Optimum unambiguous discrimination of two mixed quantum states

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We investigate generalized measurements, based on positive-operator-valued measures, and von Neumann measurements for the unambiguous discrimination of two mixed quantum states that occur with given prior probabilities. In particular, we derive the conditions under which the failure probability of the measurement can reach its absolute lower bound, proportional to the fidelity of the states. The optimum measurement strategy yielding the fidelity bound of the failure probability is explicitly determined for a number of cases. One example involves two density operators of rank \( d \) that jointly span a \( 2d \)-dimensional Hilbert space and are related in a special way. We also present an application of the results to the problem of unambiguous quantum state comparison, generalizing the optimum strategy for arbitrary prior probabilities of the states.

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Many applications in quantum communication and quantum cryptography are based on transmitting quantum systems that, with given prior probabilities, are prepared in one from a set of known mutually nonorthogonal states. Since perfect discrimination between nonorthogonal quantum states is impossible, measurement strategies for state discrimination have been developed that are optimized with respect to various criteria \([5, 6, 7, 8, 9, 10, 11]\). A complete solution, determining the state that is inferred to be in a pure state which has to be distinguished from a set of known mutually nonorthogonal states, started only recently \([5, 6]\), and of two mixed states of rank \( \eta_1 \) and \( \eta_2 = 1 - \eta_1 \), respectively, can be formalized by three positive operators \( \Pi_k \) with \( \sum_{k=0}^2 \Pi_k = I \), where \( I \) is the identity. These detection operators are defined in such a way that \( \text{Tr}(\rho\Pi_k) \) with \( k = 1, 2 \) is the probability that a system prepared in a state \( \rho \) is inferred to be in the state \( \rho_k \), while \( \text{Tr}(\rho\Pi_0) \) is the probability that the measurement fails to give a definite answer. When all detection operators are projectors, the measurement is a von Neumann measurement, otherwise it is a generalized measurement based on a positive operator-valued measure (POVM). From the detection operators \( \Pi_k \) schemes for realizing the measurement can be obtained \([12, 13]\).

It is our aim to investigate the optimum measurement strategy that minimizes the total failure probability

\[
Q = \eta_1 \text{Tr}(\rho_1 \Pi_0) + \eta_2 \text{Tr}(\rho_2 \Pi_0).
\]

From the relation between the arithmetic and the geometric mean and from the Cauchy-Schwarz-inequality \([10, 14]\) it follows that \( Q \geq 2\sqrt{\eta_1\eta_2 \text{Tr}(\rho_1 \Pi_0)\text{Tr}(\rho_2 \Pi_0)} \geq 2\sqrt{\eta_1\eta_2} \text{Max} \left\{ \text{Tr}(U\sqrt{\Pi_1}\Pi_0\sqrt{\Pi_2}) \right\} \), where \( U \) describes an arbitrary unitary transformation. The failure probability takes its absolute minimum when the two equality signs hold. This is true if and only if both the relations \( \eta_1 \text{Tr}(\rho_1 \Pi_0) = \eta_2 \text{Tr}(\rho_2 \Pi_0) \) and \( U\sqrt{\Pi_1}\Pi_0\sqrt{\Pi_2} \sim \sqrt{\Pi_2}\Pi_0 \) are fulfilled. From the first relation we conclude that the number of inconclusive results is equally distributed among the two incoming states. After multiplying the second relation with its Hermitian conjugate, the two conditions for equality can be combined to yield \( \text{Tr}(\eta_2\rho_2 - \eta_1\rho_1) = 0 \). Since in the POVM-formalism the detection operators transform a quantum state according to \( \rho \rightarrow \sum_k \sqrt{\Pi_k} \rho \sqrt{\Pi_k} \), it follows that the total failure probability is smallest when in case of failure the two density operators are transformed into states that are identical after normalization and therefore cannot be further discriminated.

