Topological gravity on plumbed V-cobordisms

Vladimir N. Efremov*†
Nikolai V. Mitskievich‡§
Alfonso M. Hernández Magdaleno‡
and
Ramona Serrano Bautista*

Abstract

An ensemble of cosmological models based on generalized $BF$-theory is constructed where the rôle of vacuum (zero-level) coupling constants is played by topologically invariant rational intersection forms (cosmological-constant matrices) of 4-dimensional plumbed V-cobordisms which are interpreted as Euclidean spacetime regions. For these regions describing topology changes, the rational and integer intersection matrices are calculated. A relation is found between the hierarchy of certain elements of these matrices and the hierarchy of coupling constants of the universal (low-energy) interactions.

PACS numbers: 0420G, 0240, 0460
1 Introduction

The principal motivation for the present paper is to relate some results in low dimensional topology to \( BF \)-theory \([1, 2]\) also known as topological gravity \([3]\). Properly speaking, we are trying to realize an analogue of the classical Abelian \( BF \)-model on plumbed (graph) V-manifolds and V-cobordisms \([4, 5, 6, 7, 8, 9]\). In addition to the usual characteristics of topological field theories (absence of local degrees of freedom, finite dimensionality of phase space, etc.), in our model as analogues of cosmological (coupling) constants there appear rational intersection forms (matrices) which are the basic topological invariants of plumbed V-manifolds \([10, 8, 9]\). Thus in our model is realized the idea that the primary values of coupling constants (which correspond to a vacuum without any local excitations) are to be looked for at the topological level (see e.g. \([11]\)).

The paper is organized as follows. Section 2 starts with a short review of the main ideas concerning plumbed V-cobordisms and V-manifolds. We reproduce definitions of basic topological invariants of these spaces, specifically, the rational and integer intersection matrices. We also formulate the important result \([8]\) concerning reciprocality (mutual inverse property) of these matrices written in natural bases.

Section 3 is dedicated to construction of Abelian \( BFE \)-models on plumbed V-cobordisms. We show that these models possess an essentially new property in comparison with usual \( BF \)-theories: the rôle of vacuum (zero-level) coupling constants is played by the topologically invariant intersection forms of plumbed V-cobordisms.

In section 4 we give a collection of non-trivial examples which realize the ideas proposed in section 3. As a starting point is taken the primary sequence of 3-dimensional Seifert fibered homology spheres (Sfh-spheres) characterized by Seifert invariants constructed of first nine prime numbers \( p_1 = 2, p_2 = 3, \ldots, p_9 = 23 \). With the help of introduced by us derivative operation acting on Sfh-spheres, we construct the basic two-parametric family of Sfh-spheres \( \Sigma^{(l)}_n \) which by means of the well-known splicing operation are pasted together into \( \mathbb{Z} \)-homology spheres \( M^{(l)}_n \). These latter ones turn out to be the boundary components of 4-dimensional plumbed V-cobordisms \( X_{D_n}^{(l)} \) together with a specific collection of lens spaces. Just on this set of cobordisms we are constructing the respective \( BFE \)-systems \( S^{(l)}_n \) with cosmological coupling matrices being, as we have proven, the representatives of rational
intersection forms of $X_D^{(l)}$. Namely these V-cobordisms can be considered as construction units in gluing together the compound V-cobordisms which describe topology changes $M_n^{(l)} \rightarrow M_{n+1}^{(l)}$ between Z-homology spheres (this is sketched in Conclusions, section 5).

Further we give results of calculation of both rational and integer intersection matrices for a finite ensemble of plumbed V-cobordisms $X_D^{(l)}$ with $n,l = 0, \ldots, 4$. So we find a relation between the diagonal elements of the integer intersection matrices and Euler numbers of the basic family of SFh-spheres $\Sigma_n^{(l)}$. These Euler numbers fairly well reproduce the hierarchy of the dimensionless low-energy coupling constants.

The standard notations $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are used for the sets of integer, real and complex numbers, respectively.

2 Plumbed 4- and 3-dimensional manifolds

In order to build a new version of the Abelian $BF$ theory on 4-dimensional pV-manifolds and pV-cobordisms whose boundaries are Z-homology spheres and lens spaces, we first give some necessary definitions essentially following the works of Saveliev [9, 12]. These ideas can be traced back to [5, 6, 13, 7].

2.1 Plumbing and splicing operations

A plumbing graph $\Gamma$ is a graph with no cycles (a finite tree) each of whose vertices $v_i$ carries an integer weight $e_i$, $i = 1, \ldots, s$. The $D^2$-bundle $Y(e_i)$ is associated to each vertex $v_i$, whose Euler class (self-intersection number of zero-section) is $e_i$. If the vertex $v_i$ has $d_i$ edges connected to it on the graph $\Gamma$, choose $d_i$ disjoint discs in the base $S^2$ of $Y(e_i)$ and call the disc bundle over the $j$th disc $B_{ij} = (D^2_j \times D^2_i)$. When two vertices $v_i$ and $v_k$ are connected by an edge, the disc bundles $B_{ij}$ and $B_{kl}$ should be identified by exchanging the base and fiber coordinates [6]. This pasting operation is called plumbing, and the resulting smooth 4-manifold $P(\Gamma)$ is known as plumbed 4-manifold. Its boundary $M(\Gamma) = \partial P(\Gamma)$ is referred to as a plumbed 3-manifold.

Since the homology group $H_1(P(\Gamma), \mathbb{Z}) = 0$, the unique non-trivial homology characteristic is $H_2(P(\Gamma), \mathbb{Z})$ which has a natural basis (set of generators) represented by the zero-sections of the plumbed bundles. All these sections are embedded 2-spheres $z_a$ where $a = 1, \ldots, r = \text{rank } H_2(P(\Gamma), \mathbb{Z})$, and they
can be oriented in such a way that the intersection (bilinear) form \( \text{(12)} \)
\[
A : H_2(P(\Gamma), \mathbb{Z}) \otimes H_2(P(\Gamma), \mathbb{Z}) \to \mathbb{Z}
\] (2.1)
will be represented by the \( r \times r \)-matrix \( A(\Gamma) = (a_{ij}) \) with the entries: \( a_{ij} = e_i \) if \( i = j \); \( a_{ij} = 1 \) if the vertex \( v_i \) is connected to \( v_j \) by an edge; and \( a_{ij} = 0 \) otherwise.

Let \( M \) be a Seifert fibered 3-manifold over \( S^2 \) with unnormalized Seifert invariants \( (a_i, b_i), \ i = 1, \ldots, n, \ a_i > 1 \). It can be obtained as the boundary of the plumbed 4-manifold \( P(\Gamma) \) where \( \Gamma \) is a star graph shown in figure 1 \( \text{(6)} \).

\[
\begin{array}{cccc}
&t_{11} & t_{12} & \cdots & t_{1m_1} \\
0 & \cdots & \cdots & \cdots & \cdots \\
&t_{n1} & t_{n2} & \cdots & t_{nm_n}
\end{array}
\]

Figure 1.

The integer weights \( t_{ij} \) in this graph are found from continued fractions \( a_i/b_i = [t_{i1}, \ldots, t_{im_i}] \); here
\[
[t_1, \ldots, t_k] = t_1 - \frac{1}{t_2 - \frac{1}{\cdots - \frac{1}{t_k}}}
\]
Note that among closed 3-manifolds \( \mathbb{Z} \)-homology spheres \( M \) are characterized by \( H_1(M, \mathbb{Z}) = 0 \).

A Seifert fibered 3-manifold \( M \) is a \( \mathbb{Z} \)-homology sphere (Sfh-sphere) iff the determinant of matrix \( A(\Gamma) \) associated with the plumbing graph in figure 1, is \( \pm 1 \) which is equivalent to the condition
\[
a \sum b_i/a_i = \pm 1
\] (2.2)
where \( a = a_1 \cdots a_n \). We fix an orientation on \( M \) by choosing \( +1 \) in (2.2).

From (2.2) it follows that \( b_i\sigma_i = 1 \mod a_i \) where \( \sigma_i = a/a_i \); thus the set of
a_i proves to be sufficient for the complete determination of any Sfh-sphere. This is why we use the standard notation Σ(a_1, . . . , a_n) for Sfh-spheres.

Lens spaces represent another case of Seifert fibered manifolds. Expanding −p/q = [t_1, . . . , t_n] into a continued fraction, we encounter L(p, q) as a boundary of the 4-manifold obtained by plumbing on the chain Γ shown in Figure 2.

\[ \cdots \]

Figure 2.

Notice that this plumbing graph simultaneously represents the lens space L(p, q^*) with −p/q^* = [t_n, . . . , t_1] where qq^* = 1 mod p. This reflects the fact that L(p, q) and L(p, q^*) are homeomorphic.

