Energy Extraction from Higher Dimensional Black Holes and Black Rings

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We analyze the energy extraction by the Penrose process in higher dimensions. Our result shows the efficiency of the process from higher dimensional black holes and black rings can be rather high compared with that in four dimensional Kerr black hole. In particular, if one rotation parameter vanishes, the maximum efficiency becomes infinitely large because the angular momentum is not bounded from above. We also apply a catastrophe theory to analyze the stability of black rings. It indicates a branch of black rings with higher rotational energy is unstable, which should be a different type of instability from the Gregory-Laflamme’s one.

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I. INTRODUCTION

In recent years, a brane world scenario has been extensively studied 123. One characteristic feature of brane world is that mini black holes might emerge in the accelerator 4. If there exist large extra dimensions, the fundamental gravity scale $M_D$ could be order of TeV, much smaller than $M_4 \sim 10^{19}$ GeV 5. Since the Schwarzschild radius of particles in the collider becomes smaller than the particle radius, mini black holes could be produced in the accelerators. If we are able to detect the production and evaporation of black holes, it would be not only the direct evidence for the Hawking radiation, but also the stepping stone to a unified theory.

For this reason, the basic study of higher dimensional black holes is now of great importance. A naive estimate shows that mini-black holes can be described by the classical solutions of the vacuum Einstein equations. We may be able to ignore the effects of brane. In the colliders, black holes would be produced with rotation in general.

The rotating black holes in $D$ dimensional space-times with $S^{D-2}$ topology were first derived by Myers and Perry 6. The black holes in higher dimensions can have arbitrarily large angular momentum unlike the case in 4-dimensional Kerr black hole. The amazing discovery is the rotating black ring solution by Emparan and Reall 7, whose topology of the event horizon is $S^1 \times S^2$. Two black rings and one MP black hole with the same mass and angular momentum can coexist. While in four dimensions, neutral charged black holes are completely specified by their mass and angular momentum, and moreover non spherical topology is forbidden 4,5. This is the first counterexample that uniqueness theorem does not hold in higher dimensions. It is then important to examine the property of the black ring and black holes in detail.

In this paper, we focus our attention on the energy extraction by the Penrose process 81011. Using such a formalism, we discuss the efficiency of the energy extraction.

The remainder of the paper is organized as follows: The geometry of the higher dimensional black holes and black ring is summarized in Section II Section III consists of the main discussions on the energy extraction from higher dimensional black holes and black rings. As for 5-dimensional objects, adopting a catastrophe theory, we analyze their stability in Section IV. Our conclusions and some remarks follow in section VII.

II. HIGHER DIMENSIONAL BLACK HOLES AND BLACK RINGS

We first summarize black hole and black ring solutions in higher dimensions, which we discuss here.

A. MP black hole solution

Myers and Perry found an exact solution of the Einstein equations in arbitrary $D$ dimensional space-time 6. It represents a rotational black hole, which is a generalization of the four-dimensional Kerr black hole. The solution is described by a different form depending on whether a space-time dimension is even or odd. Then we write the solutions in order.

1. Even dimensions ($D = 2(d + 1)$)

The metric in even dimensions is given by:

$$ds^2 = -dt^2 + r^2 d\alpha^2 + \sum_{i=1}^{d} \left( r^2 + a_i^2 \right) \left( d\mu_i^2 + \mu_i^2 d\phi_i^2 \right) + \frac{M_r}{\Pi F} \left( dt + \sum_{i=1}^{d} a_i \mu_i^2 d\phi_i \right)^2 + \frac{\Pi F}{\Pi - M_r} dt^2, \quad (2.1)$$

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where

\[ F = 1 - \sum_{i=1}^{d} \frac{\alpha_{i}^{2}}{r^{2} + a_{i}^{2}}, \quad (2.2) \]

\[ \Pi = \prod_{i=1}^{d} (r^{2} + a_{i}^{2}), \quad (2.3) \]

\[ \sum_{i=1}^{d} \mu_{i}^{2} + \alpha_{i}^{2} = 1, \quad (2.4) \]

with \( d \equiv D/2 - 1 \). The parameters \( M \) and \( a_{i} \) are related to the mass \( M \) and angular momenta \( J_{i} \) as

\[ M = \frac{D-2}{16\pi G} A_{(D-2)} M, \quad (2.5) \]

\[ J_{i} = \frac{1}{8\pi G} A_{(D-2)} M a_{i}, \quad (i = 1, \ldots, d), \quad (2.6) \]

where \( G \) and \( A_{(D-2)} \) are \( D \)-dimensional gravitational constant and the area of a unit \( (D - 2) \)-sphere, which is given by

\[ A_{(D-2)} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}, \quad (2.7) \]

respectively.

The event horizon appears where

\[ g^{rr} = \frac{\Pi - M r}{\Pi F} \quad (2.8) \]

vanishes. If at least one rotation parameter set to zero, for example \( a_{1} = 0 \), the equation for horizon is given by

\[ \Pi - M r = r^{2} \left( \prod_{i \geq 2} (r^{2} + a_{i}^{2}) - \frac{M}{r} \right) = 0. \quad (2.9) \]

In the case of \( d \geq 2 \), i.e. \( D \geq 6 \), Eq. (2.9) has a positive root independent of the magnitude of \( a_{i} \). We then find a regular black hole solution albeit the angular momenta are arbitrarily large. This is one of typical features of higher dimensional black holes.

