))$ crosses $q$. Thus, this maximum point is unique: suppose otherwise. Then, there would be another $r' \in \mathcal{E}_{p,q}$ such that there is an achronal null geodesic segment $\gamma(r', \theta)$ linking
r’ to $T(\theta)(\tau(r', \theta)) = T(\theta)(\tau(r, \theta))$ and crossing $q$. Now consider the curve segment $\gamma'(r, \theta)$ which coincides with $\gamma(r, \theta)$ from $r$ to $q$, and coincides with $\gamma(r', \theta)$ from $q$ to $T(\theta)(\tau(r, \theta))$. This segment is necessarily broken, which conflicts with the achronality of $\gamma(r, \theta)$. Exchanging the roles of $r$ and $r'$, one sees that this argument also conflicts with the achronality of $\gamma'(r', \theta)$. Notice, however, that an arbitrary $r \in \mathcal{E}_{p,q}$ need not maximize $\tau(., \theta)$ for some $\theta$. Two instances where this cannot occur are:

1. $r$ is conjugate to $q$ along a null generator of $\partial I^-(q,.\mathcal{M})$ – any future extension of this generator beyond $q$ won’t be achronal;

2. $q$ is conjugate to $s_2(r)$ along a null generator of $\partial I^+(q,.\mathcal{M})$, by the remarks made above.

The second instance, however, is excluded by our line of reasoning, because it renders impossible, by Proposition 4.4 and Theorem 4.5, to achieve a maximum value in $\mathcal{E}_{p,q}$. This cannot happen, since for every $\theta$ a maximum must exist by Proposition 3.7. The first instance can be circumvented by picking $\mathcal{O}_{p,q}$ contained, say, in a convex normal neighbourhood, which can always be done, as here $(.\mathcal{M}, \tilde{g})$ is strongly causal. One can go further and take $\mathcal{O}_{p,q}$ sufficiently small (yet nonvoid) so that every $r \in \mathcal{E}_{p,q}$ is a maximum point of $\tau(., \theta)$ for some $\theta$, as the only obstacle to this would be the second instance above, which is excluded by the above argument. All results above have a past counterpart, by exchanging $q$ with $p$ and reversing the time orientation.

Summing up, we have showed that sufficiently small $\mathcal{O}_{p,q}$ can always be precisely enveloped by wedges, by means of the prescription (16). In such a case, any point not belonging to $\overline{\mathcal{O}_{p,q}}$ lies either in the chronological future of $\partial I^-(q,.\mathcal{M})$ or in the chronological past of $\partial I^+(p,.\mathcal{M})$, and, as such, will fail to belong to some wedge enveloping $\mathcal{O}_{p,q}$. Since the points at $\partial \mathcal{O}_{p,q}$ are already excluded from the intersection by construction, one concludes that $\mathcal{O}_{p,q} = \mathcal{D}_{p,q}$ for sufficiently small $\mathcal{O}_{p,q}$. Moreover, in such a situation, each wedge in the definition (17) of $\mathcal{D}_{p,q}$ is guaranteed to be contained in some Poincaré domain. Therefore, one can even restrict to a Poincaré domain and perform the bulk reconstruction there starting from a boundary CFT in Minkowski spacetime.

4 Perspectives and open problems

For additive local quantum theories, it suffices to specify the localization of the procedures for a basis of the manifold topology. Therefore, the results in the previous Section indicate that one can completely recover the bulk quantum theory by just employing localization data from the boundary quantum theory and the Rehren bijection, and this theory is guaranteed to be causal if its holographic dual is. In situations where the boundary theory is additive, then all compactly localized bulk observables are necessarily multiples of the
identity [Reh]. In such a case, one suffices to have just wedge localization in the bulk.

The covariance issue is obviously more complicated than in the AdS case. For a proper implementation of conformal covariance in the boundary theory, two diffeomorphisms which are “asymptotic isometries” [AsMa] which differ only by a diffeomorphism which is an “asymptotic identity” (i.e., acting as the identity on the boundary) should differ, from the viewpoint of the boundary theory, only by an internal (non-geometric) symmetry. The lack of bulk isometry groups cries out for a locally covariant formalism for local quantum physics, such as the one developed in [LTFV]. Algebraic holography then maps the realization of a locally covariant quantum theory in the bulk to a globally conformally covariant quantum theory at the boundary, where the latter has, in principle, an enormous amount of internal symmetry. For the conformal group to be unitarily implementable in some GNS representation, these internal symmetries should not be generating a non-trivial cohomological obstruction. If the state associated to the GNS representation satisfies the Reeh-Schlieder property, it follows from Proposition 3.4 the discussion following it, and the work of Brunetti, Guido and Longo [BrGL] that, due to the breakdown of Haag duality, the Tomita-Takesaki modular groups associated with the diamond von Neumann algebras cannot unitarily implement the isotropy groups of the respective diamonds. From this, it follows that either (or both) (1.) The unitary representation of the conformal group cannot be of positive energy, or (2.) The conformal group is spontaneously broken. Both scenarios are of the greatest interest for further study, as well as the possibility that such a spontaneous breaking has a cohomological structure stemming from the nontrivial asymptotic identities, and possible connections with the phenomenon of holographic Weyl anomalies [HeSk]. This may even reveal an holographic encoding of bulk gravitational degrees of freedom into the modular structure of the boundary theory.

All the reasoning in Subsection 3.2 applies equally well if one wants to rebuild the bulk localization using (sufficiently small) bulk regular diamonds [GLRY, Ruzz] instead of ordinary ones. This makes it also a good starting point for studying how the superselection sector structure is holographically mapped between both theories. This problem will be attacked in forthcoming work.

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References