In section 4 we shall build universe models which are plumbed 4-manifolds (cobordisms) with boundaries whose components are homeomorphic to lens spaces or to Z-homology spheres, the latter ones being constructed by splicing of certain set of Sfh-spheres with three exceptional fibers, i.e. Σ(a_1, a_2, a_3). Therefore it is worth giving now the general definition of the splicing operation as well as the plumbed V-cobordism.

By a link \([\Sigma, K] = (\Sigma, S_1 \cup \cdots \cup S_m)\) we mean a pair consisting of oriented Z-homology sphere Σ and a collection K of disjoint oriented knots S_1, . . . , S_m in Σ. Empty links are just Z-homology spheres. Note that the links (S^3, K) where S^3 is an ordinary 3-sphere, are also allowed. If the link components S_1, . . . , S_m are fibers in Σ, then the link (Σ, K) is called a Seifert link. Let (Σ, K) and (Σ', K') be links and choose components S ∈ K and S' ∈ K'. Let also N(S) and N(S') be their tubular neighbourhoods, while m, l ⊂ ∂N(S) and m', l' ⊂ ∂N(S') be standard meridians and longitudes. The manifold Σ'' = (Σ \ intN(S)) \cup (Σ' \ intN(S')) obtained by pasting along the torus boundaries by matching m to l' and m' to l, is a Z-homology sphere. The link (Σ'', (K \ S) \cup (K' \ S')) is called the splice (splicing) of (Σ, K) and (Σ', K') along S and S'. We shall use the standard notation Σ'' = Σ \[\Sigma \cup S \Sigma'\] or simply Σ'' = Σ — Σ'. Any link which can be obtained from a finite number of Seifert links by splicing is called a graph link. Empty graph links are precisely the plumbed (graph) Z-homology spheres.

A plumbed graph Γ with added arrowhead vertices, denoted as \(\overline{\Gamma}\), repre-
sents a link \( K \) in a (plumbed) 3-manifold \( M = M(\Gamma) \) as follows: Let \( \Gamma \) be \( \overline{\Gamma} \) with all the arrows deleted, and put \( M = \partial P(\Gamma) \). Each arrowhead vertex \( v_j \) of \( \overline{\Gamma} \) is attached at some vertex of \( \Gamma \), and to this arrow we associate a fiber \( S_j \) of the bundle \( Y(e_i) \) used in the plumbing \[7, 9\].

The splicing operation on graph links can be described in terms of plumbing graphs as follows. Suppose that two graph links are represented by their plumbing diagrams \( \Gamma \) and \( \Gamma' \) (see Figure 3) with arrows attached to vertices \( e_n \) and \( e'_m \), respectively. The corresponding plumbing diagram for a spliced link is shown in Figure 4 where \( a = \det A(\Gamma_0) / \det A(\Gamma) \), while \( \Gamma \) is the plumbing graph \( \overline{\Gamma} \) with the arrow deleted, and \( \Gamma_0 \) is a portion of \( \Gamma \) obtained by removing the \( n \)th vertex weighted by \( e_n \) as well as all its adjacent edges. Another integer \( a' \) is similarly obtained from the graph \( \Gamma' \) (examples see in \[7, 9\]). The above description of splicing in terms of plumbing graphs makes it possible to treat splicing as an operation on the corresponding plumbed 4-manifold; moreover, \( M(\overline{\Gamma}) = \partial P(\Gamma) \).

![Figure 3.](image)

![Figure 4.](image)

### 2.2 Plumbed V-cobordisms and V-manifolds

A *plumbed V-cobordism* (pV-cobordism) is called plumbed 4-manifold when it is a cobordism between a plumbed 3-manifold and a disjoint union of lens
spaces. pV-cobordisms may be considered as models of elementary topology changes. They are constructed as follows.

Let \( P(\Gamma) \) be a plumbed 4-manifold corresponding to graph \( \Gamma \), and \( \Gamma^{\text{ch}} \) be a chain in \( \Gamma \) of the form shown in Figure 2. Plumbing on \( \Gamma^{\text{ch}} \) yields a submanifold \( P(\Gamma^{\text{ch}}) \) of \( P(\Gamma) \) whose boundary is a lens space \( L(p, q) \). The closure of \( P(\Gamma) \setminus P(\Gamma^{\text{ch}}) \) is a smooth compact 4-manifold \( X_D \) with oriented boundary \( -L(p, q) \sqcup \partial P(\Gamma) \) where \( \sqcup \) denotes the disjoint sum operation. Starting with several chains \( \Gamma^{\text{ch}}_i \) (\( i = 1 \) to \( I \)) in \( \Gamma \) (where \( 0 \leq I \) is the integer numbers interval from 0 to \( I \)), one can introduce a cobordism between \( \partial P(\Gamma) \) and the disjoint union \( L = \bigsqcup_{i=1}^I L(p_i, q_i) \) of several lens spaces. The chains \( \{\Gamma^{\text{ch}}_i\} \) must be disjoint in the following sense: no two chains should have a common vertex, and no edges of \( \Gamma \) should have one endpoint on one chain and another, on another one. Such cobordisms will be called pV-cobordisms. A pV-cobordism is always a smooth manifold. The notation ‘V-’ refers to the fact that each lens space \( L(p_i, q_i) \) on the boundary of \( X_D \) may be eliminated by pasting a cone \( cL(p_i, q_i) \) over the lens space. This yields the well-known V-manifold \( X \) with isolated singular points. V-manifolds of the type \( X \) are called plumbed V-manifolds (pV-manifolds).

A pV-cobordism \( X_D \) can be adequately represented by the so-called decorated plumbing graph \( \Gamma_D \) (this explains the subindex \( D \)). Such graphs are decorated with extra ovals (or circles), each enclosing exactly one chain \( \Gamma^{\text{ch}}_i \) (or vertex as a particular case of chain). (The above conditions on the chains \( \Gamma^{\text{ch}}_i \) to be disjoint translates into the following conditions on the decorating ovals: Any two 2-disks bounded by decorating ovals are disjoint, and no edge of \( \Gamma_D \) is intersected by more than one decorating oval.) Examples are considered in section 4.
2.3 Integral and rational intersection forms of pV-cobordisms

Let $X_D$ be a pV-cobordism with the boundary

$$\partial X_D = - \bigsqcup_{i=1}^L \langle p_i, q_i \rangle \sqcup M$$

where $M$ is a $\mathbb{Z}$-homology sphere. In this case $H^1(X_D, \mathbb{Z}) = 0$, and there exists the exact sequence [10, 14, 15]:

$$0 \to H^2(X_D, \partial X_D, \mathbb{Z}) \overset{j^*}{\to} H^2(X_D, \mathbb{Z}) \overset{i^*}{\to} H^2(\partial X_D, \mathbb{Z}) = \bigoplus_{i=1}^I \mathbb{Z} p_i \to 0. \quad (2.4)$$

Note that the cohomology group $H^2(X_D, \partial X_D, \mathbb{Z})$ is isomorphic to the $H^2(X, \mathbb{Z})$ where the pV-manifold is defined in (2.3).

As a consequence of the Poincaré–Lefschetz duality, the integral intersection form can be defined

$$\omega_Z : H^2(X_D, \partial X_D, \mathbb{Z}) \otimes H^2(X_D, \mathbb{Z}) \to \mathbb{Z} \quad (2.5)$$

by means of the cup product pairing [14]

$$\omega_Z(b, e) = \langle b \cup e, [X_D, \partial X_D] \rangle \quad (2.6)$$

for each $b \in H^2(X_D, \partial X_D, \mathbb{Z})$ and $e \in H^2(X_D, \mathbb{Z})$. The operation $\cup$ is known as cup product, and $[X_D, \partial X_D]$ is relative fundamental class [15, 10]. Note that the intersection form $\omega_Z$ becomes non-degenerate after factoring out the torsion subgroup $\text{Tor}(H^2(X_D, \mathbb{Z}))$ [14].

From the exactness of the sequence (2.4) and since $H^2(\partial X_D, \mathbb{Z})$ is a pure torsion, for any $e' \in H^2(X_D, \mathbb{Z})$ there exists $k \in \mathbb{Z}$ such that $i^*(ke') = 0$. Hence $ke' = j^*(b)$ for the unique $b \in H^2(X_D, \partial X_D, \mathbb{Z})$. Therefore it is natural to define the rational intersection form [10, 8]

$$\omega_Q(e', e) = \langle e' \cup e, [X_D, \partial X_D] \rangle := \frac{1}{k} \langle b \cup e, [X_D, \partial X_D] \rangle \quad (2.7)$$

for any pair $e, e' \in H^2(X_D, \mathbb{Z})$.

