Generalized Dirac-Pauli Equation and Spin Light of Neutrino in Magnetized Matter

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Abstract

We consider propagation of a massive neutrino in matter within the quantum approach based on the two equations for the neutrino field: the first one is the Dirac-Pauli equation for a massive neutrino in an external magnetic field generalized on the inclusion of effects of the background matter; the second one is the modified Dirac equation derived directly from the neutrino-matter interaction Lagrangian. On the basis of these two equations the quantum theory of a neutrino moving in the background matter is developed (the exact solutions of these equations are found and classified over the neutrino spin states, the corresponding energy spectra are also derived). Using these solutions we study within the quantum approach the spin light of neutrino (SLν) in matter with the effect of a longitudinal magnetic field being also incorporated. In particular, the SLν radiation rate and total power are derived. The use of the generalized Dirac-Pauli equation also enables us to consider the SLν in matter polarized under the influence of strong magnetic field.

1 Introduction

Recently in a series of our papers [1–3] we have developed the quasi-classical approach to the massive neutrino spin evolution in the presence of external electromagnetic fields and background matter. In particular, we have shown that the well known Bargmann-Michel-Telegdi (BMT) equation [4] of the electrodynamics can be generalized for the case of a neutrino moving in the background matter and being under the influence of external electromagnetic fields. The proposed new equation for a neutrino, which simultaneously accounts for the electromagnetic interaction with external fields and also for the weak interaction with particles of the background matter, was obtained from the BMT equation by the following substitution of the electromagnetic field tensor $F_{\mu\nu} = (E, B)$:

$$F_{\mu\nu} \rightarrow E_{\mu\nu} = F_{\mu\nu} + G_{\mu\nu},$$

(1)

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where the tensor $G_{\mu\nu} = (-P, M)$ accounts for the neutrino interactions with particles of the environment. The substitution (1) implies that in the presence of matter the magnetic $B$ and electric $E$ fields are shifted by the vectors $M$ and $P$, respectively:

$$B \rightarrow B + M, \quad E \rightarrow E - P.$$  \hspace{1cm} (2)

We have also shown how to construct the tensor $G_{\mu\nu}$ with the use of the neutrino speed, matter speed, and matter polarization four-vectors.

Within the developed quasi-classical approach to the neutrino spin evolution we have also considered [5–7] a new type of electromagnetic radiation by a neutrino moving in the background matter in the presence of electromagnetic and/or gravitational fields which we have named the "spin light of neutrino" ($SL\nu$). The $SL\nu$ originates, however, from the quantum spin flip transitions and for sure it is important to revise the calculations of the rate and total power of the $SL\nu$ in matter using the quantum theory. Note that within the quantum theory the radiation emitted by a neutrino moving in a magnetic field was also considered in [8].

In this paper we should like to present a reasonable step forward, which we have made recently [9], in the study of the neutrino interaction in the background matter and external fields. The developed quantum theory of a neutrino motion in the presence of the background matter is based on the two equations for the neutrino wave function. The first equation is obtained in the generalization of the Dirac-Pauli equation of the quantum electrodynamics under the assumption that matter effects can be introduced through the substitution (1). The second of these equations is derived directly from the neutrino interaction Lagrangian averaged over the particles of the background. In the limit of the constant matter density, we get the exact solutions of these equations, classify them over the neutrino helicity states and determine the energy spectra, that depend on the helicity.

Although the neutrino energy spectra in matter correspondent to the two equations are not equal, the difference of energies of the opposite neutrino helicity states, predicted in the linear approximation in the matter density by the two equations, are equal. Then with the use of the obtained neutrino quantum states in matter we develop the quantum theory of the $SL\nu$ and calculate the emitted photon energy, the rate and power of the radiation in matter accounting for the emitted photons polarization. The generalized Dirac-Pauli equation enables us to include the contribution from the longitudinal magnetic field and also to account for the effect of the matter polarization.