We now recall that unambiguous discrimination of two states leads to the requirement \( \rho_1 \Pi_2 = \rho_2 \Pi_1 = 0 \).
Substituting \( \Pi_0 = I - \Pi_1 - \Pi_2 \) into the inequality for the failure probability \( Q \), given above, we arrive at

\[
Q \geq 2 \sqrt{\eta_1 \eta_2} F \geq 2 \sqrt{\eta_1 \eta_2} F_0, \quad (2)
\]

where \( F = \text{Tr} \left[ (\sqrt{\rho_1} \sqrt{\rho_2})^2 \right] \) is the fidelity [12]. Using a different method, it has been found already previously by Rudolph et al. [7] that

\[
Q \geq \begin{cases} 
2 \sqrt{\eta_1 \eta_2} F = Q_0 & \text{if } F \leq \frac{1}{2}, \\
\eta_{\text{min}} + \eta_{\text{max}} F^2 & \text{otherwise}, 
\end{cases} \quad (3)
\]

with \( \eta_{\text{min}} (\eta_{\text{max}}) \) denoting the smaller (larger) of the prior probabilities. Here, in addition, we obtained the necessary and sufficient conditions that the detection operators have to fulfill in order to reach the fidelity bound \( Q_0 \). They can be summarized as

\[
\Pi_0 = I - \Pi_1 - \Pi_2 \geq 0, \quad \Pi_1 \geq 0, \quad \Pi_2 \geq 0, \quad (4)
\]

\[
\rho_1 \Pi_2 = \rho_2 \Pi_1 = 0, \quad (5)
\]

\[
\eta_1 \text{Tr}(\rho_1 \Pi_0) = \eta_1 [1 - \text{Tr}(\rho_1 \Pi_1)] = \sqrt{\eta_1 \eta_2} F, \quad (6)
\]

\[
\eta_2 \text{Tr}(\rho_2 \Pi_0) = \eta_2 [1 - \text{Tr}(\rho_2 \Pi_2)] = \sqrt{\eta_1 \eta_2} F. \quad (7)
\]

In the following we investigate the conditions under which detection operators exist that satisfy Eqs. (4) - (7). For this purpose we use the spectral representations

\[
\rho_1 = \sum_{i=1}^{d_1} |r_i\rangle\langle r_i|, \quad \rho_2 = \sum_{m=1}^{d_2} s_m |s_m\rangle\langle s_m|, \quad (8)
\]

where \( r_i, s_m \neq 0 \), and \( \langle r_i| r_m \rangle = \delta_{i,m} = \langle s_l| s_m \rangle \). Furthermore, we introduce the projection operators

\[
P_1 = \sum_{i=1}^{d_1} |r_i\rangle\langle r_i|, \quad P_2 = \sum_{m=1}^{d_2} |s_m\rangle\langle s_m|, \quad (9)
\]

and the nonnormalized states \( |r_i\rangle \equiv P_2 |r_i\rangle \). We can construct a complete orthonormal basis \( \{ |h_k\rangle \} \) in the subspace \( \mathcal{H}_{1\perp} \) spanned by the state vectors \( P_2 |r_i\rangle \), using the recursion relation \( |h_k\rangle = P_2 |r_k\rangle - \sum_{i=1}^{k-1} |h_i\rangle \langle h_i| P_2 |r_k\rangle \) and determining \( |h_k\rangle \equiv |h_k\rangle/\|h_k\| \) [14]. The dimensionality \( d_{1\perp} \) of \( \mathcal{H}_{1\perp} \) is equal to the rank of the matrix formed by the elements \( \langle r_i| P_2 |r_n\rangle \). Similarly, in the subspace \( \mathcal{H}_{1\perp} \) that is spanned by the nonnormalized vectors \( |r_i\rangle - (I - P_2)|r_i\rangle \), we can obtain an orthonormal basis \( \{ |v_i\rangle \} \) of dimension \( d_{1\perp} \). The respective projection operators into the two orthogonal subspaces are

\[
P_{1\parallel} = \sum_{k=1}^{d_{1\parallel}} |h_k\rangle\langle h_k|, \quad \quad P_{1\perp} = \sum_{i=1}^{d_{1\perp}} |v_i\rangle\langle v_i|, \quad (10)
\]