We shall use the cohomological version of the Proposition 4 from [8] which can be formulated as follows: Let $X_D$ be a pV-cobordism with $H^1(X_D, \mathbb{Z}) =$
0. If we choose a basis $b_i$ of $H^2(X_D, \partial X_D, \mathbb{Z})$ and a dual basis $e^i$ of $H^2(X_D, \mathbb{Z})$ ($i = 1, \ldots, r$, $r = \text{rank} H^2(X_D, \mathbb{Z})$), dual in the sense of

$$\omega_Z(b_i, e^j) = \delta^j_i,$$

then the integral intersection matrix $g_{ij} = \omega_Z(b_i, b_j)$ for $H^2(X_D, \partial X_D, \mathbb{Z})$ is inverse of the rational intersection matrix $\lambda^{ij} = \omega_Q(e^i, e^j)$ for $H^2(X_D, \mathbb{Z})$.

It is important that the just mentioned intersection forms (matrices) are the basic topological invariants of compact 4-manifolds. In the only cases being here under consideration, namely those of pV-cobordisms (pV-manifolds) with vanishing first cohomology groups, all (co)homology information about these pV-cobordisms (pV-manifolds) is contained in the second (co)homology groups, and hence also in the intersection matrices $g_{ij}$ and $\lambda^{ij}$. These matrices are easily calculated by means of the procedure described in detail in [9], see also [7].

### 3 Abelian BFE-theory on pV-cobordisms

In this section we propose a generalized version of topological BF-type theory (known as 4-dimensional topological gravity) on pV-cobordisms and pV-manifolds.\(^1\) The generalization is related to the fact that ranks of the 2-cohomology groups of pV-cobordisms are in general greater than one, i.e.

$$\text{rank } H^2(X_D, \partial X_D, \mathbb{Z}) = \text{rank } H^2(X_D, \mathbb{Z}) = r \geq 1.$$  

Therefore the set of basic fields should consist of $r$ (Abelian) forms $B_a$, $a = 1, \ldots, r$, as well as $r$ 2-forms $E^a$ dual to $B_a$.

This model which we call BFE-theory, will have the following important properties:

1. The rôle of vacuum (zero-level) coupling constants is played by the topologically invariant intersection forms (cosmological-constant matrices).

2. The space $\mathcal{N}$ of classical solutions (phase space) of BFE-system will be a finite-dimensional vector space

$$\mathcal{N} = H^2(X_D, \mathbb{R}) \oplus H^2(X_D, \mathbb{R})$$  

\(^1\)Due to the group isomorphism $H^2(X_D, \partial X_D, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ (see subsection 2.3), the model we propose below can be constructed in the same way on pV-cobordisms and pV-manifolds. Thus we restrict ourselves to pV-cobordisms only.
where $H^2(X_D, \mathbb{R})$ is de Rham-2-cohomology group of pV-cobordism $X_D$. (The possibility to define the discrete phase space

$$N_{\text{discr}} = H^2(X_D, \mathbb{Z}) \oplus H^2(X_D, \partial X_D, \mathbb{Z})$$

(3.2)

will be considered elsewhere.)

To realize a model with properties (1) and (2), we introduce a family of linearly independent elements $B_a$ of the cochain complex $C^2(X_D, \partial X_D, \mathbb{Z}) \otimes \mathbb{R}$. These elements (cochains) are represented by a set of Abelian 2-forms which we also denote as $B_a$. The equations of motion (3.3) and the gauge symmetries (3.8) of our model yield (see below) the property of 2-forms $B_a$ to be closed and defined up to exact forms $d\chi_a$. Thus we can assume that $B_a$ form a basis of the group $H^2(X_D, \partial X_D, \mathbb{Z}) \otimes \mathbb{R}$. Therefore the index $a$ runs from 1 to $r = \text{rank } H^2(X_D, \partial X_D, \mathbb{Z}) \otimes \mathbb{R}$. Note that the number $r$ of the basis elements $B_a$ is also equal to the number of vertices exterior to decorated chains $\Gamma_{\text{ch}}^i$ of the decorated graph $\Gamma_D$ which determines the pV-cobordism $X_D$ [9], see also examples in section 4 and the Observation 3 below.

Let us introduce a suitable set of Abelian 1-forms $A^a$, $a = 1, \ldots, r$, and a constant non-degenerate symmetric $r \times r$-matrix $\Lambda^{ab}$. Then it is natural to write the action

$$S^{(r)}_{\text{BF}} = \int_{X_D} \left\{ B_a \wedge F^a + \frac{1}{2} \Lambda^{ab} B_a \wedge B_b \right\}$$

(3.3)

which is a direct generalization of the ordinary Abelian $BF$-theory action (for $r = 1$ and with the $\Lambda$ term or cosmological constant) [1] [2] [16]

$$S^{(1)}_{\text{BF}} = \int_{X_D} \left\{ B \wedge F + \frac{\Lambda}{2} B \wedge B \right\}.$$  (3.4)

In (3.3), $F^a = dA^a$, $d$ is the exterior derivative, and in repeated indices the summation convention is applied. The constant matrix $\Lambda^{ab}$ occupies the place of cosmological constant, thus we can call it either ‘cosmological constant matrix’ or ‘coupling constant matrix’.

Observation 1. The action (3.3) has the appearance the non-Abelian $BF$-theory action

$$S_{\text{GR}} = \int \left\{ B^i \wedge F^i + \epsilon_{ij} B^i \wedge B^j + \frac{\Lambda}{2} B^i \wedge B^i \right\}$$

(3.5)
which is equivalent to general relativity \[17, 18, 19\]. While in \(S_{GR}\) the traceless symmetric 0-form \(\phi_{ij}\) is a Lagrangian multiplier, our constant matrix \(\Lambda^{ab}\), as it is shown below, coincides with the rational intersection form and is the basic topological invariant of the pV-cobordism \(X_D\). A comparison of \(3.3\) and \(S_{GR}\) shows that an immediate analogue of the cosmological constant \(\Lambda\) is trace of \(\Lambda^{ab}\). Moreover, the indices used in these two theories have different nature: in our model, \(a, b\) enumerate Abelian fields, and in the non-Abelian \(BF\)-theory, \(i, j\) are the gauge group indices.

However the proposed above model \(3.3\) possesses only cohomologically trivial solutions for \(B_a\) since it yields the equations of motion

\[
dB_a = 0, \quad (3.5)
\]

\[
F^a + \Lambda^{ab}B_b = 0. \quad (3.6)
\]

[It is clear that \(3.5\) follows from \(3.6\). Moreover, due to \(F^a = dA^a\) and the initial supposition that constant matrix \(\Lambda^{ab}\) is non-degenerate, \(3.6\) implies that \(B_a\) are exact 2-forms.] Hence \(B_a\) merely represent zero-classes in \(H^2(X_D, \partial X_D, \mathbb{Z}) \otimes \mathbb{R}\), so \(B_a\) cannot be a basis of this group.

Nevertheless, one can attain a cohomological non-triviality of \(B_a\) as solutions of dynamical equations, as well as a realization of the properties (1) and (2) mentioned in the beginning of this section. To this end it is sufficient to add a new set of 2-forms \(E^a \in C^2(X_D, \mathbb{Z}) \otimes \mathbb{R}\) to the fields used in \(3.3\). Thus we propose the following action:

\[
S_{BF E} = \int_{X_D} \left\{ (B_a + \Lambda_{ab}E^b) \wedge F^a + \frac{1}{2} \Lambda_{ab} (\Lambda^{ac}B_c - E^a) \wedge (\Lambda^{bd}B_d - E^b) \right\} \\
= \int_{X_D} \left\{ (B_a \wedge F^a + \frac{1}{2} \Lambda^{ab}B_a \wedge B_b) + \Lambda_{ab} \left( E^a \wedge F^b + \frac{1}{2} E^a \wedge E^b \right) - B_a \wedge E^a \right\}
\]

\[
(3.7)
\]

where \(\Lambda_{ab}\) is the matrix inverse to \(\Lambda^{ab}\) (\(\Lambda_{ab}\Lambda^{bc} = \delta^c_a\)).