2. Odd dimensions \((D = 2d + 1)\)

In odd dimensions, the metric of a rotating black hole is slightly changed from Eq. (2.1), which is given by

\[
\begin{align*}
\text{ds}^2 &= -dt^2 + \sum_{i=1}^{d} \left( r^2 + a_i^2 \right) \left( \sum_{i=1}^{d} \mu_i^2 r d\phi_i \right) \\
&+ \frac{M r^2}{\Pi F} \left( dt + \sum_{i=1}^{d} a_i \mu_i^2 r d\phi_i \right)^2 + \frac{\Pi F}{\Pi M r^2} dr^2,
\end{align*}
\]

with

\[ \sum_{i=1}^{d} \mu_i^2 = 1, \quad (2.11) \]

where the definition of \( \Pi \) and \( d = (D - 1)/2 \). We also find that if at least two angular momenta set to zero, the remaining angular momenta can be arbitrarily large for \( d \geq 3 \), i.e. \( D \geq 7 \) as in the case of even dimensions, because the equation for horizon is now

\[ \Pi - M r^2 = r^4 \left( \prod_{i \geq 3} (r^2 + a_i^2) - \frac{M}{r^2} \right) = 0. \quad (2.12) \]

A five-dimensional black hole is exceptional, because there is an upper bound for the angular momentum. We also write down a five-dimensional black hole solution with two rotation parameters \( a \) and \( b \) in the Boyer-Lindquist coordinates, which is given by

\[
\begin{align*}
\text{ds}^2 &= -dt^2 + \frac{\rho^2 r^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\
&+ \frac{M}{\rho^2} (dt + a \sin^2 \theta d\phi + b \cos^2 \theta d\psi)^2 \\
&+ (r^2 + a^2) \sin^2 \theta d\phi^2 + (r^2 + b^2) \cos^2 \theta d\psi^2,
\end{align*}
\]

where

\[
\rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad (2.13)
\]

\[
\Delta = (r^2 + a^2)(r^2 + b^2) - M r^2. \quad (2.14)
\]

The horizon appears where \( \Delta = 0 \), which gives the location of the horizons, i.e.

\[
r_+^2 = \frac{1}{2} \left[ M - (a^2 + b^2) \pm \sqrt{(M - (a + b)^2)(M - (a - b)^2)} \right]. \quad (2.16)
\]

The sign change of rotation parameters \( a, b \) simply reverses the direction of rotation. The condition for the existence of an event horizon is

\[ M \geq (|a| + |b|)^2. \quad (2.17) \]

The outer and inner horizons coincide when \( M = (|a| + |b|)^2 \). The area of the event horizon is given by

\[ A_H = \frac{2\pi(r_+^2 + a^2)(r_+^2 + b^2)}{r_+}. \quad (2.18) \]

The horizon vanishes if one of the angular parameters set to zero and the other approaches the extreme value (e.g. \( b = 0 \) and \( a^2 \rightarrow M \)), which corresponds to the appearance of a naked singularity. When \( (|a| + |b|)^2 \rightarrow M \) with \( a \neq 0 \) and \( b \neq 0 \), this corresponds to the extremal black hole with non-zero surface area and vanishing temperature.

B. a black ring solution

Emparan and Reall found a new exact solution of the vacuum Einstein equations in five dimensions, which is asymptotically flat, stationary and regular on and outside the event horizon \( \mathcal{I} \). This solution describes a black “hole” with the ring topology \( S^1 \times S^2 \), which is called a black ring.
The metric of a rotating black ring is written in the following form \cite{12,13}.

\[
ds^2 = -\frac{F(x)}{F(y)} \left( dt + R \sqrt{\frac{\lambda \nu(1 + \lambda)}{1 + \nu}} (1 + y) d\psi \right)^2 + \frac{R^2}{(x - y)^2} \left[ -F(x) \left( \frac{(1 + \lambda)G(y)}{(1 + \nu)^2} dy^2 + \frac{F(y)}{G(y)} dy^2 \right) + F(y)^2 \left( \frac{dx^2}{G(x)} + \frac{G(x)(1 + \lambda)}{F(x)(1 + \nu)} d\phi^2 \right) \right],
\]

where

\[
F(\xi) = 1 - \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 - \nu \xi).
\]

Here we have adopted a form of the C-metric introduced in \cite{12}.

The parameter \( R > 0 \) is interpreted as a “radius” of the ring. The parameter \( \nu \), related to a “thickness” of the ring \cite{12}, takes the value in \( 0 < \nu < 1 \). Both angular coordinates \( \phi \) and \( \psi \) have period \( 2\pi \). \( x \) and \( y \) take the value in the range

\[
-1 \leq x \leq 1, \quad -1 < y^{-1} < \lambda.
\]

\( x = +1 \) corresponds to the inside equatorial plane of the ring, while \( x = -1 \) denotes the outside of the equatorial plane. The spatial infinity corresponds to \( x \neq -1 \) and \( y \rightarrow -1 \), while the symmetric axis is given by \( x = +1 \) and \( y \rightarrow -1 \). A space-time singularity is located at \( y = \lambda^{-1} \).

The event horizon and the ergosurface are given by \( y = \nu^{-1} \) and \( y^{-1} = 0 \), respectively. We find a regular black ring solution if we impose some relation between \( \lambda \) and \( \nu \). This metric form also includes a black hole solution for a different relation of \( \lambda \) and \( \nu \). Then the parameter \( \lambda \) determines the topology of the objects, i.e.,

\[
\lambda = \begin{cases} 
1 & \text{(a black hole)} \\
\frac{2\nu}{1 + \nu^2} & \text{(a black ring).} 
\end{cases}
\]

The solution of \( \lambda = 1 \) corresponds to the five-dimensional MP black hole with one rotation parameter. In fact, Eq. \eqref{2.19} is transformed into Eq. \eqref{2.13} with \( b = 0 \), by the coordinate transformation \cite{12}.

\[
r^2 = M \frac{(1 - y)(1 - \nu x)}{(1 + \nu)(x - y)}, \quad \cos^2 \theta = \frac{(1 - y)(1 + x)}{2(x - y)},
\]

with

\[
M = \frac{4R^2}{1 + \nu}, \quad a = \frac{2\sqrt{2}\nu R}{1 + \nu}.
\]

The mass, angular momentum, surface gravity, angular velocity and horizon area are given by \cite{12}

\[
M = \frac{3\pi R^2 \lambda(\lambda + 1)}{4G} \left( \nu + 1 \right), \quad \mathcal{J} = \frac{\pi R^3 \sqrt{\lambda \nu (\lambda + 1)^{5/2}}}{(1 + \nu)^2}, \quad \kappa = \frac{1}{2R} \lambda^{1/2} (\lambda - \nu)^{1/2}, \quad \Omega_H = \frac{1}{R} \sqrt{\frac{\nu}{\lambda(1 + \lambda)}}, \quad \mathcal{A}_H = 8\pi^2 R^3 \lambda^{1/2} (\lambda + \nu)^{3/2}/(1 + \nu)^2.
\]

The horizon area vanishes as \( \nu \rightarrow 1 \), and in this limit, a naked singularity appears. The opposite limit of \( \nu \rightarrow 0 \) leads a solution without an angular momentum. This is a static black ring solution with a conical singularity \cite{14}.