where \( \rho_2 |v_i\rangle = 0 \). The operator \( P_{10} = P_{1\parallel} + P_{1\perp} \) projects onto a subspace \( \mathcal{H}_{10} \) of dimension \( d_{1\parallel} + d_{1\perp} \). Noticing that \( \text{Tr}[(P_{10} - P_1) \rho_1] = 0 \), we construct the operator

\[
\tilde{P}_1 = P_{1\parallel} + P_{1\perp} - P_1 = \sum_{j=0}^{d_1} |\tilde{r}_j\rangle\langle \tilde{r}_j|, \quad (11)
\]

where \( \rho_1 |\tilde{r}_j\rangle = 0 \). The states \( \{ |\tilde{r}_j\rangle \} \) form an orthonormal basis in the \( d_1 \)-dimensional subspace of \( \mathcal{H}_{10} \) that is spanned by all states that are orthogonal to \( P_1 \), where \( \tilde{d}_1 = d_{1\parallel} + d_{1\perp} - d_1 \). The identity is then given by

\[
I = P_{1\perp} + P_2 = P_{1\perp} + P_{1\parallel} + P_{2\perp} = P_1 + \tilde{P}_1 + P_{2\perp}. \quad (12)
\]

Here the operator \( P_{2\perp} = I - P_{1\perp} - P_{1\parallel} \) projects onto the subspace \( \mathcal{H}_2 \) spanned by those states that are orthogonal to both \( P_{1\parallel} \) and \( P_{1\perp} \), implying that \( \rho_1 P_{2\perp} = 0 \). Instead of decomposing the eigenstates of \( \rho_1 \), we might as well have started from \( |s_m\rangle = P_1 |s_m\rangle + |s_m\rangle \), obtaining instead of Eq. (12) the alternative decomposition

\[
I = P_{2\parallel} + P_1 = P_{2\parallel} + P_{2\perp} + P_1 = P_2 + \tilde{P}_2 + P_{1\perp}, \quad (13)
\]

where the projectors are defined analogously.

Now we can specify the general structure of all detection operators, \( \Pi_1 \) and \( \Pi_2 \), that describe unambiguous discrimination, i.e. satisfy Eqs. (11) and (13).

\[
\Pi_1 = \sum_{j=1}^{d_{1\parallel}} \alpha_j^t |v_j\rangle\langle v_j| + \sum_{i,j=1}^{d_{1\perp}} \alpha_{ij} |v_i\rangle\langle v_j|, \quad (14)
\]

where \( 0 \leq \alpha_j^t \leq 1 \) and \( |v_j\rangle = \sum_i u_{ji} |v_i\rangle \) with \( \{ u_{ji} \} \) being a unitary matrix. We note that \( \sum_j |v_j\rangle\langle v_j| = I \) since the eigenstates \( |v_j\rangle \) form a complete orthonormal basis in \( \mathcal{H}_{1\perp} \). For representing \( \Pi_2 \) we start from the same decomposition of the identity, and take into account that none of the eigenstates of \( \Pi_0 \) must be contained in the subspace \( \mathcal{H}_{1\parallel} \) when the failure probability is to be as small as possible. This leads to

\[
\Pi_2 = \sum_{i=1}^{d_{1\parallel}} \beta_i^t |v_i\rangle\langle v_i| + P_2' = \sum_{i,j=1}^{d_{1\perp}} \beta_{ij} |v_i\rangle\langle v_j| + I - P_{10}, \quad (15)
\]

where \( 0 \leq \beta_{ij}^t \leq 1 \) and \( |v_i\rangle = \sum_i u_{ji} |v_i\rangle = \tilde{P}_1 \). The constants \( \alpha_{ij} \) and \( \beta_{ij} \) are subject to the constraint that \( \Pi_0 \geq 0 \).