The Abelian gauge symmetries of this action (up to boundary terms \[2\])

\[
\delta A^a = d\phi^a, \quad \delta B_a = d\chi_a, \quad \delta E^a = \Lambda^{ab}d\chi_b
\]

\[
(3.8)
\]

\((\phi^a\) are 0-forms and \(\chi_a\), 1-forms) combined with equations of motion following from \(3.7\)

\[
dB_a = 0 = dE^a, \quad (3.9)
\]

11
\[ F^a = 0, \quad (3.10) \]
\[ E^a = \Lambda^{ab} B_b, \quad (3.11) \]

tell us that the space \( \mathcal{N} \) of classical solutions (phase space) of the \( BFE \) system is

\[ \mathcal{N} = \left[ H^2(X_D, \partial X_D, \mathbb{Z}) \otimes \mathbb{R} \right] \oplus \left[ H^2(X_D, \mathbb{Z}) \otimes \mathbb{R} \right] \oplus \left[ H^1(X_D, \mathbb{Z}) \otimes \mathbb{R} \right] \quad (3.12) \]

where

\[
\begin{align*}
B_a &\in H^2(X_D, \partial X_D, \mathbb{Z}) \otimes \mathbb{R} \cong H^2(X_D, \partial X_D, \mathbb{R}), \\
E^a &\in H^2(X_D, \mathbb{Z}) \otimes \mathbb{R} \cong H^2(X_D, \mathbb{R}), \\
A^a &\in H^1(X_D, \mathbb{Z}) \otimes \mathbb{R} \cong H^1(X_D, \mathbb{R}).
\end{align*}
\]

Due to the exact sequence (2.4) the following de Rham groups are isomorphic:

\[ H^2(X_D, \partial X_D, \mathbb{R}) \cong H^2(X_D, \mathbb{R}). \]

(3.14)

Taking into account this fact as well as triviality of the group \( H^1(X_D, \mathbb{R}) \), we come to the phase space (3.1) of the \( BFE \)-system described by (3.7). Note that the equation (3.10) means that all connections \( A^a \) are flat (1-forms \( A^a \) are closed) and cohomologically trivial since \( H^1(X_D, \mathbb{R}) = 0 \). Moreover, two sets of closed 2-forms \( B_a \) and \( E^a \) defined up to exact forms may be considered as a family of fundamental solutions of the \( BFE \)-system (3.7).\(^2\) These 2-forms determine two bases of \( H^2(X_D, \mathbb{R}) \) related by the non-degenerate transformation (3.11) which is nothing more than constraint equations following from the action (3.7).

Let us impose on the bases \( B_a \) and \( E^a \) the duality condition of the type (2.8) in the de Rham representation (cf. 20)

\[ \frac{1}{4\pi^2} \oint_{X_D} B_a \wedge E^b = \delta_a^b. \quad (3.15) \]

\(^2\)Due to the isomorphism (3.14) one could make no distinction between the levels of indices of \( B_a \) and \( E^a \). Initially these different levels reflected relation of these 2-forms to the groups dual in the Poincaré–Lefschetz sense; below we wouldn’t like to overlook these hereditary features.
This condition fixes the gauge 1-forms $\chi_a$ introduced in (3.8) which automatically determine the boundary terms. Then in this gauge the constraints (3.11) yield

$$\Lambda^{ab} = \frac{1}{4\pi^2} \int_{X_D} E^a \wedge E^b, \quad (3.16)$$

and

$$\Lambda_{ab} = \frac{1}{4\pi^2} \int_{X_D} B_a \wedge B_b. \quad (3.17)$$

Following [8, 21], note that the right-hand sides in (3.16) and (3.17) are rational and integer intersection forms, respectively, and they are represented by the mutually inverse intersection matrices $\Lambda^{ab}$ and $\Lambda_{ab}$ in the respective bases $E^a$ and $B_a$. In analogy with the Yang–Mills theory [21], the diagonal elements of the matrix $\Lambda^{ab}$ could be called topological charges. However due to the relation (4.12) between Euler numbers and absolute values of the diagonal elements of the inverse matrix $\Lambda_{ab}$, it is natural to consider as topological charges the inverse values of the latter ones. Thus any Abelian $BFE$-theory is characterized by the set of topological charges $1/|\Lambda_{aa}|, \ a \in \Gamma, r$.

These intersection forms are basic topological invariants of pV-cobordisms of the type $X_D$ [6, 7, 9]. Thus in our $BFE$-model the coupling (cosmological) constant matrix $\Lambda^{ab}$ is the basic topological invariant of spacetime (of Euclidean signature) described by the cobordism $X_D$, and it is the rational intersection form in the natural basis $E^a$, so that our $BFE$-system (3.7) does possess the property (1) announced in the beginning of this section.

4  $BFE$-systems on the two-parametric family of pV-cobordisms as a set of cosmological models

In this section we construct a set of cosmological models with a sequence of topology changes of 3-dimensional sections of spacetime (in Euclidean regime). Each elementary topology change is represented by a specific pV-cobordism $X_D^{(l)}_n$ which corresponds to the decorated graph $\Gamma^{(l)}_{D_n}$ (the origin of two non-negative integer parameters $n, l \in \mathbb{Z}^+$ will be explained below). On each pV-cobordism $X_D^{(l)}_n$ is defined its individual Abelian $BFE$-system
of the type (3.7) which is a pure topological ‘gravity’ with a ‘cosmological
term’ represented by a rational intersection matrix $\Lambda_{n}^{(l)}_{ab}$.

4.1 The basic set of Seifert fibered homology spheres

We construct the cobordisms $X_{D}^{(l)}_{n}$ using basic structure elements which
are plumbed 4-manifolds $P(\Gamma)$ with Seifert fibered homology spheres (Sfh-
spheres) $\Sigma(a_{1}, a_{2}, a_{3})$ (with only three exceptional fibers) as boundaries:
$\partial P(\Gamma) = \Sigma(a_{1}, a_{2}, a_{3})$. Let us remind the definition of these Sfh-spheres:
$\Sigma(a_{i}) := \Sigma(a_{1}, a_{2}, a_{3})$ is a smooth compact 3-manifold obtained by intersect-
ing the complex algebraic Brieskorn surface $z_{1}^{a_{1}} + z_{2}^{a_{2}} + z_{3}^{a_{3}} = 0$ $(z_{i} \in \mathbb{C})$
with the unit 5-dimensional sphere $|z_{1}|^{2} + |z_{2}|^{2} + |z_{3}|^{2} = 1$, where $a_{1}, a_{2}, a_{3}$
are pairwise coprime integers, $a_{i} > 1$. There exists a unique Seifert fibration
of this manifold with unnormalized Seifert invariants \[ e(\Sigma(a)) = \sum_{i=1}^{3} b_{i}/a_{i} = 1/a, \]
where $a = a_{1}a_{2}a_{3}$ and $e(\Sigma(a))$ is the well known
topological invariant of a Sfh-sphere, its Euler number.

To construct our model of universe we need a specific family of Sfh-spheres
which would be defined in following way. First, we define the derivative of a
Sfh-sphere $\Sigma(a)$ as a Sfh-sphere $\Sigma(1)(a) := \Sigma(a_{1}, a_{2}a_{3}, a+1) \equiv \Sigma(a_{1}^{(1)}, a_{2}^{(1)}, a_{3}^{(1)}).$ (4.1)
The Euler number of this Sfh-sphere is $e(\Sigma(1)(a)) = 1/a^{(1)}$ where $a^{(1)} = a_{1}^{(1)}a_{2}^{(1)}a_{3}^{(1)} = a(a+1)$. By induction, we define the derivative $\Sigma(l)(a) = \Sigma(a_{1}^{(l)}, a_{2}^{(l)}, a_{3}^{(l)})$ of $\Sigma(a)$ of any order $l$. In particular, there holds the recurrent
relation
$\quad a^{(l)} = a^{(l-1)}(a^{(l-1)} + 1)$ (4.2)
for a product of three Seifert invariants $a^{(l)} = a_{1}^{(l)}a_{2}^{(l)}a_{3}^{(l)}$. Second, we define a
sequence of Sfh-spheres which we shall call primary sequence. Let $p_{i}$ be the
$i$th prime number in the set of the positive integers $\mathbb{N}$, e.g., $p_{1} = 2$, $p_{2} = 3$, . . . .
The primary sequence of Sfh-spheres is defined as
$\quad \{ \Sigma(q_{i}, p_{i+1}, p_{i+2}) | i \in \mathbb{N} \}$ (4.3)
where $q_{i} = p_{1} \cdots p_{i}$. Finally, to the end of constructing our model of universe,
we include in this sequence as its first two terms the usual 3-dimensional
spheres $S^{3}$ with Seifert fibrations (Sf-spheres) determined by the mappings
\[ h_{pq} : S^3 \to S^2 \] in their turn defined as \[ h_{pq}(z_1, z_2) = \frac{z_1^p}{z_2^q}, \quad p, q \in \mathbb{N} \] Recall that \( S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1 \} \) and \( \frac{z_1^p}{z_2^q} \in \mathbb{C} \cup \{\infty\} \cong S^2 \). We denote these two Sf-spheres as \( \Sigma(1, 1, 2) \), \( p = 1, q = 2 \) and \( \Sigma(1, 2, 3) \), \( p = 2, q = 3 \). In these notations we use an additional number (unit) which corresponds to an arbitrary regular fiber. This will enable us to take derivatives of Seifert fibrations on \( \Sigma(1, 1, 2) \) and \( \Sigma(1, 2, 3) \) by the same rule \((4.1)\) as for other members of the sequence \((4.3)\).

