Klein-Gordon particles in mixed vector-scalar inversely linear potentials

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Abstract
The problem of a spinless particle subject to a general mixing of vector and scalar inversely linear potentials in a two-dimensional world is analyzed. Exact bounded solutions are found in closed form by imposing boundary conditions on the eigenfunctions which ensure that the effective Hamiltonian is Hermitian for all the points of the space. The nonrelativistic limit of our results adds a new support to the conclusion that even-parity solutions to the nonrelativistic one-dimensional hydrogen atom do not exist.
The problem of a particle subject to an inversely linear potential in one spatial dimension ($\sim |x|^{-1}$), known as the one-dimensional hydrogen atom, has received considerable attention in the literature (for a rather comprehensive list of references, see [1]). This problem presents some conundrums regarding the parities of the bound-state solutions and the most perplexing is that one regarding the ground state. Loudon [2] claims that the nonrelativistic Schrödinger equation provides a ground-state solution with infinite eigenenergy and a related eigenfunction given by a delta function centered about the origin. This problem was also analyzed with the Klein-Gordon equation for a Lorentz vector coupling and there it was revealed a finite eigenenergy and an exponentially decreasing eigenfunction [3]. By using the technique of continuous dimensionality the problem was approached with the Schrödinger, Klein-Gordon and Dirac equations [4]. The conclusion in this last work reinforces the claim of Loudon. Furthermore, the author of Ref. [4] concludes that the Klein-Gordon equation, with the interacting potential considered as a time component of a vector, provides unacceptable solutions while the Dirac equation has no bounded solutions at all. On the other hand, in a more recent work [4] the authors use connection conditions for the eigenfunctions and their first derivatives across the singularity of the potential, and conclude that only the odd-parity solutions of the Schrödinger equation survive. The problem was also sketched for a Lorentz scalar potential in the Dirac equation [5], but the analysis is incomplete. In a recent work [6] it was shown that the problem of a fermion under the influence of a general scalar potential for nonzero eigenenergies can be mapped into a Sturm-Liouville problem. Next, the key conditions for the existence of bound-state solutions were settled for power-law potentials, and the possible zero-mode solutions were shown to conform with the ultrarelativistic limit of the theory. In addition, the solution for an inversely linear potential was obtained in closed form. The effective potential resulting from the mapping has the form of the Kratzer potential [7]. It is noticeable that this problem has an infinite number of acceptable bounded solutions, nevertheless it has no nonrelativistic limit for small quantum numbers. It was also shown that in the regime of strong coupling additional zero-energy solutions can be obtained as a limit case of nonzero-energy solutions. The ideas of supersymmetry had already been used to explore the two-dimensional Dirac equation with a scalar potential [8]-[9], nevertheless the power-law potential has been excluded of such discussions.

The Coulomb potential of a point electric charge in a 1+1 dimension, con-
sidered as the time component of a Lorentz vector, is linear ($\sim |x|$) and so it provides a constant electric field always pointing to, or from, the point charge. This problem is related to the confinement of fermions in the Schwinger and in the massive Schwinger models [10]-[11], and in the Thirring-Schwinger model [12]. It is frustrating that, due to the tunneling effect (Klein’s paradox), there are no bound states for this kind of potential regardless of the strength of the potential [13]-[14]. The linear potential, considered as a Lorentz scalar, is also related to the quarkonium model in one-plus-one dimensions [15]-[16]. Recently it was incorrectly concluded that even in this case there is solely one bound state [17]. Later, the proper solutions for this last problem were found [18]-[20]. However, it is well known from the quarkonium phenomenology in the real 3+1 dimensional world that the best fit for meson spectroscopy is found for a convenient mixture of vector and scalar potentials put by hand in the equations (see, e.g., [21]). The same can be said about the treatment of the nuclear phenomena describing the influence of the nuclear medium on the nucleons [22]-[30]. The mixed vector-scalar potential has also been analyzed in 1+1 dimensions for a linear potential [31] as well as for a general potential which goes to infinity as $|x| \to \infty$ [32]. In both of those last references it has been concluded that there is confinement if the scalar coupling is of sufficient intensity compared to the vector coupling.