We introduce a dimensionless reduced spin parameter \( j \) by

\[
j^2 = \frac{27\pi}{32G} \mathcal{J}^2.
\]

Using Eqs. \eqref{2.20} and \eqref{2.21}, we find

\[
j^2 = \begin{cases} 
\frac{2\nu}{1 + \nu} & \text{(a black hole)} \\
\frac{(1 + \nu)^3}{8\nu} & \text{(a black ring).}
\end{cases}
\]

There exist two black rings and a black hole with the same mass and spin in the range of \( 27/32 < j^2 < 1 \). Two rings are distinguished by their area (entropy) \cite{11}.

### III. ENERGY EXTRACTION

In this section, we discuss the Penrose process \cite{3,11}, by which we can extract a rotation energy from a black hole or a black ring. In the Penrose process, an incident particle with the \( D \)-momentum \( p_{(0)}^\mu \) is supposed to split into two fragments (first and second particles with the \( D \)-momenta \( p_{(1)}^\mu \) and \( p_{(2)}^\mu \) in the ergoregion. One of them crosses the event horizon, while the other escapes to infinity.

In order to calculate the efficiency of energy extraction by the Penrose process, we consider a very simple case, that is, all particles are confined on one plane, which we call an “equatorial” plane. In higher dimensions, there are several planes on which a particle’s trajectory can be confined. At the point of split, the total \( D \)-momentum is conserved as

\[
p_{(0)}^\mu = p_{(1)}^\mu + p_{(2)}^\mu.
\]

The momenta of three particles \( p_{(I)}^\mu \) (\( I = 0, 1, 2 \)) are non-spacelike and hence should lie inside a local light cone.

The orbit of the particle moving on a plane is described by two dimensional coordinates, i.e. radial and one angular coordinates \( (r \text{ and } \phi) \). Then we can write the momentum of a particle along the geodesic \( \gamma \) as

\[
p_{\gamma} = p^\mu \frac{\partial}{\partial x^\mu} = p^t \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial v} + \Omega \frac{\partial}{\partial \phi} \right),
\]
Suppose that the first particle crosses the horizon with an asymptotic infinity observer (Ω) takes the value in the range of Ω < Ω ≤ Ω⁺, where

\[ \Omega \pm = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} \]  

with

\[ \omega = \frac{g_{t\phi}}{g_{\phi\phi}} \]  

which denotes the angular velocity of a locally nonrotating observer at a given radius r. The conservation of energy (E = -pᵗ X) and angular momentum (L = pᵗ Ω) are written as

\[ p^{(0)}_t X^{(0)} = p^{(1)}_t X^{(1)} + p^{(2)}_t X^{(2)} \]  

\[ p^{(0)}_t \Omega^{(0)} = p^{(1)}_t \Omega^{(1)} + p^{(2)}_t \Omega^{(2)} \]  

(3.8)  

(3.9)

Then the angular velocity with respect to an asymptotic infinity observer at a given radius r is

\[ \Omega_{(1)} = \Omega^- \]  

\[ \Omega_{(2)} = \Omega^+ \]  

(3.14)

and from Eqs. 3.10 and 3.11,

\[ \Omega_{(0)} = \frac{-g_{t\phi}(1 + g_{tt}) + \sqrt{(1 + g_{tt})(g_{t\phi}^2 - g_{tt} g_{\phi\phi})}}{g_{\phi\phi} + g_{t\phi}^2} \]  

(3.15)

The maximum efficiency of the Penrose process is achieved when the radial velocities \( v^{(1)} \) vanish, assuming \( p^{(1)}_t \) and \( p^{(2)}_t \) are null.

The maximum efficiency is then given by

\[ \eta_{\text{max}} = \chi_{\text{max}} - 1 = \frac{(\Omega_{(0)} - \Omega^-)(g_{tt} + g_{t\phi} \Omega^+)}{(\Omega^+ - \Omega^-)(g_{tt} + g_{t\phi} \Omega_{(0)}) - 1} \leq \frac{g_{\phi\phi}(\sqrt{1 + g_{tt}^2} + g_{t\phi}^2)}{2g_{\phi\phi}\sqrt{1 + g_{tt}^2}} - 1 \]  

(3.16)

where the equality holds when the split occurs at the horizon.

In four dimensions, the rotation parameter of Kerr black hole has an upper bound a ≤ M, then we find

\[ \eta_{\text{max}} = \chi_{\text{max}} - 1 = \frac{1}{2} \left( \sqrt{\frac{2M}{r_+}} - 1 \right) \leq \frac{1}{2}(\sqrt{2} - 1) \sim 20.7\% \]  

(3.17)

The equality holds when the split occurs at the event horizon and the black hole is extreme (a = M). Then we find the maximum efficiency of the Penrose process in the Kerr space-time is about 20.7%. This recovers the results in [14, 15].

Now we discuss the efficiency of the Penrose process in higher dimensions for each solution in order.
A. Black Hole

1. Even dimensions \((D = 2(d + 1))\)

The coordinates \(\mu_i\) and \(\alpha\) in the metric (2.1), are written explicitly by colatitude angle \(\theta_i\) as follows:

\[
\begin{align*}
\mu_1 &= \sin \theta_1 \\
\mu_2 &= \cos \theta_1 \sin \theta_2 \\
\vdots \\
\mu_d &= \cos \theta_1 \cos \theta_2 \cdots \sin \theta_d \\
\alpha &= \cos \theta_1 \cos \theta_2 \cdots \cos \theta_d.
\end{align*}
\] (3.18)

Suppose that the orbits of particles are constrained on the “equatorial” plane \(\theta_1 = \theta_2 = \cdots = \theta_d = \pi/2\). Note that since each coordinate \(\mu_i\) is an equal footing, we can exchange the numbering of \(\mu_i\). Then we find \(d\) “equatorial” planes, on which the orbits of particles are confined. As a result, we can replace \(a_1\) in the result obtained below with any other rotation parameter \(a_i\).