Now we form the family of manifolds corresponding to the first nine primary Sfh-spheres and their derivatives up to the fourth order,

\[ \{\Sigma^{(l)}(q_{i-1}, p_i, p_{i+1}) \mid i \in 0, 1, l \in 0, 1, \ldots, 4\}. \quad (4.4) \]

Note that the subfamily corresponding to \( i = 0, 1 \) is built of the ordinary spheres \( S^3 \) with fixed Seifert fibrations. In order to include the Sf-spheres \( \Sigma(1, 1, 2) \) and \( \Sigma(1, 2, 3) \) in this family, one has to put \( q_{-1} = q_0 = p_0 = 1 \).

E.g., for the well known Poincaré homology sphere \( \Sigma(p_1, p_2, p_3) = \Sigma(2, 3, 5) \), the sequence of derivatives is

\[
\begin{align*}
\Sigma^{(1)}(2, 3, 5) &= \Sigma(2, 15, 31), \\
\Sigma^{(2)}(2, 3, 5) &= \Sigma(2, 465, 931), \\
\Sigma^{(3)}(2, 3, 5) &= \Sigma(2, 432915, 865831), \\
\Sigma^{(4)}(2, 3, 5) &= \Sigma(2, 374831227365, 749662454731).
\end{align*}
\]

The calculation results for Euler numbers of Seifert structures of Sf- and Sfh-spheres in the family \((4.4)\) are given in the table\( \square \) We find that for the

\[
\begin{array}{|c|c|c|c|c|}
\hline
i \setminus l & 0 & 1 & 2 & 3 \\
\hline
0 & 0.5 & 0.166 & 2.38 \times 10^{-5} & 5.33 \times 10^{-6} \\
1 & 0.166 & 2.38 \times 10^{-2} & 5.33 \times 10^{-4} & 3.06 \times 10^{-6} \\
2 & 3.33 \times 10^{-2} & 1.07 \times 10^{-3} & 1.15 \times 10^{-6} & 1.33 \times 10^{-12} \\
3 & 4.76 \times 10^{-3} & 2.26 \times 10^{-5} & 5.09 \times 10^{-10} & 2.59 \times 10^{-19} \\
4 & 4.33 \times 10^{-4} & 1.87 \times 10^{-7} & 3.51 \times 10^{-14} & 1.23 \times 10^{-27} \\
5 & 3.33 \times 10^{-5} & 1.11 \times 10^{-9} & 1.23 \times 10^{-18} & 1.51 \times 10^{-36} \\
6 & 1.96 \times 10^{-6} & 3.84 \times 10^{-12} & 1.47 \times 10^{-23} & 2.17 \times 10^{-46} \\
7 & 1.03 \times 10^{-7} & 1.06 \times 10^{-14} & 1.13 \times 10^{-28} & 1.28 \times 10^{-56} \\
8 & 4.48 \times 10^{-9} & 2.01 \times 10^{-17} & 4.04 \times 10^{-34} & 1.64 \times 10^{-67} \\
\hline
\end{array}
\]
subfamily

\[
\left\{ \Sigma^{(l)} = \Sigma^{(l)}(q_{2l-1}, p_{2l}, p_{2l+1}) \, \bigg| \, l \in \{0, 4\} \right\}
\]  

(4.5)

the Euler numbers (the boldface numbers) reproduce fairly well the experimental hierarchy of dimensionless low-energy coupling (DLEC) constants of fundamental interactions, see table 2.

Table 2: Euler numbers vs. experimental DLEC constants.

<table>
<thead>
<tr>
<th>l</th>
<th>(e \left( \Sigma^{(l)}_l \right))</th>
<th>Interaction</th>
<th>(\alpha_{\text{exper}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5</td>
<td>strong</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(1.07 \times 10^{-3})</td>
<td>electromagnetic</td>
<td>(7.20 \times 10^{-3})</td>
</tr>
<tr>
<td>2</td>
<td>(3.51 \times 10^{-14})</td>
<td>weak</td>
<td>(3.04 \times 10^{-12})</td>
</tr>
<tr>
<td>3</td>
<td>(2.17 \times 10^{-46})</td>
<td>gravitational</td>
<td>(2.73 \times 10^{-46})</td>
</tr>
<tr>
<td>4</td>
<td>(2.70 \times 10^{-134})</td>
<td>cosmological</td>
<td>&lt; (10^{-120})</td>
</tr>
</tbody>
</table>

Notes: 1. The dimensionless strong interaction constant is \(\alpha_{\text{st}} = G/\hbar c\), \(G\) characterizes the strength of the coupling of the meson field to the nucleon. 2. The fine structure (electromagnetic) constant is \(\alpha_{\text{em}} = e^2/\hbar c\). 3. The dimensionless weak interaction constant is \(\alpha_{\text{weak}} = (G_F/\hbar c)(m_e \cdot c^2/\hbar)^2\), \(G_F\) being the Fermi constant (\(m_e\) is mass of electron). 4. The dimensionless gravitational coupling constant is \(\alpha_{\text{gr}} = G_N m_e^2/\hbar c\), \(G_N\) being the Newtonian gravitational constant. 5. The cosmological constant \(\Lambda\) multiplied by the squared Planckian length is \(\alpha_{\text{cosm}} = \Lambda G_N \hbar c^3\). The mentioned dimensionless constants (except the cosmological one) are also known as Dyson numbers.

Observation 2. Usually one calls as fundamental (universal) such an interaction which is essentially characterized by only one coupling constant. For example, the weak interaction is universal since it is characterized only by the Fermi constant \(G_F\) (if effects of mixing different fundamental particles are not taken into account). In this sense each interaction given in table 2 is really characterized by a single coupling constant and is universal, though only at low energies \(E_{\text{low}} \ll M_W \approx 80\) GeV. At higher energies a unification of interactions takes place, and the collection of fundamental interactions changes. The collection of coupling constants changes too. In our model this
Table 3: Euler number of \((n, t)\)-family of Sf- and Sfh-spheres.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(4)</th>
<th>(3)</th>
<th>(2)</th>
<th>(1)</th>
<th>(0)</th>
<th>(-1)</th>
<th>(-2)</th>
<th>(-3)</th>
<th>(-4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.4</td>
<td>0.2</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>1.7</td>
<td>1.6</td>
<td>1.6</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.4</td>
<td>1.3</td>
<td>1.2</td>
</tr>
<tr>
<td>3</td>
<td>2.1</td>
<td>2.0</td>
<td>2.0</td>
<td>1.9</td>
<td>1.9</td>
<td>1.8</td>
<td>1.7</td>
<td>1.6</td>
<td>1.5</td>
</tr>
<tr>
<td>4</td>
<td>2.6</td>
<td>2.5</td>
<td>2.4</td>
<td>2.3</td>
<td>2.2</td>
<td>2.1</td>
<td>2.0</td>
<td>1.9</td>
<td>1.8</td>
</tr>
</tbody>
</table>

The agreement of the calculated hierarchy of the DLEC constants and the experimental data suggests the idea to construct a model of universe with spacelike sections obtained by splicing of Sf- and Sfh-spheres, see subsection 4.2 and [23] (compact locally homogeneous universes with spatial sections homeomorphic to Seifert fibrations were considered in [24]). To this end we primarily have to reduce and reparametrize the family (4.4). First, in accordance with (4.5), we eliminate the Sf- and Sfh-spheres with odd numbers \(i\) introducing a new parameter \(n \in \{0, 4\}\) related to \(i = 2n\). Then (in certain cases) it is convenient also to use another parameter \(t = n - l, \ t \in [-4, 4]\). The resulting family of Sf- and Sfh-spheres is

\[
\begin{align*}
\{ \Sigma_n^{(l)} & = \Sigma \left( a_1^{(l)}, a_2^{(l)}, a_3^{(l)} \right) \mid n \in \{0, 4\}, l \in \{0, 4\} \} := \\
\{ \Sigma^{(n-t)} & (q_{2n-1}, p_{2n}, p_{2n+1}) \mid n \in \{0, 4\}, t \in \{-4, 4\} \},
\end{align*}
\]

which contains (4.3) as a subset for \(t = 0, \ i.e. \) when \(n = l\). The Euler numbers of this family of Sfh-spheres are given in the table 4.