Motivated by the success found in Ref. [6], the problem of a fermion in the background of an inversely linear potential by considering a convenient mixing of vector and scalar Lorentz structures was re-examined [33]. The problem was mapped into an exactly solvable Sturm-Liouville problem of a Schrödinger-like equation with an effective Kratzer potential. The case of a pure scalar potential with their isolated zero-energy solutions, already analyzed [6], was obtained as a particular case. In the present paper the same problem is analyzed for a spinless particle. These new results favour the conclusion that even-parity solutions to the nonrelativistic one-dimensional hydrogen atom do not exist.

In the presence of vector and scalar potentials the 1+1 dimensional time-independent Klein-Gordon equation for a spinless particle of rest mass $m$ reads

$$-\hbar^2 c^2 \frac{d^2 \psi}{dx^2} + (mc^2 + V_s)^2 \psi = (E - V_v)^2 \psi$$

(1)

where $E$ is the energy of the particle, $c$ is the velocity of light and $\hbar$ is the Planck constant. The vector and scalar potentials are given by $V_v$ and $V_s$,.
respectively. The subscripts for the terms of potential denote their properties under a Lorentz transformation: \( v \) for the time component of the 2-vector potential and \( s \) for the scalar term. It is worth to note that the Klein-Gordon equation is covariant under \( x \to -x \) if \( V_v(x) \) and \( V_s(x) \) remain the same.

In the nonrelativistic approximation (potential energies small compared to \( mc^2 \) and \( E \simeq mc^2 \)) Eq. (1) becomes

\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_v + V_s \right) \psi = \left( E - mc^2 \right) \psi
\]

Eq. (2) shows that \( \psi \) obeys the Schrödinger equation with binding energy equal to \( E - mc^2 \), and without distinguishing the contributions of vector and scalar potentials.

It is remarkable that the Klein-Gordon equation with a scalar potential, or a vector potential contaminated with some scalar coupling, is not invariant under \( V \to V + \text{const.} \); this is so because only the vector potential couples to the positive-energies in the same way it couples to the negative-ones, whereas the scalar potential couples to the mass of the particle. Therefore, if there is any scalar coupling the absolute values of the energy will have physical significance and the freedom to choose a zero-energy will be lost. It is well known that a confining potential in the nonrelativistic approach is not confining in the relativistic approach when it is considered as a Lorentz vector. It is surprising that relativistic confining potentials may result in nonconfinement in the nonrelativistic approach. This last phenomenon is a consequence of the fact that vector and scalar potentials couple differently in the Klein-Gordon equation whereas there is no such distinction among them in the Schrödinger equation. This observation permit us to conclude that even a “repulsive” potential can be a confining potential. The case \( V_v = -V_s \) presents bounded solutions in the relativistic approach, although it reduces to the free-particle problem in the nonrelativistic limit. The attractive vector potential for a particle is, of course, repulsive for its corresponding antiparticle, and vice versa. However, the attractive (repulsive) scalar potential for particles is also attractive (repulsive) for antiparticles. For \( V_v = V_s \) and an attractive vector potential for particles, the scalar potential is counterbalanced by the vector potential for antiparticles as long as the scalar potential is attractive and the vector potential is repulsive. As a consequence there is no bounded solution for antiparticles. For \( V_v = 0 \) and a pure scalar attractive potential, one finds energy levels for particles and antiparticles arranged symmetrically.
about \( E = 0 \). For \( V_v = -V_s \) and a repulsive vector potential for particles, the scalar and the vector potentials are attractive for antiparticles but their effects are counterbalanced for particles. Thus, recurring to this simple standpoint one can anticipate in the mind that there is no bound-state solution for particles in this last case of mixing.