Substituting the metric (2.1) at the horizon \(r_+\) into Eq. (3.10), we find the maximum efficiency \(\eta_{\text{max}}\) as

\[
\eta \leq \eta_{\text{max}} = \frac{1}{2} \left( \sqrt{1 + \frac{a^2}{r_+^2}} - 1 \right). \tag{3.19}
\]

The black hole event horizon \(r_+\) is given by

\[
\prod_i (r_i^2 + a_i^2) - M r_+ = 0. \tag{3.20}
\]

In six or higher dimensions, the angular parameter \(a_1\) can take arbitrarily large value if at least one remaining rotation parameter is zero. Thus in this case, we are able to extract quite a lot of energy from black holes.

We shall see this fact more explicitly. For simplicity, we first consider the case of \(D = 6\). In this case, we have two rotation parameter. We set \(a_1 = a, a_2 = \beta a\) \((a > 0, \beta > 0)\). \(\beta\) denotes the ratio of the second rotation parameter to the first one. Then the equation for the horizon leads

\[
(x_+^2 + 1)(x_+^2 + \beta^2) - M x_+ = 0, \tag{3.21}
\]

where \(x_+ = r_+/a, \tilde{M} = M/a^3\). The horizon is larger than the critical value

\[
x_{c\text{r}}^2 \equiv \frac{1}{6} \left[ -(1 + \beta^2) + \sqrt{(1 + \beta^2)^2 + 12 \beta^2} \right], \tag{3.22}
\]

The rotation parameter is limited, when \(\beta\) is fixed, as

\[
a^3 \leq \frac{x_{c\text{r}}}{(x_{c\text{r}}^2 + 1)(x_{c\text{r}}^2 + \beta^2)} M. \tag{3.23}
\]

From Eq. (3.19), we obtain the maximum efficiency in terms of \(\beta\) as

\[
\eta_{\text{max}} = \frac{1}{2} \left( \sqrt{1 + \frac{a^2}{x_{c\text{r}}^2}} - 1 \right) = \frac{1}{2} \left( \sqrt{\frac{1}{2\beta^2} (1 + 3\beta^2 + \sqrt{(1 + \beta^2)^2 + 12 \beta^2})} - 1 \right). \tag{3.24}
\]

Suppose that the orbits of particles are constrained on the “equatorial” plane \(\theta_1 = \theta_2 = \cdots = \theta_d = \pi/2\). Note that since each coordinate \(\mu_i\) is an equal footing, we can exchange the numbering of \(\mu_i\). Then we find \(d\) “equatorial” planes, on which the orbits of particles are confined. As a result, we can replace \(a_1\) in the result obtained below with any other rotation parameter \(a_i\).

\begin{table}[h]
\centering
\caption{The maximum efficiency \(\eta_{\text{max}}\) of the Penrose process in MP black hole with \(D = 6\). \(a_1\) and \(a_2\) are two rotation parameters. \(\beta = a_2/a_1\) and \(r_+\) is the horizon radius.}
\begin{tabular}{|c|c|c|c|}
\hline
\(a_1\) & \(a_2\) & \(a_1 \gg a_2\) & \(a_1 \ll a_2\) \\
\hline
\(\beta\) & \(\rightarrow \infty\) & \(1\) & \(0\) \\
\(r_+\) & \(r_+ > a_1\) & \(r_+ > \frac{a_1}{\sqrt{3}}\) & \(r_+ > \beta a_1\) \\
\(a_1\) & \(a_1 \leq \left( \frac{M}{2\beta^2} \right)^{1/3}\) & \(a_1 \leq \left( \frac{3\sqrt{3}M}{16} \right)^{1/3}\) & \(a_1 \leq \left( \frac{M}{2\beta} \right)^{1/3}\) \\
\(\eta_{\text{max}}\) & \(\frac{1}{2} (\sqrt{2} - 1)\) & \(\frac{1}{2}\) & \(\frac{1}{2\beta} (\rightarrow \infty)\) \\
\hline
\end{tabular}
\end{table}

We summarize some typical cases in Table 1. We find the maximum efficiency diverges as \(1/(2\beta)\) when the second rotation parameter \(a_2\) decreases to zero, i.e. \(a_2 = \beta a_1 \rightarrow 0(\beta \rightarrow 0)\). On the other hand, even if the first rotation parameter \(a_1\) is very small as \(a_1 = a_2/\beta \rightarrow 0(\beta \rightarrow \infty)\), the efficiency does not vanish. It gives the same efficiency as that in four dimensional black hole. Although this value is not so large, we obtain the large efficiency if \(a_2\) is large enough by putting the particles on the different equatorial plane, i.e. \((\mu_2, \phi_2)\)-plane. This result is also obtained just by the exchange of coordinates \((\mu_i, \phi_i) (i = 1, 2)\). As the result, if one rotation parameter is enough large, we can extract rotational energy by any amount by the Penrose process.

One interesting observation is the case of the same rotation parameters : \(a_1 = a_2 = a(\beta = 1)\). In this case, the horizon radius is limited as \(r_+ > a/\sqrt{3}\). The efficiency is finite, i.e. \(\eta_{\text{max}} = 1/2\).

This is true for any even dimensions. If all rotation parameters are the same, i.e. \(a_i = a(i = 1, \cdots, d)\), we find the maximum efficiency by

\[
\eta_{\text{max}} = \frac{1}{2} \left( \sqrt{2d - 1} \right). \tag{3.25}
\]

In this case the horizon and rotation parameters are limited as

\[
r_+ > \frac{a}{\sqrt{2d - 1}} \tag{3.26}
\]

\[
a < \left( \frac{(2d - 1)^{-d-1/2}}{(2d)^{d-1}} \right)^{1/(2d-1)} M^{1/(2d-1)}. \tag{3.27}
\]

The efficiency becomes larger as \(d\) increases, but not so much.