Parameter \(t\) in our model is the discrete cosmological ‘time’, \(t = 0\) labelling the present state of the universe where an observer can determine the DLEC constants \(\alpha_l^{(l)}\) of the five \((l \in \{0, 4\}\) fundamental interactions (see table 2) remember that in this case \(l = n\)). The relation (4.2) readily yields good estimates of the DLEC constants \(\alpha_l^{(l)} = e^{\left( \Sigma_l^{(l)} \right)} \simeq (q_{2l+1})^{-2l}\). Remember that \(q_{2l+1} = p_1 \cdots p_{2l+1}\) is product of the first \(2l + 1\) prime numbers in \(\mathbb{N}\). Note that for \(l = 5\) there would be \(\alpha_5^{(5)} \simeq 1.4 \cdot 10^{-357}\) which is too small to be identified with a certain experimentally determined coupling constant, thus we impose the restriction \(n, l \in \{0, 4\}\) (see however [23] where this restriction is lifted when the open discrete cosmological models are described). Our hypothesis is that also in other \((t \neq 0)\) columns of table 3, the Euler numbers
can be related to the coupling constants as $e \left( \Sigma_{n}^{(l)} \right) \sim \alpha_{n}^{(l)}$. In the next subsection we discuss the realization of this hypothesis in our $BFE$-model and introduce a new concept of $(n, l)$-preinteractions which play the role of fundamental interactions in our purely topological approach. In the framework of our model, any low-energy $(n, l)$-preinteraction (which can be labeled, or better baptized, on the basis of the coupling constants hierarchy, see table 2) can be traced to its counterparts at $t \neq 0$ when one of the parameters ($n$ or $l$) not changes. E.g., the counterparts of the ‘cosmological’ $(4,4)$-preinteraction are $(4, l)$-preinteractions with $l \in \mathbb{Z}$, and $(n, 4)$-preinteractions ($n \in \mathbb{Z}$).

Though this approach leads to a hierarchy of the DLEC constants, it yields neither a description of the spacetime structure of universe, nor other its features, therefore we pass to framing a more constructive universe model in terms of 4-dimensional plumbed cobordisms with boundary components glued by splicing of $Sf$- and $Sfh$-spheres.

4.2 Construction of $pV$-cobordisms and respective $BFE$-systems

Let $\Gamma_{D}^{(l)}_{n}$ be the decorated graph shown in Figure 5.

This graph corresponds to the result of plumbing of elementary manifolds $P(\Gamma_{m}^{(l)})$ with boundaries homeomorphic to $Sfh$-spheres $\Sigma_{m}^{(l)}$, $m \in \mathbb{Z}$, minus the plumbed manifolds $P(\Gamma_{i}^{ch})$ corresponding to decorated chains $\Gamma_{i}^{ch}$, $i \in \mathbb{T}$. The notation $\setminus P(\Gamma_{i}^{ch})$ in the Figure 5 means subtraction of the 4-manifold $P(\Gamma_{i}^{ch})$ with the boundary $L(p_{i}, q_{i})$ from the 4-manifold $P(\tilde{\Gamma}_{n}^{(l)})$ where $\tilde{\Gamma}_{n}^{(l)}$ is the graph $\Gamma_{D}^{(l)}_{n}$ without decoration of chains. In other words,
the graph $\Gamma_D^{(l)}$ determine the pV-cobordism

$$X_D^{(l)} = P(\Gamma_D^{(l)}) = P(\widetilde{\Gamma}_n^{(l)}) \setminus \bigcup_{i=1}^{I} P(\Gamma_i^c)$$  \hspace{1cm} (4.7)$$

between $\mathbb{Z}$-homology sphere

$$M_n^{(l)} = \Sigma_0^{(l)} \longrightarrow \Sigma_1^{(l)} \longrightarrow \cdots \longrightarrow \Sigma_n^{(l)}$$  \hspace{1cm} (4.8)$$

obtained by consecutive splicing of Sfh-spheres $\Sigma_m^{(l)}$, $m \in \overline{0,n}$, and the set of lens spaces $\bigsqcup_{i=1}^{I} L(p_i, q_i)$. Thus

$$\partial X_D^{(l)} = -\bigcup_{i=1}^{I} L(p_i, q_i) \bigcup M_n^{(l)}.$$  \hspace{1cm} (4.9)$$

see section 2.1 and [9] about the connection between plumbing and splicing operations.

In order to calculate explicitly the intersection matrices $\Lambda_n^{(l)\text{ab}}$ and $\Lambda_n^{(l)\text{ab}}$, we apply a special case of splicing (4.8). Let us call as $\Sigma_m^{(l)} = \Sigma(a_1^{(l)} m, a_2^{(l)} m, a_3^{(l)} m)$ the $m$-level Sfh-sphere with given parameter $l$ [see (1.6) for definitions]. We shall use here splicing only between Sfh-spheres with the same (fixed) $l$ (though this is not obvious at the first sight, it is an important point of our algorithm), hence we sometimes omit $l$ for brevity, e.g. $a_j^{(l)} m = a_j m$, $j = 1, 2, 3$.

We also suppose that splicing is performed on the exceptional fibers $S_{a_3 m-1}$ and $S_{a_2 m}$, $m \in \overline{0, n}$, i.e. we consider a special case of (4.8) which reads

$$M_n^{(l)} = \Sigma_0^{(l)} S_{a_{30}} S_{a_{21}} \Sigma_1^{(l)} S_{a_{31}} S_{a_2} \cdots S_{a_{3 n-1}} S_{a_{2n}} \Sigma_n^{(l)}.$$  \hspace{1cm} (4.10)$$

This splicing diagram corresponds to the decorated graph shown in Figure 6 which is a special case of that shown in Figure 5.

Now we calculate rational intersection forms $\Lambda_n^{(l)\text{ab}}$ of pV-cobordisms $X_D^{(l)}$ which correspond to graphs $\Gamma_D^{(l)}$ for $n, l \in \overline{0, 4}$. These matrices are obtained in the natural basis $E_n^{(l)\text{a}}$ of $H^2 \left( X_D^{(l)}, \mathbb{R} \right)$ which is Poincaré dual to the natural basis $\left[ z_n^{(l)\text{a}} \right]$ of $H_2 \left( X_D^{(l)}, \partial X_D^{(l)}, \mathbb{R} \right)$ defined in [8] and briefly described
in subsection 2.1. Note that the duality of these bases is defined by the pairing operation as

\[ \langle E_n^{(l)a}, [z_n^{(l)b}] \rangle = \int_{z_n^{(l)b}} E_n^{(l)a} = \delta^a_b \]  

(4.11)

where the relative cycles \( \text{mod} \partial X_D^{(l)} \) \( z_n^{(l)b} \) are representatives of cohomological classes \([z_n^{(l)b}]\). Here \( \text{rank } H^2 \left( X_D^{(l)} n, \mathbb{R} \right) = \text{rank } H_2 \left( X_D^{(l)} n, \partial X_D^{(l)} n, \mathbb{R} \right) = n + 1 \). Hence the set of Abelian forms of the (3.7)-type \( BF E \)-system corresponding to the cobordism \( X_D^{(l)} n \), is \( \left( A_n^{(l)a}, B_n^{(l)a}, E_n^{(l)a} \right), a \in 0, n \).

Observation 3. The calculation of \( \Lambda_n^{(l)ab} \) was given in practical terms by Saveliev in [9]. A vertex \( v \) and a decorating oval (or circle) are called adjacent if the oval intersects an edge one of whose endpoints is \( v \). The generators of the cohomology group \( H^2 \left( X_D^{(l)} n, \mathbb{Z} \right) \) correspond to the vertices of \( \Gamma_D^{(l)} n \) outside of all decorating ovals. Given such a vertex \( v_a \) weighted by an integer \( e^a \), we have (see subsection 4.1 for basic definitions)

\[ \Lambda_n^{(l)aa} = e^a - \sum d_i, \quad d_i = \text{det } A \left( \Gamma_i^{\text{pch}} \right) / \text{det } A \left( \Gamma_i^{\text{ch}} \right). \]

Here \( A(\Gamma) \) is an integer intersection matrix (2.1) corresponding to the graph \( \Gamma \). The summation goes over all indices \( i \) such that \( v_a \) is adjacent to the oval containing \( \Gamma_i^{\text{ch}} \). As \( \Gamma_i^{\text{pch}} \) is denoted the portion of the chain \( \Gamma_i^{\text{ch}} \) obtained by by removing the vertex of \( \Gamma_i^{\text{ch}} \) adjacent to \( v_a \) and deliting all its adjacent edges. (Note that the determinant of an empty graph is equal to 1.) For any two generating vertices \( v_a \) and \( v_b \) connected by an edge inside \( \Gamma_D^{(l)} n \) away
from the decorating ovals \( \Lambda^{(l)ab}_n = 1 \). If these two vertices are adjacent to the same decorating oval enclosing a chain \( \Gamma^{ch}_k \), we have
\[
\Lambda^{(l)ab}_n = 1 / \det A (\Gamma^{ch}_k).
\]

Note that we use unnormalized Seifert invariants \([13]\) and subsection 2.1, thus all generating vertices \( v_a \) have in our diagrams weights \( e^a = 0 \), see figure 6.