Now let us focus our attention on scalar and vector potentials in the form

\[
V_s = -\frac{\hbar c q_s}{|x|}, \quad V_v = -\frac{\hbar c q_v}{|x|}
\]

where the coupling constants, \( q_s \) and \( q_v \), are dimensionless real parameters. In this case Eq. (1) becomes

\[
H_{\text{eff}} \psi = -\frac{\hbar^2}{2m} \psi'' + V_{\text{eff}} \psi = E_{\text{eff}} \psi
\]

where

\[
E_{\text{eff}} = \frac{E^2 - m^2 c^4}{2mc^2}, \quad V_{\text{eff}} = -\frac{\hbar c q_{\text{eff}}}{|x|} + \frac{A}{x^2}
\]

and

\[
q_{\text{eff}} = \frac{mc^2 q_s + Eq_v}{mc^2}, \quad A = \frac{\hbar^2}{2m} \left( q_s^2 - q_v^2 \right)
\]

Therefore, one has to search for bounded solutions of a particle in an effective Kratzer-like potential. The Klein-Gordon eigenvalues are obtained by inserting the effective eigenvalues in (5).

Before proceeding, it is useful to make some qualitative arguments regarding the Kratzer-like potential and its possible solutions. The effective Kratzer-like potential is unable to bind particles on the condition that either \( q_{\text{eff}} < 0 \) and \( A \geq 0 \), or \( q_{\text{eff}} = 0 \) and \( A > 0 \), because the effective potential is repulsive everywhere. For \( q_{\text{eff}} < 0 \) and \( A < 0 \), there appears a double-barrier potential structure with singularity given by \(-1/x^2\). In such a case the effective potential could be a binding potential even with \( E_{\text{eff}} > 0 \). In all the remaining cases the effective potential could bind the particle only with \( E_{\text{eff}} < 0 \), corresponding to Klein-Gordon eigenvalues in the range \(-mc^2 < E < +mc^2\). For \( q_{\text{eff}} > 0 \) and \( A > 0 \) there is a double-well potential structure, and for \( q_{\text{eff}} > 0 \) and \( A \leq 0 \) as well as for \( q_{\text{eff}} = 0 \) and \( A < 0 \), there appears a singularity at the origin given by \(-1/x^2 \) \((-1/|x|\) if \( A = 0 \)). It is worthwhile to note at this point that the singularity \(-1/x^2\) will never expose the particle to collapse to the center [34] on the condition that
$A$ is never less than the critical value $A_c = -\hbar^2/(8m)$, i.e., $q_s^2 \geq q_v^2 - 1/4$. This last observation allows us to get rid of the incongruous possibility of bounded solutions for $q_{\text{eff}} < 0$ and $A < 0$ mentioned above, and for $q_{\text{eff}} = 0$ and $A < 0$ yet. Therefore, one can foresee that only for $q_{\text{eff}} > 0$, i.e., $q_s > -q_v E/(mc^2)$, can the potential hold bound-state solutions, and that the eigenenergies in the range $|E| > mc^2$ correspond to the continuum.

The Schrödinger equation with the Kratzer-like potential is an exactly solvable problem and its solution, for an attractive inversely linear term plus a repulsive inverse-square term in the potential, can be found on textbooks [34]-[36]. Since we need solutions involving repulsive as well as attractive terms in the potential, the calculation including this generalization is presented. The spectrum will be uniquely determined by ensuring the Hermiticity of the effective Hamiltonian given by (4). This means that we will demand normalizable and orthogonal eigenfunctions. As a bonus, the appropriate boundary conditions on the Klein-Gordon wave functions will be proclaimed.

Defining the quantities $z$ and $B$,

\[ z = \frac{2}{\hbar} \sqrt{-2E_{\text{eff}}} |x|, \quad B = q_{\text{eff}} c \sqrt{-\frac{1}{2E_{\text{eff}}}} \quad (7) \]

and using (4)-(5) one obtains the equation

\[ \psi'' + \left( -\frac{1}{4} + \frac{B}{z} - \frac{2A}{\hbar z^2} \right) \psi = 0 \quad (8) \]

Now the prime denotes differentiation with respect to $z$. The normalizable asymptotic form of the solution as $z \to \infty$ is $e^{-z^2/2}$. As $z \to 0$, when the term $1/z^2$ dominates, the solution behaves as $z^s$, where $s$ is a solution of the algebraic equation

\[ s(s - 1) - \frac{2A}{\hbar^2} = 0 \quad (9) \]

viz.