2. Odd dimensions \((D = 2d + 1)\)

The efficiency is obtained by almost the same fashion as that in even dimensions. The coordinates \(\mu_i\) are explicitly
written such that
\[
\mu_1 = \sin \theta_1 \\
\mu_2 = \cos \theta_1 \sin \theta_2 \\
\vdots \\
\mu_{d-1} = \cos \theta_1 \cos \theta_2 \cdots \sin \theta_{d-1} \\
\mu_d = \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{d-1}.
\]
(3.28)

If the orbit particle is constrained on the “equatorial” plane \(\{\theta_1 = \theta_2 = \cdots = \theta_{d-1} = \pi/2\}\), in five dimensions [12], we obtain the efficiency of the Penrose process from Eqs. (3.10) and (2.10),
\[
\eta = \frac{1}{2} \left( \sqrt{1 + \frac{a_1^2}{r_+^2}} - 1 \right),
\]
(3.29)
where the horizon radius \(r_+\) is given by
\[
\prod_i (r_+^2 + a_i^2) - M r_+^2 = 0.
\]
(3.30)

As in the case of even dimensions, \(a_i\) can be arbitrarily large if \(d \geq 7\) and at least two rotation parameters vanish, we could have more productive energy extraction from a higher dimensional black hole than from four dimensional Kerr black hole. However, if all rotation parameters are the same as \(a_i = a (i = 1, \cdots, d)\), we find the maximum efficiency by
\[
\eta_{\text{max}} = \frac{1}{2} \left( \sqrt{d} - 1 \right).
\]
(3.31)

In this case the horizon and rotation parameters are limited as
\[
r_+ > \frac{a}{\sqrt{d-1}}
\]
(3.32)
\[
a < \left( \frac{(d-1)^{d-1}}{d!} \right)^{1/2(d-1)} M^{1/(2(d-1))}.
\]
(3.33)

The efficiency becomes larger as \(d\) increases, but not so much.

We shall give the detail in five dimensions with two rotation parameters \(a = a_1\) and \(b = a_2\). The equation for the horizon is now
\[
(x_+^2 + 1)(x_+^2 + \beta^2) - \hat{M} x_+^2 = 0,
\]
(3.34)
where \(x_+ = r_+/a\), \(\beta = b/a\), and \(\hat{M} = M/a^2\). We find that the horizon radius is limited as \(x_+ \geq \beta^{1/2}\). Then we obtain the maximum efficiency as
\[
\eta_{\text{max}} = \frac{1}{2} \left( \sqrt{1 + \frac{\beta}{\beta}} - 1 \right).
\]
(3.35)

This result is summarized in Table 2.

### B. Black Ring

Since infinity is at \(x = y = -1\), we consider to inject a particle from \(x = -1\) direction. Writing down the geodesic equation, we can see the particle is constrained \(x = -1\) plane through the orbit. In Appendix A we summarize a motion of a particle on the equatorial plane in a black ring spacetime.

The maximum efficiency of the process is given from Eq. (3.10) as
\[
\eta_{\text{max}} = \frac{1}{2} \left( \sqrt{1 + \nu(1 + \lambda)} - 1 \right)
\]
(3.36)

Since \(\nu\) takes value in the range \(0 < \nu < 1\), the efficiency is greater than that from 4-dimensional Kerr black hole. This black hole solution corresponds to MP black hole in five dimensions with one rotation parameter \(a\). In the limit of \(\nu \rightarrow 1\), we find maximum efficiency \((\eta_{\text{max}} \rightarrow \infty)\). Compared a black ring with a black hole, the efficiency of a black ring is larger than that of a black hole for the same value of \(\nu\). However, if we evaluate them for the same reduced angular momentum, i.e. \(j\), we find the opposite result (see Fig. 2) in the common range of \(27/32 < j^2 < 1\). From Fig. 2, we also find an interesting feature for a black ring. The efficiency for larger angular momentum \((j^2 > 1)\) decreases with respect to \(j\) to some finite constant. The larger angular momentum does not provide the larger efficiency. We will discuss its reason later.

### IV. IRREDUCIBLE MASS AND ROTATIONAL ENERGY

The relationship between the black hole mechanics and thermodynamics is now well established [10]. The integrated mass formula holds for higher dimensional MP black holes and also black rings [12].
\[
M = \frac{d - 2}{d - 3} \left( \frac{\kappa}{8 \pi G} A + \Omega_R J \right).
\]
(4.1)

The equation (4.1) in Kerr space-time is first found by Smarr [17]. The mass of the neutral black hole consists

| Table II: The maximum efficiency \(\eta_{\text{max}}\) of the Penrose process in MP black hole with \(D = 5\). \(a\) and \(b\) are two rotation parameters. \(\beta = b/a\) and \(r_+\) is the horizon radius. |
|---------------------|---------------------|---------------------|
| \(\beta\)          | \(r_+\)            | \(a \ll b\)        |
| \(\rightarrow \infty\) | \(r_+ > \sqrt{3}a\) | \(1\)               |
| \(a \leq \frac{\sqrt{M}}{\beta}\) | \(r_+ > a\)        | \(\rightarrow 0\)   |
| \(a \leq \sqrt{M}\beta\) | \(a \leq \frac{\sqrt{M}}{2}\) | \(a \leq \sqrt{M}\) |
| \(\eta_{\text{max}}\) | \(\frac{1}{4\beta}(\rightarrow 0)\) | \(\frac{1}{2}\)       |
| \(\beta = \frac{b}{a}\) | \(\frac{(\sqrt{2} - 1)}{\sqrt{2}}\) | \(\frac{1}{2\sqrt{3}}(\rightarrow \infty)\) |

...
of the surface energy and the rotational energy \[17\],
\[
\varepsilon_S \equiv \frac{1}{8\pi G} \int \kappa(A, J = 0) dA, \quad (4.2)
\]
\[
\varepsilon_R \equiv \int \Omega_H(A, J) dJ. \quad (4.3)
\]

The surface energy \(\varepsilon_S\) is often referred as the irreducible mass \(M_{\text{irr}}\), which cannot be decreased by any classical process. For the Kerr black hole, the irreducible mass is
\[
M_{\text{irr}} = \frac{1}{2}(a^2 + J^2)^{1/2}. \quad (4.4)
\]

It takes maximal value \(M/\sqrt{2}\) when the solution is extremal \(a = M\). Hence the rotational energy which we can extract at most is
\[
\varepsilon_R/M = \left(1 - \frac{1}{\sqrt{2}}\right) \sim 29.3\% \quad \text{(Kerr).} \quad (4.5)
\]

By the Penrose process, we can extract about 29.3% of the initial mass energy \(M_0(A_0, J_0)\). A Schwarzschild black hole would be eventually left with its mass \(M_f = M_{\text{irr}}(M_0, J_0)\).