Clearly, when \( P_1 = P_{1\parallel} = I \), and consequently also \( P_2 = P_{2\parallel} = I \), it follows that \( \Pi_1 = \Pi_2 = 0 \) and \( \Pi_0 = I \), yielding a unit failure probability that makes error-free discrimination impossible. We therefore require that \( P_{1\parallel} \neq 0 \), or \( P_{2\parallel} \neq 0 \), respectively, which, because of normalization, is equivalent to

\[
\text{Tr}(P_1 \rho_2) < \text{Tr}(P_1 \rho_2), \quad \text{Tr}(P_2 \rho_1) < \text{Tr}(P_2 \rho_1). \quad (16)
\]

Before studying the optimum measurement, let us consider the von Neumann measurements for unambiguous discrimination. If \( \alpha_{ij} \neq 0 \) for all \( j \) and \( \beta_{ij} \neq 1 \) for all \( i \), it follows that \( \Pi_1 = 0 \) and \( \Pi_2 = \tilde{P}_1 + P_{2\perp} \). Hence \( \Pi_0 = P_1 \), with the failure probability \( Q_{N1} = \eta_1 + \eta_2 \text{Tr}(P_1 \rho_2) \).

Another von Neumann measurement is generated when \( \alpha_{ij} = 1 \) for all \( j \) and \( \beta_{ij} = 0 \) for all \( i \), giving \( \Pi_1 = P_{1\parallel} \) and \( \Pi_2 = P_{2\perp} \). Then \( \Pi_0 = P_{1\parallel} \), with the failure probability

\[
Q_{N1\parallel} = \eta_1 \text{Tr}(P_2 \rho_1) + \eta_2 \text{Tr}(P_{1\parallel} \rho_2), \quad (17)
\]
where the relation $\text{Tr}(P_{1\|}\rho_1) = 1 - \text{Tr}(P_{1\perp}\rho_1) = \text{Tr}(P_2\rho_1)$ has been applied. In this measurement the state is unambiguously found to be $\rho_1$ when a detector click occurs in a direction orthogonal to all eigenstates of $\rho_2$. On the other hand, for a click in a direction orthogonal to both $P_{1\|}$ and $P_{1\perp}$, the state is determined to be $\rho_2$ with certainty, and in the rest of cases the result is inconclusive. So far we relied on Eq. (12). Based on the complementary decomposition of the identity, Eq. (13), we obtain an alternative pair of von Neumann measurements. These yield the failure probabilities $Q_{N12} = \eta_2 + \eta_1 \text{Tr}(P_2\rho_1)$ and

$$Q_{N21} = \eta_2 \text{Tr}(P_1\rho_2) + \eta_1 \text{Tr}(P_2\|\rho_1).$$

(18)

Obviously $Q_{N2\|} \leq Q_{N1}$ and $Q_{N1\|} \leq Q_{N2}$.

We now return to the optimum measurement. Since the von Neumann measurements can be performed for arbitrary given parameters, the optimized failure probability certainly obeys the inequality

$$Q_{\text{opt}} \leq \text{Min}\{Q_{N1\|}, Q_{N2\|}\}.$$  

(19)

According to Eqs. (10) and (11) the absolute minimum of the failure probability, $Q_0 = 2\sqrt{\eta_2/\eta_1}F$, is reached if and only if the two conditions $\text{Tr}(\rho_1\Pi_0)/F = \sqrt{\eta_2/\eta_1}$ and $F/\text{Tr}(\rho_2\Pi_0) = \frac{\sqrt{\eta_2/\eta_1}}{\eta_2/\eta_1}$ are fulfilled. However, due to the structure of the operators $\Pi_1$ and $\Pi_2$, the possible values of $\text{Tr}(\rho_k\Pi_k)$, for $k = 1, 2$, have a lower bound. In particular,

$$\text{Tr}(\rho_1\Pi_0) \geq 1 - \text{Tr}(P_{1\perp}\rho_1) = \text{Tr}(P_2\rho_1),$$

(20)

$$\text{Tr}(\rho_2\Pi_0) \geq \text{Tr}(P_{1\|}\rho_2) - \text{Tr}(P_1\rho_2) = \text{Tr}(P_2\rho_2),$$

(21)

where in the first equation the equality sign holds when $\alpha_j' = 1$ in Eq. (14), and in the second equation the equality is reached when $\beta_i' = 1$ in Eq. (15). Therefore we obtain that the condition

$$\frac{\text{Tr}(P_2\rho_1)}{F} \leq \frac{\sqrt{\eta_2/\eta_1}}{\eta_2/\eta_1} \leq \frac{\text{Tr}(P_1\rho_2)}{F},$$

(22)

is necessary, i.e. the fidelity bound, $Q = Q_0$, can only be reached in part or in the whole of this interval.