2 Neutrino wave function and energy spectrum in matter

In this section we discuss the two quantum equations for a massive neutrino wave function in the background matter. The two wave functions and energy spectra, that correspond to these two equations, are not the same. However, the results for the $SL\nu$ photon energy, the rate and the power obtained on the basis of these two equations in the lowest approximation over the matter density (see Section 3), are equal.
2.1 Modified Dirac-Pauli equation for neutrino in matter

To derive a quantum equation for the neutrino wave function in the background matter we start with the well-known Dirac-Pauli equation for a neutral fermion with non-zero magnetic moment. For a massive neutrino moving in an electromagnetic field $F_{\mu\nu}$ this equation is given by

$$
\left(i\gamma^\mu \partial_\mu - m - \frac{\mu}{2} \sigma^{\mu\nu} F_{\mu\nu}\right)\Psi(x) = 0,
$$

where $m$ and $\mu$ are the neutrino mass and magnetic moment\(^1\), $\sigma^{\mu\nu} = i/2(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$.\(^2\)

Now let us consider the case of a neutrino moving in the presence of matter without any electromagnetic field in the background. Our goal is to study the spin-light photon emission in the process of the neutrino transition between the two quantum states with opposite helicities in the presence of matter. Since the calculation of the $SL\nu$ rate and power are performed below (in Section 3) within the lowest approximation over the density of the background matter, we are interested now in the difference of energies of the two neutrino states with opposite helicities in the presence of matter. The quantum equation for the neutrino wave function, appropriate for this task, can be obtained from (3) with application of the substitution \(^1\) which now becomes

$$
F_{\mu\nu} \rightarrow G_{\mu\nu}.
$$

Thus, we get the quantum equation for the neutrino wave function in the presence of the background matter in the form \([9]\)

$$
\left(i\gamma^\mu \partial_\mu - m - \frac{\mu}{2} \sigma^{\mu\nu} G_{\mu\nu}\right)\Psi(x) = 0,
$$

that can be regarded as the modified Dirac-Pauli equation. The generalization of the neutrino quantum equation for the case when an electromagnetic field is present, in addition to the background matter, is discussed below in Section 2.2. Here we should like to note that Eq.(5) is derived under the assumption that the matter term is small. This condition is similar to the condition of smallness of the electromagnetic term in the Dirac-Pauli equation \([3]\) in the electrodynamics.

The detailed discussion on the evaluation of the tensor $G_{\mu\nu}$ is given in \([1–3]\). We consider here the case of the electron neutrino moving in the unpolarized matter composed of the only one type of fermions of a constant density. For a background of only electrons we get

$$
G^{\mu\nu} = \gamma^{(1)} \rho^{(1)} n \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -\beta_3 & \beta_2 \\
0 & \beta_3 & 0 & -\beta_1 \\
0 & -\beta_2 & \beta_1 & 0
\end{pmatrix},
\gamma = (1 - \beta^2)^{-1/2},
\rho^{(1)} = \frac{\tilde{G}_F}{2\sqrt{2} \mu},
$$

$$
\tilde{G}_F = G_F(1 + 4 \sin^2 \theta_W),
$$

\(^1\)For the recent studies of a massive neutrino electromagnetic properties, including discussion on the neutrino magnetic moment, see Ref. \([10]\)
where $\beta = (\beta_1, \beta_2, \beta_3)$ is the neutrino three-dimensional speed, $n$ denotes the number density of the background electrons, $G_F$ is the Fermi constant, and $\theta_W$ is the Weinberg angle. Note that the neutrino magnetic moment simplifies in Eq. (5). From (39) and the two equations, (3) and (5), it is possible to see that the term $M = \gamma \rho(1)n\beta$ in Eq. (5) plays the role of the magnetic field $B$ in Eq. (3). Therefore, the Hamiltonian form of (5) is

$$i \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \hat{H}_G \Psi(\mathbf{r}, t),$$

where

$$\hat{H}_G = \hat{\alpha} \mathbf{p} + \hat{\beta} m + \hat{V}_G,$$

and

$$\hat{V}_G = -\tilde{G}_F \frac{n}{2\sqrt{2} m} \hat{\beta} \mathbf{p},$$

here $\mathbf{p}$ is the neutrino momentum.