Calculating these quantities using the formula in [18] for five dimensional black hole and black ring, we find
\[
M_{\text{irr}} = \frac{3(2\pi^2 A^2)^{1/3}}{16\pi G} = \frac{3\pi \lambda^{1/3}(1 + \lambda)^{1/3}(\lambda - \nu)}{2^{5/3}G(1 + \nu)^{1/3}(1 - \nu)^{2/3}}, \quad (4.6)
\]
we find the rotational energy \(\varepsilon_R = M - M_{\text{irr}}\) is
\[
\varepsilon_R \equiv \frac{\varepsilon_R}{M} = \begin{cases} \frac{1}{2} - \frac{1}{1 + \nu} \frac{1}{1 + \nu}^{1/3} & (\text{a black hole}) \\ 1 - (\nu(1 - \nu)/2)^{1/3} & (\text{a black ring}). \end{cases} \quad (4.7)
\]

Fig. 3 is the plot for \(\varepsilon_R = \varepsilon_R/M\) against \(J^2\). The rotational energy is monotonically increasing function of \(J^2\) for MP black holes with one angular momentum, while it has a cusp at \(J^2 = 27/32\) for a black ring solution. This point gives a lower bound of angular momentum of a black ring. We also find a similar structure for the horizon area (entropy)-angular momentum relation (see Fig. 3 in [18]), in which we have also a cusp at the same point. The higher entropy branch should be relatively stable, while the lower entropy branch may be unstable, i.e. it contains at least one unstable mode. Note that the higher rotational energy branch corresponds to the lower entropy branch. The appearance of such a cusp indicates change of stability. Such a behavior of stability could be understood by a catastrophe theory [19].

V. CATASTROPHE THEORY AND STABILITY

Catastrophe theory is a mathematical tool to explain a change of stability in nature. In some phenomena, a state changes discontinuously in spite of a gradual change of the state parameters. The stability changes of colored black holes, for example, are explained by a catastrophe theory [19, 20]. The stability analysis via a catastrophe theory seems to have one-one correspondence to the linear perturbations.

To see this more precisely, we introduce a potential function, which is a functional of a control parameter and a state variable. We here assume that a control parameter is a radius of a black ring \(R^2 = 3\pi R^2/2GM\) (This can be driven by solving (2.26) for \(R\)), a state variable is the reduced angular momentum \(J^2\) and a potential function is the reduced rotational energy \(\varepsilon_R\). In Fig. 4 we depict the equilibrium space \(V\). We find that there are two smooth curves: One (the fine line) corresponds to a set of black hole solutions and the other (the solid line) to that of black rings. The projection of the equilibrium space onto
the control plane (Fig. 3) is called a catastrophe map \( \chi_V : \mathcal{V} \to \mathbb{R}^2 \). At the point of the bifurcation \((\bar{R}^2, j^2, \varepsilon_R) = (25/12, 27/32, 1/2)\), the mapping \( \chi_V \) becomes singular. If such a singular point exist, the stability changes there.

![Figure 4](image)

**FIG. 4**: Two smooth curves in the equilibrium space \( \mathcal{V} = (\bar{R}^2, j^2, \varepsilon_R) \): One (the fine line) corresponds to a set of black hole solutions and the other (the solid line) to that of black rings.

Since the irreducible mass is proportional to the two third powers of the area \( (\mathcal{M}_{1rr} \propto A^{2/3}_{H}) \), the lower branch of the rotational energy has higher entropy, which is thermodynamically stable configuration. When we compare the rotational energy of a black hole \( (\varepsilon_{R(BH)}) \) and that of a black ring in the lower branch \( (\varepsilon_{R(BR(-))}) \), we find that

\[
\varepsilon_{R(BH)} < \varepsilon_{R(BR(-))} \quad \text{for} \quad \frac{27}{32} < j^2 < \frac{8}{9},
\]

\[
\varepsilon_{R(BH)} > \varepsilon_{R(BR(-))} \quad \text{for} \quad \frac{8}{9} < j^2 < 1.
\]

As a result, a black hole might be unstable for larger angular momentum, while a black ring might be unstable for smaller angular momentum. One may expect that there is a phase transition from a black ring to a black hole when the object loses its angular momentum. However, a catastrophe theory cannot predict it because two curves in the equilibrium space are connected at a singular point \((\bar{R}^2, j^2, \varepsilon_R) = (2, 1, 1)\) but not smoothly. We do not predict any stability for the equilibrium curve with such a singular point in a catastrophe theory. Since two objects are topologically different, classical perturbations will also not judge which spacetime is more stable.

We should also mention about Gregory-Laflamme (GL) instability of a black ring [21]. When the angular momentum gets large, a ring becomes very thin, which phase corresponds to the lower branch in Fig. 3. Then we expect the similar instability to GL instability of a black string [21] because in the limit \( R \to \infty, \nu, \lambda \to 0 \) [12], a black ring solution approaches a boosted black string solution (see Appendix B). This GL instability seems to be different from the above instability discussed in a catastrophe theory, which we shall call the small entropy (SE) instability because the smaller horizon area (entropy) branch would be unstable. The upper branch of a black ring is unstable in the sense of SE instability, while the lower branch (at least for large \( j^2 \)) becomes unstable in the sense of GL instability. Therefore, it is not so clear what kind of stable configuration is achieved for a black ring.