The interval specified by Eq. (22) is not empty only when $\text{Tr}(P_2\rho_1)\text{Tr}(P_1\rho_2) \leq F^2$. For two density operators that violate this inequality, the failure probability $Q_0$ cannot be achieved for any values of the prior probabilities of the states, and the conditions (10) and (11) are then of no help for determining the optimum measurement. Moreover, our result shows that in general the lower bound $Q_0$ can only be reached in an interval of the ratio $\eta_2/\eta_1$ that is smaller than the interval given in Eq. (9), since $\text{Tr}(P_2\rho_1)/F \geq F$ and $F/\text{Tr}(P_1\rho_2) \leq 1/F$. The latter relations follow from the general inequalities

$$\text{Tr}(P_2\rho_1)\text{Tr}(P_{1\|}\rho_2) \geq F^2, \quad \text{Tr}(P_1\rho_2)\text{Tr}(P_{1\perp}\rho_1) \geq F^2,$$

(23)

that can be readily inferred from Eqs. (17), (18) and (22).

The parameter intervals in Eqs. (9) and (22) coincide when $\text{Tr}(P_1\rho_2) = \text{Tr}(P_2\rho_1) = F^2$. This condition is fulfilled, e.g., for density operators of the form $\rho_1 = \sum_{i=1}^d r_i|\chi_i\rangle\langle\chi_i|$ and $\rho_2 = \sum_{i=1}^d r_i|\beta_i\rangle\langle\beta_i|$, with $\langle r_i|\beta_i\rangle = b_i\delta_{ij}$, where the corresponding eigenvalues are identical. The fidelity is then found to be $F = |b|_i$.

Another simplification arises when $P_{1\|}$ and $P_{1\perp}$ are one-dimensional projectors, $d_{1\|} = d_{1\perp} = 1$. In this case equality holds in Eqs. (23), (17) which implies that $F^2 = \text{Tr}(P_2\rho_1)\text{Tr}(P_{1\perp}\rho_2) \geq \text{Tr}(P_2\rho_1)\text{Tr}(P_1\rho_2)$, where Eq. (16) has been taken into account. Hence again for any two density operators the necessary condition (22) is fulfilled for a certain range of the ratio $\eta_2/\eta_1$. At the lower limit of this range, i.e., for $\sqrt{\eta_2/\eta_1} = \text{Min}(\eta_2/\eta_1)$, we can write

$$2\sqrt{\eta_2/\eta_1}F = \eta_1 F^2 + \eta_2 \text{Tr}(P_1\rho_2) = Q_{N1\|},$$

and similarly we find that at the upper limit $2\sqrt{\eta_2/\eta_1}F = Q_{N2\|}$. Thus, if $Q = Q_0$ in the entire range in Eq. (22), the complete solution for the optimum measurement is known.

In general, in order to find the optimum measurement strategy that yields the failure probability $Q_0$, we have to determine the parameters $\alpha_{ij}$ and $\beta_{ij}$ in Eqs. (13) and (14) that satisfy the necessary and sufficient conditions (11) - (17). In the following we apply this method to a number of special cases.