\[ s = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{8A}{\hbar^2}} \right) = \frac{1}{2} \pm \sqrt{q_s^2 - q_v^2 + \frac{1}{4}} \quad (10) \]

Notice that $A \geq A_c$. The boundary conditions on the eigenfunctions comes into existence by demanding that the effective Hamiltonian given (4) is Hermitian, viz.
\[ \int_0^\infty dx \psi_n^{*}(H_{\text{eff}}\psi_n) = \int_0^\infty dx (H_{\text{eff}}\psi_n)^{*}\psi_n' \quad (11) \]

In passing, note that a necessary consequence of Eq. (11) is that the eigenfunctions corresponding to distinct effective eigenvalues are orthogonal. Recalling that \( \psi_n(\infty) = 0 \), it can be shown that (11) is equivalent to

\[ \lim_{x \to 0} \left( \frac{\psi_n^*}{dx} \frac{d\psi_n}{dx} - \frac{d\psi_n^*}{dx} \frac{\psi_n'}{dx} \right) = 0 \quad (12) \]

There results that the allowed values for \( s \) are restricted to \( s \geq 1/2 \) and \( q_s^2 \geq q_v^2 - \frac{1}{4} \quad (13) \)

The constraint over \( s \) implies that only the positive sign in (10) must be considered. The solution for all \( z \) can be expressed as \( \psi(z) = z^s e^{-z/2} w(z) \), where \( w \) is solution of Kummer’s equation \[ zw'' + (b - z)w' - aw = 0 \quad (14) \]

with

\[ a = s - B, \quad b = 2s \quad (15) \]

Then \( w \) is expressed as \( M(a, b, z) \) and in order to furnish normalizable \( \psi \), the confluent hypergeometric function must be a polynomial. This demands that \( a = -n \), where \( n \) is a nonnegative integer in such a way that \( M(a, b, z) \) is proportional to the associated Laguerre polynomial \( L^{b-1}_{n}(z) \), a polynomial of degree \( n \). This requirement, combined with the first equation of (15), also implies into quantized Klein-Gordon eigenvalues:

\[ E = mc^2 - \frac{q_s q_v}{(s+n-1)^2} \pm \sqrt{1 - \frac{q_s^2 - q_v^2}{q_s^2 (s+n-1)^2}}, \quad n = 1, 2, 3, \ldots \quad (16) \]

The Klein-Gordon eigenfunctions on the half-line are given by

\[ \psi(z) = Nz^s e^{-z/2} L^{2s-1}_{n-1}(z) \quad (17) \]

where \( N \) is a constant related to the normalization. Since \( B \) is a positive number,

\[ q_s > -q_v \frac{E}{mc^2} \quad (18) \]
as advertized by the preceding qualitative arguments.

Eqs. \((13)\) and \((18)\) can be used to achieve the constraints on the coupling constants as well as on the allowed signs of \(E\). For instance, there are only positive (negative) energy solutions when a \(q_v = +q_s\) \((q_v = +q_s)\) and \(0 < q_s < 1\). Likewise, there are only positive (negative) energy solutions if \(q_v\) is positive (negative) and \(q_s < 0\). In all the other circumstances the spectra may acquiesce both signs of eigenenergies. Anyway, \(E \to -E\) and \(\psi\) is invariant as \(q_v \to -q_v\).

Now that we know the solution of the problem on the half-line, we will start to analyze some illustrative particular cases.

1) \(q_v = 0\). In the case of a pure scalar potential Eq. \((18)\) demands that \(q_s > 0\). The energy levels are given by

\[
E = \pm mc^2 \sqrt{1 - \left( \frac{q_s}{s + n - 1} \right)^2}
\]

(19)

so that the energy levels for particles and antiparticles are symmetric about \(E = 0\). The nonrelativistic limit of the theory, a regime of weak coupling \((q_s \ll 1)\), furnishes \(E - mc^2 \approx -mc^2q_s^2/(2n^2)\). On the other hand, in the regime of strong coupling, i.e., for \(q_s \gg 1\), one has \(E \approx \pm mc^2 (n - 1)/q_s\) and as the coupling becomes extremely strong the lowest effective eigenvalues end up close to zero. One sees clearly that the eigenvalues for a zero-energy solution can be obtained only as a limit case of a nonzero-energy solution.