**VI. CONCLUDING REMARKS**

In this paper, we have discussed the energy extraction via the Penrose process from higher dimensional black holes and black rings. The result is that we can extract much more energy from higher dimensional black holes than from 4-dimensional Kerr black hole. Although we can gain at most about 20.7% energy of incident particle in 4-dimensional Kerr space-time, the efficiency could be rather amplified in higher dimensions. In particular, if one rotation parameter vanishes, the maximum efficiency becomes infinitely large because the angular momentum is not bounded from above. We also apply a catastrophe theory to analyze the stability of black rings. Our analysis indicates a branch of black rings with higher rotational energy is unstable, which should be a different type of instability from the Gregory-Laflamme’s one. The consequence might enable us to distinguish the 4-dimensional Kerr black hole from higher dimensional ones by their energy extraction rates.

However we have to be more careful. The Penrose process tells us only possibility. If we are interested in the microscopic process such as the Hawking evaporation of a black hole or a superradiance, we have to put some field in some background spacetime and then quantize it. The super-radiant mechanism in the MP black hole has been investigated in [22, 23]. In the 5-dimensional black hole, if one rotation parameter is much smaller (but not zero) than the other, super-radiant modes play an important role, by which two rotation parameters turn eventually to be almost equal [22]. For the black hole with the same rotation parameters, it turns out that the thermal radiation is much important than the superradiance even if the black hole is maximally rotating. This result would be consistent with our result, i.e. the efficiency of energy extraction can be infinitely large if one rotation parameter is very small, on the other hand it is finite for the case with the same rotation parameters.

As for a black ring, so far we do not know so much. Although the efficiency of energy extraction from a black ring can be infinitely large in the limit \( \nu \to 1 \) [30, 31], the effective potential becomes larger and larger in this
limit. When we quantized some field in this background, the particles created by a quantum process may not be able to escape to infinity. If it is the case, the energy extraction via quantum process is not likely. To evaluate it, we have to investigate the superradiance and Hawking radiation. The evaporation mechanism and evolution of a black ring would tell us the real conclusion. However, we have unfortunately not succeeded to separate the variables of geodesic motion and the Klein-Gordon equation in a black ring space-time. What we can do so far is to analyze it in the limit of \( j^2 \to \infty \) as \( M/R \) fixed, which corresponds to a boosted black string spacetime. We find that no superradiance occurs in this highly rotating spacetime. The detail is given in Appendix B. This result is again consistent with our result, i.e. the efficiency of a black ring approaches \( \epsilon \to 0 \), where the efficiency reaches some finite constant even when \( j \to \infty \).

The stability of the black ring as well as higher dimensional rotating black hole, has not yet been established. Only the stability of Schwarzschild black hole is proved in higher dimensions as well [23]. An interesting thermodynamics approach to stability shows that they are unstable in specific limit [12, 21]. A black ring approaches a boosted black string solution in the limit of \( j^2 \to \infty \) [22]. Black strings and membranes are shown to be unstable [23]. It would be interesting to perform a stability analysis by the metric perturbations and compare the result with a thermodynamics viewpoint.

As we demonstrated in section IV, the appearance of cusp in Fig. 4 indicates that there is a stability change at that point via a catastrophe theory. This shows that a black ring has another unstable mode which is different from the GL instability. It may be interesting to clarify what kind of instabilities exist in a black ring spacetime.

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APPENDIX A: THE EFFECTIVE POTENTIAL OF A TEST PARTICLE IN A BLACK RING SPACETIME

We discuss here about a particle motion in a black ring spacetime. We assume that a particle moves on the “outside” equatorial plane, i.e. \( x = -1 \). Because \( y \) coordinate is singular on the ergosurface, we introduce another coordinate \( z \), which is defined by \( z = - \tanh^{-1} y \). Using this coordinate, we write down the effective potential on the plane \( x = -1 \) for a particle with the energy \( E \) and angular momentum \( L \), which is obtained from \( p^\mu p_\mu = -m^2 \) (\( m^2 = 0 \) for a null particle), as

\[
\frac{d^2z}{d\tau^2} = \varepsilon(AE^2 - 2BE + C) = \varepsilon A(E - V_+)(E - V_-),
\]

where

\[
V_\pm = \frac{B \pm \sqrt{B^2 - AB}}{A},
\]

and

\[
\varepsilon = \frac{1 - \tanh z}{R^2(1 + \lambda)^2(\tanh z + 1)(1 + \tanh z)^2}, \quad A = \lambda \nu(\tanh^2 z - 3 \tanh z + 4) + (\tanh z + \lambda + \nu),
\]

\[
B = \frac{L^2}{R}\sqrt{\frac{\lambda \nu}{1 + \lambda}(1 + \nu)(1 - \tanh z)^2},
\]

\[
C = \frac{L^2}{R^2}(1 + \nu)^2(1 + \lambda) \tanh z(\tanh z - 1) - m^2(1 + \lambda)(1 + \tanh z)(\tanh z + \nu).
\]

In Fig. 6 we depicts the effective potential for a null particle against \( z = -\tanh^{-1} y \). In this coordinates, asymptotic infinity is at \( z \to +\infty \). The negative energy states exist in the ergoregion \( -\tanh^{-1} \nu < z < 0 \).

APPENDIX B: NO SUPERRADIANCE OF A BLACK RING IN THE LIMIT OF \( j^2 \to \infty \)

We show that superradiance does not occur from a very large black ring. A black ring solution approaches a boosted black string solution in the limit of \( j^2 \to \infty \) as \( M/R \) is fixed [12]. In this limit,

\[
R \to \infty, \quad \lambda, \nu \to 0,
\]