Let us now determine the energies of the two different neutrino helicity states in matter. For the stationary states of Eq. (7) we get

$$\Psi(\mathbf{r}, t) = e^{-i(Et - \mathbf{p}\mathbf{r})}u(\mathbf{p}, E),$$

where $u(\mathbf{p}, E)$ is independent on the spacial coordinates and time. Upon the condition that Eq. (5) has a non-trivial solution, we arrive to the energy spectrum of different helicity states in the background matter [9]:

$$E = \sqrt{\mathbf{p}^2 + m^2 \left(1 - \frac{\alpha \mathbf{p}}{m}\right)^2},$$

where

$$\alpha = \frac{1}{2\sqrt{2}} \tilde{G}_F \frac{n}{m}.$$

It is important that the energy (11) in the background matter depends on the state of the neutrino longitudinal polarization (helicity), i.e. the negative-helicity and positive-helicity neutrinos with equal momentum $\mathbf{p}$ have different energies.

Note that in the relativistic energy limit the negative-helicity neutrino state is dominated by the left-handed chiral state ($\nu_- \approx \nu_L$), whereas the positive-helicity state is dominated by the right-handed chiral state ($\nu_+ \approx \nu_R$). For the relativistic neutrinos one can derive, using Eq. (9), the probability of the neutrino spin oscillations $\nu_L \leftrightarrow \nu_R$ with the correct form of the matter term [1–3] (for the further details see Section 2.2).

The procedure, similar to one used for the derivation of the solution of the Dirac equation in vacuum, can be adopted for the case of the neutrino moving in matter. We apply this procedure to Eq. (7) and arrive to the final form of the wave function of a neutrino...
moving in the background matter [9]:

\[
\Psi_{p,s}(r,t) = \frac{e^{-i(\xi t - pr)}}{2L^{\frac{3}{2}}} \begin{pmatrix}
\sqrt{1 + \frac{m - s\alpha p}{E}} & \sqrt{1 + \frac{s\alpha_p}{p}} e^{i\delta} \\
s\sqrt{1 - \frac{m - s\alpha p}{E}} & \sqrt{1 - \frac{s\alpha_p}{p}} e^{i\delta}
\end{pmatrix},
\]

(13)

where \( L \) is the normalization length and \( \delta = \arctan \frac{p_y}{p_x} \). In the limit of vanishing density of matter, when \( \alpha \rightarrow 0 \), the wave function of Eq. (13) transforms to the solution of the Dirac equation in the vacuum.

Calculations on the basis of the modified Dirac-Pauli equation (5) enables us to reproduce, to the lowest order of the expansion over the matter density, the correct energy difference between the two neutrino helicity states in matter. Therefore, the quantum theory of the \( SL\nu \) in the lowest approximation over the matter density can be developed using this equation (see Section 3). However, in order to derive the correct absolute values for the two neutrino helicity states in matter, we investigate in Section 2.3 the neutrino quantum states on the basis of the modified Dirac equation that we obtain from the corresponding neutrino interaction Lagrangian.

### 2.2 Modified Dirac-Pauli equation in magnetized matter

We should like to note that it is easy to generalize the Dirac-Pauli equation (3) (or (5)) for the case when a neutrino is moving in the magnetized background matter [9]. For this case (i.e., when the effects of the presence of matter and a magnetic field on neutrino have to be accounted for) the modified Dirac-Pauli equation is

\[
\left\{ i\gamma^\mu \partial_\mu - m - \frac{\mu}{2} \sigma^{\mu\nu}(F_{\mu\nu} + G_{\mu\nu}) \right\} \Psi(x) = 0,
\]

(14)

where the magnetic field \( \mathbf{B} \) enters through the tensor \( F_{\mu\nu} \). If a constant magnetic field present in the background and a neutrino is moving parallel (or anti-parallel) to the field vector \( \mathbf{B} \), then the corresponding neutrino energy spectrum can be obtained within the procedure discussed above in the previous section. In particular, the neutrino energy and wave function in the magnetized matter can be obtained from (11) and (13) by the following redefinition

\[
\alpha \rightarrow \alpha' = \alpha + \frac{\mu B_\parallel}{p},
\]

(15)

where \( B_\parallel = (\mathbf{B} \cdot \mathbf{p})/p \). Thus, the neutrino energy in this case reads [9]

\[
E = \sqrt{p^2 + m^2 \left( 1 - s \frac{\alpha p + \mu B_\parallel}{m} \right)^2}.
\]

(16)
For the relativistic neutrinos the expression of Eq. (16) gives, in the linear approximation over the matter density and the magnetic field strength, the correct value (see [1, 3]) for the energy difference of the two opposite helicity states in the magnetized matter:

$$\Delta_{\text{eff}} = \tilde{G}_F \sqrt{2} n + 2 \mu B_{\parallel} \gamma.$$ \hspace{1cm} (17)

that confirms our previous result of refs. [1, 3].