To begin with, we consider as an example the subset of rational intersection matrices \( \Lambda^{(l)ab}_n \) corresponding to \( n = l \), i.e. the matrices \( \Lambda^{(l)ab}_l \), \( l \in \{0, 4\} \), \( a, b \in \{0, 4\} \), calculated with the use of Saveliev’s algorithm \([9]\) (see also \([7]\)):
\[
\begin{align*}
\left( \Lambda^{(0)}_0 \right) &= (\Lambda_{\text{strong}}) = \begin{bmatrix} 0.5 \end{bmatrix} \\
\left( \Lambda^{(1)}_1 \right) &= (\Lambda_{\text{elmag}}) = \begin{bmatrix} 9.5 \times 10^{-2} & -1.3 \times 10^{-2} \\
-1.3 \times 10^{-2} & 6.1 \times 10^{-4} \end{bmatrix} \\
\left( \Lambda^{(2)}_2 \right) &= (\Lambda_{\text{weak}}) = \begin{bmatrix} 9.8 \times 10^{-3} & -1.3 \times 10^{-4} & 0 \\
-1.3 \times 10^{-4} & 1.6 \times 10^{-6} & -6.7 \times 10^{-12} \\
0 & -6.7 \times 10^{-12} & 3.9 \times 10^{-17} \end{bmatrix} \\
\left( \Lambda^{(3)}_3 \right) &= (\Lambda_{\text{grav}}) = \begin{bmatrix} 1.9 \times 10^{-4} & -1.8 \times 10^{-8} & 0 & 0 \\
-1.8 \times 10^{-8} & 1.8 \times 10^{-12} & -1.4 \times 10^{-21} & 0 \\
0 & -1.4 \times 10^{-21} & 1.2 \times 10^{-27} & -2.2 \times 10^{-40} \\
0 & 0 & -2.2 \times 10^{-40} & 4.1 \times 10^{-53} \end{bmatrix} \\
\left( \Lambda^{(4)}_4 \right) &= (\Lambda_{\text{cosm}}) = \begin{bmatrix} 1.0 \times 10^{-7} & -4.9 \times 10^{-16} & 0 & 0 & 0 \\
-4.9 \times 10^{-16} & 2.4 \times 10^{-24} & -5.5 \times 10^{-41} & 0 & 0 \\
0 & -5.5 \times 10^{-41} & 1.5 \times 10^{-54} & -1.2 \times 10^{-76} & 0 \\
0 & 0 & -1.2 \times 10^{-76} & 4.7 \times 10^{-92} & -6.9 \times 10^{-119} \\
0 & 0 & 0 & -6.9 \times 10^{-119} & 1.0 \times 10^{-145} \end{bmatrix}
\end{align*}
\]

Note that all numbers in these matrices are rational; they are given here up to two significant digits.
In our model, this subset of matrices is associated with the contemporary \((t = n - l = 0)\) stage of universe, and with the low-energy sector of fundamental interactions (see the boldface column in table 3). Note that the matrix elements \(\Lambda_l^{(t)}\) \((t = l - 1)\) (boldface numbers) are subjected to an hierarchy similar to that of the Euler numbers \(e\left(\sum_n\right)\). Thus we shall name the \((l, l)\)-preinteraction (corresponding to \(\Lambda_l^{(ab)}\)) in the same way as the fundamental interaction labelled by the same parameter \(l\) in table 2. It is clear that the rational intersection matrices \(\Lambda_n^{(ab)}\) contain more numerical information about \((n, l)\)-preinteractions than the Euler numbers \(e\left(\sum_n\right)\). To disentangle this information, it is worth passing to the inverse matrices \(\Lambda_l^{(0)}\)

being integer intersection matrices:

\[
(\Lambda_l^{(0)})^{-1} =
\begin{bmatrix}
-2 & 10 & -660 & 2.4 \times 10^6 & -1.6 \times 10^{12} \\
10 & -30 & 1980 & -7.1 \times 10^6 & 4.9 \times 10^{12} \\
-660 & 1980 & -2310 & 8.3 \times 10^6 & -5.7 \times 10^{12} \\
2.4 \times 10^6 & -7.1 \times 10^6 & 8.3 \times 10^6 & -5.1 \times 10^5 & 3.5 \times 10^{11} \\
-1.6 \times 10^{12} & 4.9 \times 10^{12} & -5.7 \times 10^{12} & 3.5 \times 10^{11} & -2.2 \times 10^8
\end{bmatrix}
\]

\[
(\Lambda_l^{(1)})^{-1} =
\begin{bmatrix}
-6 & 124 & -2.8 \times 10^5 & 1.4 \times 10^{11} & -3.2 \times 10^{19} \\
124 & -930 & 2.1 \times 10^6 & -1.1 \times 10^{12} & 2.4 \times 10^{20} \\
-2.8 \times 10^5 & 2.1 \times 10^6 & -5.3 \times 10^6 & 2.7 \times 10^{12} & -6.1 \times 10^{20} \\
1.4 \times 10^{11} & -1.1 \times 10^{12} & 2.7 \times 10^{12} & -2.6 \times 10^{11} & 5.8 \times 10^{19} \\
-3.2 \times 10^{19} & 2.4 \times 10^{20} & -6.1 \times 10^{20} & 5.8 \times 10^{19} & -5.0 \times 10^{16}
\end{bmatrix}
\]

\[
(\Lambda_l^{(2)})^{-1} =
\begin{bmatrix}
-42 & 11172 & -1.9 \times 10^9 & 2.2 \times 10^{17} & -2.1 \times 10^{28} \\
11172 & -8.7 \times 10^5 & 1.5 \times 10^{11} & -1.7 \times 10^{19} & 1.6 \times 10^{30} \\
-1.9 \times 10^9 & 1.5 \times 10^{11} & -2.9 \times 10^{13} & 3.2 \times 10^{21} & -3.1 \times 10^{32} \\
2.2 \times 10^{17} & -1.7 \times 10^{19} & 3.2 \times 10^{21} & -6.8 \times 10^{22} & 6.6 \times 10^{33} \\
-2.1 \times 10^{28} & 1.6 \times 10^{30} & -3.1 \times 10^{32} & 6.6 \times 10^{33} & -2.5 \times 10^{33}
\end{bmatrix}
\]

\[
(\Lambda_l^{(3)})^{-1} =
\]

22
(4.12)
section 3. Inverse ones of the matrices $\Lambda_{4ab}^{(l)}$ were called cosmological (or coupling) constant matrices. Now this name acquires justification not only in the framework of the generalized BF-theory, but also from the viewpoint of a comparison with the experimental low-energy coupling constants. In fact, the boldface-written elements of the matrices $\Lambda_{4ab}^{(l)}$ correspond to coupling constants at $t = n - l = 0$ in the sense that $\alpha_{t}^{(l)} \sim e \left( \Sigma_{l}^{(l)} \right) = 1/|\Lambda_{4ll}^{(l)}|$. Hierarchically, to them are also related the next near-diagonal elements $\Lambda_{4l,l-1}^{(l)} = \Lambda_{4l-1,l}^{(l)}$.

These relations permit us to conclude that our rather exotic cosmological model may be associated with the really existing universe. To put it more strictly, the experimentally observed coupling constants hierarchy may be determined by topological invariants (intersection matrices) of cobordisms $X_{D_{n}^{(l)}}$ which we relate to Euclidean spacetime regions describing topology changes.

It is worth now speaking in more details about this evolutionary scheme (to the extent admissible in our model). Insofar as the $BFE$-system does not possess local degrees of freedom, the evolution ought to be understood as a sequence of topological (phase) transitions resulting to changes of the set of topological invariants $\Lambda_{n}^{(l)} := (\Lambda_{n}^{(ab)}, \Lambda_{n}^{(l)})$ as well as of $BFE$-system of forms $S_{n}^{(l)} := \left( A_{n}^{(a)} B_{n}^{(l)} E_{n}^{(l)a} \right)$. We consider the $BFE$-theory as a topological analogue of gravitation with cosmological constant [but without other (phenomenological) kinds of matter], thus our model is purely cosmological: One may imagine a table similar to table 3; now each cell characterized by the parameters $(n, t)$ or $(n, t = n - t)$, will be related to a cosmological model which involves the respective homology spheres $M_{n}^{(l)}$ as spacelike sections, and the $BFE$-system $S_{n}^{(l)}$. Instead of the Euler numbers $e \left( \Sigma_{n}^{(l)} \right) = 1/\alpha_{n}^{(l)}$, in this new table there should appear the intersection (cosmological constant) matrices $\Lambda_{n}^{(l)}$. The latter ones of course contain much more numerical information than a mere set of coupling constants $\alpha_{n}^{(l)} \sim e \left( \Sigma_{n}^{(l)} \right)$. We shall call the collection $U_{n}^{(l)} = \left\{ X_{D_{n}^{(l)}}^{(l)}; S_{n}^{(l)}; \Lambda_{n}^{(l)} \right\}$ primary $(n, l)$-universe, or $(n, l)$-preuniverse (reminiscent of pregeometry of John A. Wheeler).