2) \(q_v = q_s\). The coupling constants are restricted to positive values, and the energy levels given by

\[
E = mc^2 \frac{n^2 - q_s^2}{n^2 + q_s^2}
\]

(20)

are pushed down from the upper continuum so that these energy levels correspond to bound states of particles. In this case there are no energy levels for antiparticles. All the Klein-Gordon eigenvalues are positive if \(q_s < 1\), and some negative eigenvalues arise if \(q_s > 1\). One has \(E - mc^2 \approx -2mc^2q_s^2/n^2\) as long as \(q_s \ll 1\).

3) \(q_v = -q_s\). Eq. \((18)\) demands that \(q_s > 0\). Only the energy levels emerging from the lower continuum, the energy levels for antiparticles, survive:

\[
E = -mc^2 \frac{n^2 - q_s^2}{n^2 + q_s^2}
\]

(21)
Note that $E \approx mc^2$ only in the strong-coupling regime.

4) $q_s = 0$. In the case of a pure vector potential Eqs. (L3) and (L8) demand that $|q_v| \leq 1/2$ and $\varepsilon(E) = \varepsilon(q_v)$, where $\varepsilon$ stands for the sign function. It follows that

$$E = \varepsilon(q_v) \frac{mc^2}{\sqrt{1 + \left(\frac{q_v}{s+n-1}\right)^2}} \quad (22)$$

In this circumstance, the energy spectrum consists of energy levels either for particles ($q_v > 0$) or for antiparticles ($q_v < 0$). The nonrelativistic limit of the theory, a regime of weak coupling ($0 < q_v \ll 1/2$), furnishes $E - mc^2 \approx -mc^2q_v^2/(2n^2)$.

There are no bounded solutions for particles in mixture $q_v = -q_s$ and the nonrelativistic limit is not viable, as expected. For all the other particular cases one sees that the regime of weak coupling runs in the appropriate nonrelativistic limit: the energy levels for particles are given by the nonrelativistic Coulomb potential. As a matter of fact, this nonrelativistic limit is always feasible in the regime of weak coupling provided $q_s > -q_v$. In all the circumstances, there is no atmosphere for the spontaneous production of particle-antiparticle pairs. No matter the signs of the potentials or how strong they are, the particle and antiparticle levels neither meet nor dive into the continuum. Thus there is no room for the production of particle-antiparticle pairs. This all means that Klein’s paradox never comes to the scenario.

The Klein-Gordon eigenenergies are plotted in Fig. 1 for the four lowest bound states as a function of $q_v/q_s$ ($q_s > 0$). Starting from the minimum possible value for $q_v/q_s$ (that one consonant with (L3) and (L8)), when only antiparticles levels show their face, one sees that those levels tend to sink at the continuum of negative energies while the particle levels emerge from the continuum of positive energies. When $q_v/q_s = 0$, the levels corresponding to particles and antiparticles are disposed symmetrically about $E = 0$. Another noteworthy point is that for $q_v/q_s \geq 1$ ($q_v/q_s \leq 1$) there are only particle (antiparticle) levels. On the other side, Figs. 2 and 3 show the eigenenergies as a function of $q_s/q_v$. In both of these last figures one can perceive the tendency of the spectra to be symmetric about $E = 0$ as $q_s/q_v \rightarrow \infty$. In particular, notice in Fig. 2 that there are only particle levels in the event that $-1 < q_s/q_v < +1$.