### 2.3 Modified Dirac equation for neutrino in matter

The absolute value of the energy of the neutrino helicity states in the presence of matter can be obtained on the basis of the modified Dirac equation that we derive below directly from the neutrino interaction Lagrangian. For definiteness, we consider again the case of the electron neutrino propagating through moving and polarized matter composed of only electrons (the electron gas). The generalizations for the other flavour neutrinos and also for more complicated matter compositions are just straightforward.

Assume that the neutrino interactions are described by the extended standard model supplied with $SU(2)$-singlet right-handed neutrino $\nu_R$. We also suppose that there is a macroscopic amount of electrons in the scale of a neutrino de Broglie wave length. Therefore, the interaction of a neutrino with the matter (electrons) is coherent. In this case the averaged over the matter electrons addition to the vacuum neutrino Lagrangian, accounting for the charged- and neutral-current interactions, can be written in the form

$$\Delta L_{\text{eff}} = -f^\mu \left( \bar{\nu} \gamma_\mu \frac{1 + \gamma^5}{2} \nu \right), \quad f^\mu = \frac{G_F}{\sqrt{2}} \left( (1 + 4 \sin^2 \theta_W) j^\mu - \lambda^\mu \right),$$ \hspace{1cm} (18)

where the electrons current $j^\mu$ and electrons polarization $\lambda^\mu$ are given by

$$j^\mu = (n, n\mathbf{v}),$$ \hspace{1cm} (19)

and

$$\lambda^\mu = \left( n(\zeta \mathbf{v}), n \zeta \sqrt{1 - v^2} + \frac{n \mathbf{v} (\zeta \mathbf{v})}{1 + \sqrt{1 - v^2}} \right).$$ \hspace{1cm} (20)

The Lagrangian accounts for the possible effect of the matter motion and polarization. Here $n$, $\mathbf{v}$, and $\zeta$ ($0 \leq |\zeta|^2 \leq 1$) denote, respectively, the number density of the background electrons, the speed of the reference frame in which the mean momentum of the electrons is zero, and the mean value of the polarization vector of the background electrons in the above mentioned reference frame. The detailed discussion on the determination of the electrons polarization can be found in [1–3].

From the standard model Lagrangian with the extra term $\Delta L_{\text{eff}}$ being added, we derive the following modified Dirac equation [9] for a neutrino moving in the background matter,

$$\left\{ i \gamma_\mu \partial^\mu - \frac{1}{2} \gamma_\mu (1 + \gamma_5) f^\mu - m \right\} \Psi(x) = 0.$$ \hspace{1cm} (21)
This is the most general equation of motion of a neutrino in which the effective potential \( V_\mu = \frac{1}{2}(1 + \gamma_5)f_\mu \) accounts for both the charged and neutral-current interactions with the background matter and also for the possible effects of the matter motion and polarization. It should be noted here that the modified effective Dirac equations for a neutrino with various types of interactions with the background environment were used previously in [12–18] for the study of the neutrino dispersion relations and derivation of the neutrino oscillation probabilities in matter. If we neglect the contribution of the neutral-current interaction and possible effects of motion and polarization of the matter then from (21) we can get corresponding equations for the left-handed and right-handed chiral components of the neutrino field derived in [13]. The similar equation for a neutrino in the background of non-moving and unpolarized neutrons was also used in [19, 20].