First we consider two density operators of rank $d$ in a 2$d$-dimensional joint Hilbert space. In such a case $P_{1\perp}^2 = 0$ and the identity can be alternatively expressed as $I = P_1 + P_1$ or $I = P_{1\perp} + P_2$ which means that $P_{1\|} = P_2$, $P_{1\perp} = P_1$, and $P_{1\perp} = P_1$. We start from Eqs. (8) with $d_1 = d_2 = d$ and assume that $|s_i\rangle = (|\chi_i\rangle + |\beta_i\rangle)/\sqrt{2}$, and $|r_i\rangle = (|\chi_i\rangle - |\beta_i\rangle)/\sqrt{2}$ ($i = 1, \ldots, d$). Then we obtain $F = \sum_{i=1}^d \sqrt{r_i}s_i/2$ and $\text{Tr}(P_2\rho_2) = \text{Tr}(P_2\rho_1) = 1/2$. It is important to note that in general there exist sets of eigenvalues $r_i$ and $s_i$ where $F^2 < 1/4$ and the necessary condition, Eq. (22), cannot be fulfilled. In the following, however, we restrict ourselves to the special case that $r_i = s_i$ for $i = 1, \ldots, d$, for which $F = 1/\sqrt{2}$. The necessary condition for the lower bound $Q_0$ to be achievable then reads $\frac{1}{\sqrt{2}} \leq \frac{\sqrt{\eta_2/\eta_1}}{\eta_2/\eta_1} \leq \frac{1}{\sqrt{2}}$. Further, we find the solutions $\alpha_{ij} = \alpha \delta_{ij}$ and $\beta_{ij} = \beta \delta_{ij}$ ($i, j = 1, \ldots, d$), where $\alpha = 2 - \sqrt{2\eta_2/\eta_1}$ and $\beta = 2 - \sqrt{2\eta_2/\eta_1}$. $\Pi_0$ has two eigenvalues, $\lambda_0 = 0$ and $\lambda_1 = 2 - \alpha - \beta$, each with a fold degeneracy. Thus the optimum $\Pi_0$ is always an operator of rank $d$. Note that $2\sqrt{2} - 2 \leq \lambda_1 \leq 1$ in the whole interval $F \leq \frac{\sqrt{\eta_2/\eta_1}}{\eta_2/\eta_1} \leq \frac{1}{\sqrt{2}}$. Hence in this parameter interval the optimum detection operators yielding the lower bound $Q_0$ are $\Pi_1 = \alpha P_{1\perp}$ and $\Pi_2 = \beta P_{1\perp}$. At the upper and lower limits of the interval the measurement turns into the von Neumann measurements that give the failure probabilities $Q_{N1\|} = Q_{N2}$ and $Q_{N2\|} = Q_{N1}$, respectively. Since in our example $\text{Tr}(P_2\rho_1) = \text{Tr}(P_1\rho_2) = F^2$, we find that $Q_{N1} = \eta_1 F^2$ and $Q_{N2} = \eta_2 F^2$. Thus we derived a measurement strategy that yields the equality sign in Eq. (8) for two mixed states.

In our next examples we focus on the case $d_{1\|} = d_{2\|} = 1$. First we assume that the density operators given in Eq. (8) have arbitrary ranks $d_1$ and $d_2$, and that $\langle r_i|s_m\rangle = a_i \delta_{im} \delta_{m,1}$ with $|a| < 1$. This yields
\[ F = \sqrt{s_1 r_1} |a|, \quad \text{Tr}(P_2 \rho_1) = F^2/s_1, \quad \text{Tr}(P_1 \rho_2) = F^2/r_1 \]
and \( d_{1\parallel} = d_{2\parallel} = 1 \). For the parameter range specified in Eq. (22) we obtain the optimum detection operators

\[
\Pi_1 = \left(1 - \sqrt{\frac{\eta_2}{\eta_1}} \frac{F}{r_1} \right) |\bar{\psi}_1\rangle \langle \bar{\psi}_1| + \sum_{i=2}^{d_1} |r_i\rangle \langle r_i| 
\]
\[
\Pi_2 = \left(1 - \sqrt{\frac{\eta_1}{\eta_2}} \frac{F}{s_1} \right) |\bar{\psi}_1\rangle \langle \bar{\psi}_1| + \sum_{m=2}^{d_2} |s_m\rangle \langle s_m|, \quad (25)
\]
where we introduced \(|\bar{\psi}_1\rangle = |r_1\rangle - a|s_1\rangle \) and \(|\bar{\psi}_1\rangle = |s_1\rangle - a^*|r_1\rangle \).