Figs. 4 and 5 illustrate the behavior of the position probability density
associated to the Klein-Gordon ground-state eigenfunction, $|\psi|^2$, on the positive side of the $x$-axis for the positive-energy solutions. The normalization was obtained by numerical computation. In Fig. 4 one can note that the position probability density for the particle (antiparticle) states is more concentrated near the origin for $q_v > 0$ ($q_v < 0$). In the limit $q_v \to q_s$ ($q_v \to -q_s$) the position probability densities corresponding to the antiparticles (particles) spread without bound and only those ones corresponding to the particles (antiparticles) are left. Comparison of the different curves in Fig. 5 (the case of a pure vector coupling) shows that the position probability density is notably smaller for smaller values of $q_v$, and that the best localization of the particle is reached for the highest possible value for the coupling constant.

Since the inversely linear potential given by (3) is invariant under reflection through the origin ($x \to -x$), eigenfunctions of the wave equation given by (1) with well-defined parities can be found. Those eigenfunctions can be constructed by taking symmetric and antisymmetric linear combinations of $\psi$. These new eigenfunctions possess the same Klein-Gordon eigenvalue, then there is a two-fold degeneracy. Nevertheless, the matter is a little more complicated because the effective potential presents a singularity. Recall that $\psi$ vanishes at the origin but its first derivative does not, so the symmetric combination of $\psi$ presents a discontinuous first derivative at the origin. In fact, the second-order differential equation given by (1) implies that $\psi'$ can be discontinuous wherever the potential undergoes an infinite jump. In the specific case under consideration, the effect of the singularity of the potential can be evaluated by integrating (1) from $-\delta$ to $+\delta$ and taking the limit $\delta \to 0$. The connection condition among $\psi'(+\delta)$ and $\psi'(-\delta)$ can be summarized as

$$\psi'(+\delta) - \psi'(-\delta) = -\frac{c q_{\text{eff}}}{2\hbar} \int_{-\delta}^{+\delta} dx \frac{\psi}{|x|}$$  \hspace{1cm} (23)

Substitution of (17) into (23) allows us to conclude that $\psi'(+\delta) = \psi'(-\delta)$ in all the circumstances so that we are forced to conclude that the Klein-Gordon eigenfunction must be an odd-parity function. Therefore, the bound-state solutions are nondegenerate.

We have succeeded in searching for exact Klein-Gordon bounded solutions for massive particles by considering a mixing of vector-scalar inversely linear potentials in 1+1 dimensions. For $q_s^2 > q_v^2$, there exist bound-state solutions for particles and antiparticles. As for $q_s^2 \leq q_v^2$, there are bound-state solutions either for particles or for antiparticles. Invariably, the spectrum is nondegenerate and the eigenfunction behaves as an odd-parity function. In addition,
for the special case $q_s > -q_v$ the theory presents a definite nonrelativistic limit ($|q_s|, |q_v| \ll 1$ and $E \simeq mc^2$), as one should expect.

Beyond its intrinsic importance as a new solution for a fundamental equation in physics, the problem analyzed in this paper favors the conclusion that even-parity solutions to the nonrelativistic one-dimensional hydrogen atom do not exist.

Acknowledgments

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References


Figure 1: Klein-Gordon eigenvalues for the four lowest energy levels as a function of $q_v/q_s$. The full thick line stands for $n = 1$, the full thin line for $n = 2$, the heavy dashed line for $n = 3$ and the light dashed line for $n = 4$ ($m = c = q_s = 1$).
Figure 2: Klein-Gordon eigenvalues for the four lowest energy levels as a function of $q_s/q_v$. The full thick line stands for $n = 1$, the full thin line for $n = 2$, the heavy dashed line for $n = 3$ and the light dashed line for $n = 4$ ($m = c = q_v = 1/3$).
Figure 3: The same as in Fig. 2, for $q_v = 1$. 
Figure 4: $|\psi|^2$ as a function of $x$, corresponding to the positive-ground-state energy ($n = 1$). The full thick line stands for $q_v/q_s = -1/2$, the full thin line for $q_v/q_s = 0$ and the dashed line for $q_v/q_s = +1/2$ ($m = c = \hbar = q_s = 1$).
Figure 5: The same as in Fig. 4, but $q_s = 0$. The full thick line stands for $q_v = 0.5$, the full thin line for $q_v = 0.4$ and the dashed line for $q_v = 0.3$. 