Upon the condition that the equation (21) has a non-trivial solution, we arrive to the energy spectrum of a neutrino moving in unpolarized background matter at rest:

\[
E_\varepsilon = \varepsilon \sqrt{p^2 \left(1 - s\alpha \frac{m}{p}\right)^2 + m^2 + \alpha m}.
\] (22)

The quantity \( \varepsilon = \pm 1 \) splits the solutions into the two branches that in the limit of the vanishing matter density, \( \alpha \rightarrow 0 \), reproduce the positive and negative-frequency solutions, respectively. Note that again the neutrino energy in the background matter depends on the state of the neutrino longitudinal polarization, i.e. in the relativistic case the left-handed and right-handed neutrinos with equal momenta have different energies.

Although the obtained neutrino energy spectrum (22) does not reproduce the one of Eq. (11), an equal result for the energy difference \( \Delta E = E(s = -1) - E(s = +1) \) of the two neutrino helicity states can be obtained from both of the spectra in the low matter density limit \( \alpha \frac{m}{E_0} \ll 1 \):

\[
\Delta E \approx 2m\alpha \frac{p}{E_0},
\] (23)

where we use the notation \( E_0 = \sqrt{p^2 + m^2} \). It should be also noted that for the relativistic neutrinos the energy spectrum for the neutrino helicity states of Eq. (22) in the low density limit reproduces the correct energy values for the neutrino left-handed and right-handed chiral states:

\[
E_{\nu_L} \approx E(s = -1) \approx E_0 + \frac{\tilde{G}_F}{\sqrt{2}} n,
\] (24)

and

\[
E_{\nu_R} \approx E(s = -1) \approx E_0,
\] (25)

as it should be for the active left-handed and sterile right-handed neutrino in matter.

For the wave function of a neutrino in the background matter given by Eq. (21) we get:
\[ \Psi_{\epsilon,p,s}(\mathbf{r}, t) = \frac{e^{-i(E_{\epsilon}t - \mathbf{p}\cdot\mathbf{r})}}{2L_{\parallel}^3} \left( \begin{array}{c} \sqrt{1 + \frac{m}{E_{\epsilon} - \alpha m}} \sqrt{1 + \frac{s p_1}{p}} e^{i\delta} \\ s \varepsilon \sqrt{1 - \frac{m}{E_{\epsilon} - \alpha m}} \sqrt{1 - \frac{s p_1}{p}} e^{i\delta} \\ \varepsilon \sqrt{1 - \frac{m}{E_{\epsilon} - \alpha m}} \sqrt{1 - \frac{s p_1}{p}} e^{i\delta} \end{array} \right), \] (26)

Obviously, in the limit of vanishing density of matter, when \( \alpha \to 0 \), the wave function (26) transforms to the solution of the Dirac equation for a neutrino in the vacuum.

### 3 Spin light of neutrino in matter and magnetic field

The proposed quantum equations (5), (14) and (21) for a neutrino moving in the background matter establish a basis for a new method in the investigation of different processes with participation of neutrinos in the presence of matter and external electromagnetic fields. As an example, we should like to study of the \(<SL\nu>\) in the magnetized matter and develop the quantum theory of this effect.

Within the quantum approach, the corresponding Feynman diagram of the \(<SL\nu>\) in matter is the standard one-photon emission diagram with the initial and final neutrino states described by the "broad lines" that account for the neutrino interaction with matter and the external electromagnetic field. From the usual neutrino magnetic moment interaction, it follows that the amplitude of the transition from the neutrino initial state \( \psi_i \) to the final state \( \psi_f \), accompanied by the emission of a photon with a momentum \( k^\mu = (\omega, \mathbf{k}) \) and a polarization \( e^* \), can be written in the form

\[ S_{fi} = -\mu \sqrt{\frac{2\pi}{\omega L^3}} 2\pi \delta(E_f - E_i + \omega) \int d^3 x \bar{\psi}_f(\mathbf{r})(\hat{\Gamma}e^*) e^{i\mathbf{k}\cdot\mathbf{r}} \psi_i(\mathbf{r}), \] (27)

where

\[ \hat{\Gamma} = i\omega \left\{ \left[ \mathbf{\Sigma} \times \mathbf{\sigma} \right] + i\gamma^5 \mathbf{\Sigma} \right\}. \] (28)