This solution can be applied to the problem of quantum state comparison \cite{1}, where two identical quantum objects are each prepared either in the state \(|\psi_1\rangle\), or in the state \(|\psi_2\rangle\), and where we wish to determine unambiguously whether the states are equal or different. The task amounts to distinguishing the two-particle states \(\rho_1 = \frac{1}{2} \left( |\psi_1, \psi_1\rangle \langle \psi_1, \psi_1| + |\psi_2, \psi_2\rangle \langle \psi_2, \psi_2| \right)\) and \(\rho_2 = \frac{1}{2} \left( |\psi_1, \psi_2\rangle \langle \psi_1, \psi_2| + |\psi_2, \psi_1\rangle \langle \psi_2, \psi_1| \right)\), where \(F = |\langle \psi_1 | \psi_2 \rangle|\). Upon determining the eigenstates, we find that the structure of \(\rho_1\) and \(\rho_2\) corresponds to the one treated in the above special example, with \(r_1 = s_1 = (1 + F^2)/2\) and \(|a| = 2F/(1 + F^2)\). The minimum failure probability in unambiguous quantum state comparison follows to be

\[ Q_{\text{opt}} = \begin{cases} 
\frac{2\sqrt{\eta_2 F}}{\eta_{\max} \sqrt{1 + F^2}} + \eta_{\min} \frac{1 + F^2}{2} & \text{if} \quad \frac{\eta_{\min}}{\eta_{\max}} \geq 2F \sqrt{1 + F^2} \\
\text{otherwise} & \end{cases} \quad (26)
\]
Here \(\eta_{\min} (\eta_{\max})\) is the smaller (larger) of the values \(\eta_1 = p_1^2 + p_2^2\) and \(\eta_2 = 2p_1p_2\), where \(p_1\) and \(p_2\) are the prior probabilities of the states \(|\psi_1\rangle\) and \(|\psi_2\rangle\), respectively.

As our final example we mention the problem of discriminating a pure state, \(\rho_1 = |r_1\rangle \langle r_1|\), from a mixed state \(\rho_2\), or from a set of pure states, respectively, that has been introduced as quantum state filtering \cite{3,17}. In this case \(\text{Tr}(P_1 \rho_2) = F^2\) and \(\text{Tr}(P_2 \rho_1) = \|r_1\|^2\). In the parameter interval given by Eq. (22) the optimum detection operators take the form \(\Pi_1 = \left(1 - \sqrt{\frac{\eta_2}{\eta_1}} \frac{F}{r_1} \right) |\bar{\psi}_1\rangle \langle \bar{\psi}_1| + \Pi_2\), and the previous solution for the minimum failure probability in optimum unambiguous quantum state filtering \cite{3,6} is readily regained.

In summary, we performed a detailed analysis of the probabilistic measurement for unambiguous discrimination between two arbitrary mixed quantum states. We derived general analytical relations that depend on five quantities characterizing the mutual relationship of the density operators of the states. These quantities are the expressions \(\text{Tr}(P_1 \rho_2)\) and \(\text{Tr}(P_2 \rho_1)\), as well as \(\text{Tr}(P_2 \rho_1)\) and \(\text{Tr}(P_2 \rho_1)\) and, most importantly, the fidelity \(F\). We also showed that the method developed in this paper can be used to find complete analytical solutions that describe the optimum measurement for special cases.

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References:

[2] The support of a density operator is the Hilbert space spanned by its eigenvectors with nonzero eigenvalues. The rank of the density operator is equal to the dimension of the support.
[15] Note that \(\rho_2 |r_1\rangle \rangle = \|r_1\|^2 \rho_2 |h_1\rangle \rangle\) when \(d_{1\parallel} = 1\). Therefore we can write \(\sqrt{\rho_1} \rho_2 \sqrt{\rho_1} = \text{Tr}(P_1 \rho_2)\) as \(|\langle h_1 | \rho_2 | h_1 \rangle|\), where \(|\langle h_1 | \rho_2 | h_1 \rangle|\) is a normalized pure state. Hence \(F = \text{Tr}(P_1 \rho_2) \text{Tr}(P_2 \rho_1)\)^{1/2}.