FIG. 6: The effective potential for a black ring. Here we used a new coordinate \( z = -\tanh^{-1} y \). The horizon corresponds to \( z = -\tanh^{-1} \nu \) and the infinity to \( z = \infty \). The plot is for \( \nu = 1/2 \) and the vertical line is normalized by \( L/R \).
with keeping $R\lambda$ and $R\nu$ constant. Introducing new co-
ordinates $r, \theta, \omega$ and parameters $r_H, \sigma$ as
\[
  r = -RF(y)/y, \quad \cos \theta = x, \quad \varpi = R\nu, \\
  R\lambda = r_H \cosh^2 \sigma, \quad R\nu = r_H \sinh^2 \sigma.
\] (B2)
and taking the above limit of a black ring solution \[B1\], we obtain a boosted black string solution
\[
ds^2 = -\tilde{f} \left( dt - \frac{r_H \sinh 2\sigma}{2r f} d\varpi \right)^2 + \frac{f}{\tilde{f}} d\varpi^2 + \frac{1}{\tilde{f}} dr^2 + r^2 d\Omega_2^2
\] (B3)
with
\[
f = 1 - \frac{r_H}{r}, \quad \tilde{f} = 1 - \frac{r_H \cosh^2 \sigma}{r}, \quad (B4)
\]
d\[B5\]
which \[B5\] is the tortoise coordinate
\[
  \sigma = 0 \text{ corresponds to a (static) black string solution. The horizon exists at } r = r_H. \text{ The Killing vector } \xi = \partial_t \text{ becomes null at the horizon. The ergoregion also exists (} r_H < r < r_H \cosh^2 \sigma).\]
We consider a massless free scalar field $\Phi$ in a boosted black string spacetime \[B8\]. The basic equation for $\Phi$ is
\[
  \Box \Phi = \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g} g^{\mu\nu} \partial_{\nu} \Phi) = 0, \quad (B6)
\]
which is separable in the spacetime \[B8\]. Here we set
\[
  \Phi = \frac{F(r)}{r} Y_{lm}(\theta, \phi) e^{ikz} e^{-i\omega t} \quad (B7)
\]
where $Y_{lm}$ denotes spherical harmonics. Then, we find the radial equation of \[B9\] as
\[
d^2 F \over dr^2 + V(r) F = 0, \quad (B8)
\]
with
\[
  V(r) = f \left( \frac{\omega^2}{f} - \frac{f'}{r} - \frac{l(l+1)}{r^2} \right) \\
  - \tilde{f} \left( k + \frac{r_H \sinh 2\sigma}{2rf} \omega \right)^2, \quad (B9)
\]
where $r_*$ is the tortoise coordinate
\[
  \frac{dr_*}{dr} = 1/f, \quad r_* = r + r_H \ln |r - r_H|. \quad (B10)
\]
The potential function $V$ asymptotically approaches some constants as
\[
  V(r) \sim \begin{cases} 
  \omega_\infty^2 & (r \to \infty) \\
  \omega_H^2 & (r \to r_H), \quad (B11)
\end{cases}
\]
where
\[
  \omega_\infty = (\omega^2 - k^2)^{1/2}, \quad \omega_H = \cosh \sigma (\omega - k \tanh \sigma). \quad (B12)
\]
To discuss the scattering of the scalar wave in this spacetime, we consider an incoming wave with a unit amplitude $F_{\omega lm k}$, which asymptotic forms are given by
\[
  F_{\omega lm k} \sim \begin{cases} 
  e^{-i\omega \rho_*} + A_{\omega lm k} e^{i\omega \rho_*} & (r \to \infty) \\
  B_{\omega lm k} e^{-i\omega \rho_*} & (r \to r_H) \quad (B14)
\end{cases}
\]
Since the Wronskian is constant with respect to $r_*$ for the solutions \[B8\] and their complex conjugates, we find
\[
  1 - |A_{\omega lm k}|^2 = \frac{\omega_H}{\omega_\infty} |B_{\omega lm k}|^2 \quad (B15)
\]
Since the wave we are discussing is travelling at infinity, we have a constraint that $\omega_\infty > 0$, which gives $\omega > k$. Then the condition for the superradiance ($|A_{\omega lm k}| > 1$) is
\[
  \omega_H < 0 \Leftrightarrow \omega < k \tanh \sigma, \quad (B16)
\]
which leads to $\omega < k$. This is not consistent with our previous condition for $\omega$. Hence we come to the conclusion that superradiance does not occur for a boosted black string.

The superradiance occurs by the difference between minimum energy of a particle at the event horizon and that at infinity. In the Kerr black hole, due to the inertia frame dragging, the minimum energy at the horizon is raised by the rotation of a black hole from 0 to $m\Omega_H$, where $m$ is a magnetic quantum number. Hence, if $0 < \omega < m\Omega_H$, we have superradiance. In the present case, the minimum energy of a particle at the horizon cannot be raised enough high compared with that at infinity (see Fig. 1).

To see this more precisely, we write down the radial equation for a massless particle, i.e. the equation $p_\mu p^\mu = 0$ leads to
\[
  \dot{r}^2 = \tilde{f} E^2 + \frac{r_H \sinh 2\sigma}{r} L_\varpi - \frac{f L_\varpi^2}{r^2} + \frac{f}{r^2} L_\phi^2 \quad (B17)
  \quad = \tilde{f} (E - U_+)(E - U_-), \quad (B18)
\]
with
\[
  \tilde{f} = 1 + \frac{r_H \sinh^2 \sigma}{r}, \quad (B19)
\]
\[
  U_\pm = \frac{1}{\tilde{f}} \left[ \frac{r_H \sinh 2\sigma}{2r} L_\varpi \pm \sqrt{\left( \frac{r_H \sinh 2\sigma}{2r} L_\varpi \right)^2 + \tilde{f} \left( \frac{f L_\varpi^2}{r^2} + \frac{f}{r^2} L_\phi^2 \right)} \right], \quad (B20)
\]
where $E, L_\varpi$, and $L_\phi$ represent the energy, the $\varpi$- and $\phi$- component of the angular momentum of a particle, respectively. The effective potential $U_\pm$ asymptotes the value
\[
  U_\pm \sim \begin{cases} 
  \pm L_\varpi & (r \to \infty) \\
  L_\varpi \tanh \sigma & (r \to r_H), \quad (B21)
\end{cases}
\]
which gives the minimum energy of a particle at infinity or that at horizon. Fig. 7 shows the typical behavior of the effective potential \( U_{\pm}(r) \) for a particle. \( U_{+}(r) \) gives the minimum energy of the particle at \( r \).

Although there is no superradiance in a boosted black string spacetime, the Penrose process is still possible. In fact, the maximum efficiency for a black ring is finite (\( \eta_{\text{max}} = (\sqrt{2} - 1)/2 \)) even in the limit of \( j^2 \to \infty \).

**FIG. 7:** The effective potential \( U_{\pm} \) for a boosted black string. We set \( L_\phi = 2 \), \( L_\phi = 6 \), \( \sigma = 1 \).

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