Here \( \mathbf{\sigma} = \frac{k}{\omega} \) is the unit vector pointing in the direction of the emitted photon propagation. The delta-function stands for the energy conservation. Performing the integrations over the spatial co-ordinates, we can recover the delta-functions for the three components of the momentum. In the lowest order of the expansion over the density of the background matter, the properties of the \(<SL\nu>\) (in particular, the rate and radiation power), obtained on the basis of Eqs. (5) and (21), are the same. This is because, as it has been already mentioned, the difference of the energies of the two neutrino helicity states, calculated with use of Eqs. (5) and (21), are equal. The additional effect from a magnetic field (if it is also present in the background environment) can be accounted for if one describes the neutrino on the basis the modified Dirac-Pauli equation of Eq. (14). Note that the \(<SL\nu>\) in the presence of a magnetic field can have interesting applications for magnetized astrophysical media.
Let us suppose that the weak interaction of the neutrino with the electrons of the background is indeed weak. Thus, in wide ranges of densities of matter and strengths of the magnetic field that are appropriate for the astrophysical applications, we can expand the energy \( \frac{\alpha' \mu m}{E_0} \ll 1 \) and in the linear approximation get for the emitted photon energy

\[
\omega = (s_f - s_i) \alpha' m \frac{\beta}{1 - \beta \cos \theta},
\]

where \( \theta \) is the angle between the direction of the neutrino speed \( \beta \).

From the above consideration it follows that the only possibility for the \( SL\nu \) to appear is provided in the case when the neutrino initial and final states are characterized by \( s_i = -1 \) and \( s_f = +1 \), respectively. Thus we conclude, on the basis of the quantum treatment of the \( SL\nu \) in the magnetized matter, that in this process the relativistic left-handed neutrino is converted to the right-handed neutrino and the emitted photon energy is given by

\[
\omega = \frac{1}{1 - \beta \cos \theta} \omega_0,
\]

where we use the notation

\[
\omega_0 = \frac{G_F}{\sqrt{2}} n\beta + 2 \frac{\mu B_\parallel}{\gamma}.
\]

Note that the photon energy depends on the angle \( \theta \) and also on the value of the neutrino speed \( \beta \). In the case of \( \beta \approx 1 \) and \( \theta \to 0 \) we confirm the estimation for the emitted photon energy in the background matter obtained in [5]. If the effect of the background matter is subdominant and the main contribution to the \( SL\nu \) is given by the magnetic field term, then from Eq.(30) we obtain the corresponding result of ref. [8] where the \( SL\nu \) in the presence of a magnetic field was considered.

For the spin light transition rate in the lowest order approximation over the parameter \( \frac{\alpha' \mu m}{E_0} \) we get [9]

\[
\Gamma_{SL} = \mu^2 \omega_0^3 \int S \sin \theta \frac{1}{(1 - \beta \cos \theta)^4} d\theta,
\]

where \( S = (\cos \theta - \beta)^2 + (1 - \beta \cos \theta)^2. \)

The corresponding expression for the radiation power is

\[
I_{SL} = \mu^2 \omega_0^4 \int S \sin \theta \frac{1}{(1 - \beta \cos \theta)^5} d\theta.
\]

Performing the integrations in Eq.(32) over the angle \( \theta \), we obtain for the rate

\[
\Gamma_{SL} = \frac{8}{3} \mu^2 \omega_0^3 \gamma^2.
\]
For the total radiation power from Eq. (34) we get,

$$I_{SL} = \frac{8}{3} \mu^2 \omega_0^4 \gamma^4. \quad (36)$$

From the obtained result of Eq. (35) by switching off the contribution from the magnetic field one can get the $SL\nu$ rate in matter (see also [7]). This result exceeds the value of the neutrino spin light rate derived in [5] by a factor of two because here the neutrinos in the initial state are totally left-handed polarized, whereas in [5] the case of initially unpolarized neutrinos (i.e., an equal mixture of the left- and right-handed neutrinos) is considered.