From table 3 one can see that at $t = n - l = 0$ values of the Euler numbers
\( e \left( \Sigma_{l}^{(l)} \right) \) (boldface numbers) well represent the hierarchy of the DLEC constants (see also table 2). Since the information on the Euler numbers \( e \left( \Sigma_{n}^{(l)} \right) \) is contained in the intersection matrices \( \Lambda_{n}^{(l)} \), it is possible to conclude that the ensemble of \((n, l)\)-preuniverses \( \left\{ U_{n}^{(l)} \middle| n = l \right\} \) should correspond to the basic vacuum state of the present stage of the composite universe with five \((l \in 0, 4)\) BFE-systems. It is remarkable that this ensemble of preuniverses contains information about the hierarchy of dimensionless low-energy coupling constants of the real fundamental interactions (see boldface numbers in the intersection matrices given above). This hierarchy has in our model a purely topological origin, and it springs up before any local degrees of freedom are introduced. Thus the coupling constants which in most field theories have (semi-)phenomenological character, in our model are topological invariants describing the global properties of the spacetime (at least, in the Euclidean regime).

For other values of \( t \in -4, 4 \), the ensembles of \((n, l)\)-preuniverses
\[
\left\{ U_{n}^{(l)} \middle| n - l = t \right\}
\] (4.13)
describe the composite-universe states both of the ‘past’ \((t < 0)\) and the ‘future’ \((t > 0)\). Thus in our model, the ‘real’ universe is a superposition of \((n, l)\)-preuniverses at any fixed discrete time parameter \( t \). Let us identify the BFE-system \( S_{n}^{(l)} \) with the unique ‘fundamental interaction’ \([(n, l)-preinteraction] acting in the \((n, l)\)-preuniverse. Then from the modified version of table 3 (which we described above only verbally) it follows that the number of \((n, l)\)-preinteractions in the superposition of \((n, l)\)-preuniverses \(\{4, 13\}\) is growing from 1 to 5 when \( t \) changes from \(-4 \) to 0. The further growth of \( t \) from 0 to 4 results in decrease of the number of \((n, l)\)-preinteractions to one. Thus in our model there exists the possibility to realize the idea of unification of interactions, but in a rather unusual form (instead of successive symmetry breakdowns in the gauge theory, in our model a sequence of topology changes takes place).

The above-described scheme corresponds to a closed model of universe. Some details of the universe evolution with ‘inflationary’ stages and a possible treatment of unification ideas in closed and open cosmological models, are given in [23] on the basis of \( T_{0}\)-discrete space approach.
5 Conclusions

Let us now summarize the basic features of our model and some pending problems.

The ensemble of preuniverses $U^{(l)}_n$ proposed in this paper involves $BFE$-systems $S^{(l)}_n$ which possess the basic characteristic features of ordinary $BF$-systems $[1]$; this means that all physical fields may be gauged away locally. Thus the phase spaces (sets of classical solutions) are finite dimensional spaces [of the type of (3.1)] of zero-modes and have a purely topological sense. But $BFE$-models also involve the intersection matrices $\Lambda^{(l)}_n$ as analogues of coupling constants of fundamental interactions. The latter ones are however described at the purely topological level, thus we call them $(n,l)$-preinteractions. Since the intersection matrices are basic topological invariants of pV-cobordisms $X^{(l)}_{D_n}$, in our version of $BF$-theory coupling constants lose their usual phenomenological character and acquire the status of topological invariants of the spacetime manifold on which the $BFE$-system $S^{(l)}_n$ is constructed. Insofar as the experimentally observed ‘running coupling constants’ do depend on local characteristics of interactions (such as energy density), these ‘constants’ have even in the low-energy case their values different from those calculated in our model (see table 2). However already at the topological vacuum level (i.e. in the complete absence of local degrees of freedom of all fields including the gravitational one) the information about the hierarchy of (at least) the DLEC constants of real fundamental interactions is contained.

This situation may be interpreted as a generalization of the Mach principle in the sense of a determining influence of the global topological (cosmological) characteristics of the universe on the local properties of universal interactions ‘switched on’ in this universe. It is appropriate to note that in our model all (pre)interactions bear ‘cosmological traces’, that is, each preinteraction is forming a certain preuniverse $U^{(l)}_n$ where it is the only one which is switched on. The spacetime topology of this preuniverse is completely determined by the cosmological constant matrix $\Lambda^{(l)}_n$ representing the rational intersection form of the pV-cobordism $X^{(l)}_{D_n}$. The real universe involving several interactions, is treated as a superposition of preuniverses $U^{(l)}_n$ with $n-l = t = \text{const}$. The problem still is how to determine ‘ordinary’ fields with their local degrees of freedom in conformity with the topological structure of
The preinteractions unification concept qualitatively differs from the usual scheme of unification accepted in gauge theories. For example, the set of \((n, l)\)-preinteractions found for \(t = n - l\) is replaced by another set of \((n', l')\)-preinteractions. If \(t \leq 0\) and \(n' - l' = t - 1\), the latter set contains one preinteraction less than the former one. The number of preinteractions decreases by a shift to the left from \(t = 0\) in table 3 (or in the analogue of this table verbally described in subsection 4.2) as well.

Elementary pV-cobordisms \(X_D^{(l)}\) and \(X_{D_{n+1}}^{(l)}\) can be pasted into cobordisms describing topology changes between \(\mathbb{Z}\)-homology spheres. This is accompanied by creation and annihilation of certain sets of disjoint lens spaces \(L_{\text{out}}\) and \(L_{\text{in}}\). Pasting of these pV-cobordisms is naturally performed along sets of pairwise homeomorphic lens spaces \(L_n^{(l)} \subset \partial X_D^{(l)}\) and \(L_{n+1}^{(l)} \subset \partial X_{D_{n+1}}^{(l)}\) yielding the pV-cobordism

\[
X_{D_{n,n+1}}^{(l)} = -X_D^{(l)} \bigcup X_{D_{n+1}}^{(l)}.
\]

The boundary of this cobordism,

\[
\partial X_{D_{n,n+1}}^{(l)} = - \left( M_n^{(l)} \bigcup L_{\text{in}} \right) \bigcup \left( M_{n+1}^{(l)} \bigcup L_{\text{out}} \right),
\]

contains both \(\mathbb{Z}\)-homology spheres \(M_n^{(l)} \subset \partial X_D^{(l)}\), \(M_{n+1}^{(l)} \subset \partial X_{D_{n+1}}^{(l)}\) and sets of mutually non-homeomorphic lens spaces \(L_{\text{in}}, L_{\text{out}}\). Thus the pV-cobordism \(X_{D_{n,n+1}}^{(l)}\) describes the topology change

\[
M_n^{(l)} \bigcup L_{\text{in}} \longrightarrow M_{n+1}^{(l)} \bigcup L_{\text{out}}.
\]

Here one confronts, however, with the still open problem of the junction of the BF\(E\)-systems \(S_n^{(l)}\) and \(S_{n+1}^{(l)}\) which are defined on pV-cobordisms \(X_D^{(l)}\) and \(X_{D_{n+1}}^{(l)}\) respectively.

It is interesting that the intersection matrices \(\Lambda_n^{(l)}\) always have signature \((- + \cdot \cdot \cdot +)\) for any values of \(n\) and \(l\). This may hint at the possibility to construct a discrete model of a spacetime based on Lorentz-signature lattices spanned on eigenvectors of intersection matrices, the dimensionality of any lattice being \(n + 1\). Realization of this approach should be based on a study of the discrete phase space (3.2) containing richer cohomological information about the pV-cobordisms \(X_D^{(l)}\) than the real vector space (3.1).
Finally, note that in addition to the direct analogues of coupling constants [that is, topological charges (4.12)] the intersection matrices $\Lambda(l)_n$ contain a large amount of numerical information about $(n,l)$-preuniverses, so that these matrices could be considered as their numerical ‘code’, maybe (see tables 2 and 3) a ‘code’ of our proper universe as well. This information is encoded in the topology of the ordinary 3-sphere’s ‘nearest relatives’, namely in topology invariants of $\mathbb{Z}$-homology spheres being spacelike sections of spacetime manifolds.

In fact, we are greatly baffled by the strange results to which led an application of quite an abstract and fundamental part of mathematics, the algebraic topology, and we feel it to be appropriate to conclude this paper with comforting and reassuring words of Eugene P. Wigner: “...the mathematical formulation of the physicist’s often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena. This shows that the mathematical language has more to commend it than being the only language which we can speak; it shows that it is, in a very real sense, the correct language” [26].

Acknowledgments

We thank Nikolai Saveliev for kind interest to our work and for helpful advices and questions.

References