4 Summary and conclusion

We have developed the quantum approach to description of a neutrino moving in the background matter on the basis of the generalized Dirac-Pauli and modified Dirac equations. In the low matter density limit the two equations give equal values for the energy difference of the opposite neutrino helicity states in matter. The use of the generalized Dirac-Pauli equation also enables us to account for the effect of the longitudinal magnetic field. In the derivation of the Dirac-Pauli equation (see (5) and (14)) it has been supposed that the matter parameter is small,

$$\alpha \frac{m}{E_0^2} = \frac{1}{2\sqrt{2}} \tilde{G}_F \frac{np}{E_0^2} \ll 1. \quad (37)$$

However, due to the fact that even for the extremely dense matter with $n = 10^{37} \text{ cm}^{-3}$ one gets $\frac{1}{2\sqrt{2}} \tilde{G}_F n \sim 1 \text{ eV}$, the restriction (37) does not forbid to use the generalized Dirac-Pauli equation even for a very dense medium of neutron stars if the neutrino has the relativistic energy. It should be noted that in the derivation of the modified Dirac equation (21) no any restrictions of this kind has been made.

On the basis of these two equations the quantum treatment of a neutrino moving in the presence of the background matter has been realized with the effect of the longitudinal magnetic field being incorporated. Within the developed quantum approach, the emission rate and power of the $SL\nu$ in magnetized matter has been calculated accounting for the emitted photons polarization. The existence of the neutrino-self polarization effect in the process of the spin light radiation in the background matter and magnetic field has been shown. The photon energy, in the case when the both effects of the background matter and longitudinal magnetic field are important, has been derived for the first time. The photon energy depends on the density of matter, the value of the neutrino magnetic moment, the strength of the magnetic field and also on the direction of the neutrino propagation in respect to the magnetic field $B$ orientation. The $SL\nu$ radiation and the corresponding neutrino self-polarization effect, due to the significant dependence on the matter density and the magnetic field strength, are expected to be important in different astrophysical dense media and in the early Universe.
In conclusion, let us consider the case when the background magnetic field is strong enough so that the following condition is valid

$$B > \frac{p_F}{2e},$$

where $$p_F = \sqrt{\mu^2 - m_e^2}$$, $$\mu$$ and $$m_e$$ are, respectively, the Fermi momentum, chemical potential, and mass of electrons. Then all of the electrons of the background occupy the lowest Landau level (see, for instance, [11]), therefore the matter is completely polarized in the direction opposite to the unit vector $$\mathbf{B}/B$$. From the general expression for the tensor $$G_{\mu\nu}$$ (see the second paper of [1]) we get

$$G_{\mu\nu} = \gamma n \left\{ \rho^{(1)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta_3 & \beta_2 & 0 \\ \beta_3 & 0 & -\beta_1 & 0 \\ 0 & -\beta_2 & \beta_1 & 0 \end{pmatrix} + \rho^{(2)} \begin{pmatrix} 0 & -\beta_2 & \beta_1 & 0 \\ \beta_2 & 0 & -\beta_0 & 0 \\ -\beta_1 & \beta_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\rho^{(2)} = -\frac{G_F}{2\sqrt{2}\mu}.$$  

On the basis of the modified Dirac-Pauli equation (14) with the tensor $$G_{\mu\nu}$$ given by (39) it is possible to consider the $$SL\nu$$ in the case when the neutrino is moving in the completely polarized matter parallel (or anti-parallel) to the magnetic field vector $$\mathbf{B}$$. The neutrino energy and wave function in such a case can be obtained from (11) and (13) by the following redefinition

$$\alpha \rightarrow \tilde{\alpha} = \alpha \left[ 1 - \frac{\text{sign} \left( \frac{B_\parallel}{B} \right)}{1 + \sin^2 4\theta_W} + \frac{\mu B_\parallel}{p} \right].$$

The second term in brackets in Eq.(40) accounts for the effect of the matter polarization. It follows, that the effect of the matter polarization can reasonably change the total matter contribution to the neutrino energy (16). The emitted $$SL\nu$$ photon energy is determined in this case by (30) with

$$\omega_0 = \frac{G_F}{\sqrt{2} \mu \beta} \left[ (1 + \sin^2 4\theta_W) - \text{sign} \left( \frac{B_\parallel}{B} \right) \right] + 2\frac{\mu B_\parallel}{\gamma}.$$  

Consequently, the effect of the matter polarization significantly influence the rate and radiation power of the $$SL\nu$$.

References
