Quasi-planar steep water waves

V. P. Ruban∗
Landau Institute for Theoretical Physics, 2 Kosygin Street, 119334 Moscow, Russia
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A new description for highly nonlinear potential water waves is suggested, where weak 3D effects are included as small corrections to exact 2D equations written in conformal variables. Contrary to the traditional approach, a small parameter in this theory is not the surface slope, but it is the ratio of a typical wave length to a large transversal scale along the second horizontal coordinate. A first-order correction for the Hamiltonian functional is calculated, and the corresponding equations of motion are derived for steep water waves over an arbitrary inhomogeneous quasi-1D bottom profile.

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I. INTRODUCTION

The problem of water waves is one of the classical fields of the hydrodynamics, and it has been studied extensively over many years. Starting from the middle of 90-s, in the theory of two-dimensional (2D) potential flows of an ideal fluid with a free surface, the so called conformal variables have been actively employed [1–5]. With these variables, highly nonlinear equations of motion for planar water waves can be written in an exact and compact form containing integral operators diagonal in the Fourier representation. Such integrodifferential equations are very suitable for numerical simulation, because effective computer programs for the discrete fast Fourier transform (FFT) are now available (see, e.g., [6]).

Recently, the exact 2D description has been generalized to the case of a highly space- and time-inhomogeneous bottom profile [9, 10]. However, the real water waves are never ideally two-dimensional. Therefore there is a need of a theory, which could describe strongly nonlinear, even breaking waves and, from the other hand, it would take into account 3D effects, at least as weak corrections. In present work such a highly nonlinear weakly 3D theory is suggested as an extension of the exact 2D theory. It should be emphasized that existing approximate nonlinear evolution equations for water waves, for example the famous Kadomtsev-Petviashvily equation, equations of Boussinesq type [11], or the equations obtained by Matsuno [12], are valid just for weakly nonlinear water waves, but not for overturning or breaking waves.

It is a well known fact that a very significant difficulty in the 3D theory of potential water waves is the general impossibility to solve the Laplace equation for the velocity potential \( \varphi(x, y, q, t) \),

\[
\varphi_{x x} + \varphi_{y y} + \varphi_{q q} = 0,
\]

in the flow region \(-H(x, q) \leq y \leq \eta(x, q, t)\) between the (static for simplicity) bottom and a time-dependent free surface, with the given boundary conditions

\[
\varphi|_{y=\eta(x,q,t)} = \psi(x,q,t), \quad (\partial \varphi / \partial n)|_{y=-H(x,q)} = 0.
\]

(Here \( x \) and \( q \) are the horizontal Cartesian coordinates, \( y \) is the vertical coordinate, while the symbol \( z \) will be used for the complex combination \( z = x + iy \)). Therefore a compact expression is absent for the Hamiltonian functional of the system,

\[
\mathcal{H}\{\eta, \psi\} = \frac{1}{2} \int dx dq \int_{-H(x,q)}^{\eta(x,q,t)} \left( \varphi_x^2 + \varphi_y^2 + \varphi_q^2 \right) dy
\]

\[
+ \frac{g}{2} \int \eta^2 dx dq \equiv \mathcal{K}\{\eta, \psi\} + \mathcal{P}\{\eta\},
\]

(the sum of the kinetic energy of the fluid and the potential energy in the vertical gravitational field \( g \)). The Hamiltonian determines the canonical equations of motion (see [13–15], and references therein)

\[
\eta_t = \frac{\delta \mathcal{H}}{\delta \psi}, \quad -\psi_t = \frac{\delta \mathcal{H}}{\delta \eta}
\]

in accordance with the variational principle \( \delta \int \tilde{\mathcal{L}} dt = 0 \), where the Lagrangian is

\[
\tilde{\mathcal{L}} = \int \psi \eta_t dx dq - \mathcal{H}.
\]

In the traditional approach, the problem is partly solved by an asymptotic expansion of the kinetic energy \( \mathcal{K} \) on a small parameter — the slope of the surface (see [13, 15], and references therein). As the result, a weakly nonlinear theory arises, which is not good to describe large-amplitude waves (see [16] for a discussion about the limits of such theory). The theory developed in present work is based on another small parameter — the ratio of a typical length of the waves propagating along the \( x \)-axis, to a large scale along the transversal horizontal direction, denoted by \( q \). Thus, we define \( \varepsilon = (l_x / l_q)^2 \ll 1 \) and note: the less this parameter, the less our flow differs from a purely 2D flow. The profile \( y = \eta(x,q,t) \) of the

*Electronic address: ruban@itp.ac.ru
free surface, the boundary value of the velocity potential \( \psi(x, q, t) \equiv \varphi(x, \eta(x, q, t), q, t) \), and a given bottom profile \( R(x, q) \) are allowed to depend strongly on the coordinate \( x \), while the derivatives over the coordinate \( q \) will be supposed small: \( |\eta_q| \sim \epsilon^{1/2}, |\psi_q| \sim \epsilon^{1/2}, |H_q| \sim \epsilon^{1/2} \).

The paper is organized as follows. Sec. II is devoted to a general description of the present approach. In Sec. III, an explicit expression for the first-order correction \( K^{(1)} \) is obtained, thus we can take into account, in the main approximation, weak 3D effects.

II. GENERAL IDEA OF THE METHOD

In the same manner as in the exact 2D theory \([9, 10]\), instead of the Cartesian coordinates \( x \) and \( y \), we use curvilinear conformal coordinates \( u \) and \( v \), which make the free surface and the bottom effectively flat:

\[
x + iy \equiv z = z(u + iv, q, t), \quad -\infty < u < +\infty, \quad 0 \leq v \leq 1,
\]

where \( z(w, q, t) \) is an analytical on the complex variable \( w = u + iv \) function without any singularities in the flow region \( 0 \leq v \leq 1 \). The bottom corresponds to \( v = 0 \), while on the free surface \( v = 1 \). The boundary value of the velocity potential is \( \varphi|_{v=1} \equiv \psi(u, q, t) \). In the case of a non-horizontal curved bottom, it is convenient to represent the conformal mapping \( z(w, q, t) \) as a composition of two conformal mappings \( w \mapsto \zeta \mapsto z \), similarly to works \([9, 10]\):

\[
z(w, q, t) = \mathcal{Z}(\zeta(w, q, t), q).
\]

Here the intermediate function \( \zeta(w, q, t) \) possesses the property \( \text{Im} \zeta(u + 0i, q, t) = 0 \), thus resulting in the important relation

\[
\zeta(u + i, q, t) = \xi(u, q, t) = (1 + i\hat{R})\rho(u, q, t),
\]

where \( \rho(u, q, t) \) is a purely real function, and \( \hat{R} = i \tanh \hat{k} \) (here \( \hat{k} \equiv -i\hat{\partial}_u \)) is the anti-Hermitian operator, which is diagonal in the Fourier representation: it multiplies the Fourier-harmonics \( \rho_k(u, q, t) \equiv \int \rho(u, q, t)e^{-iku}du \) by \( R_k = i \tanh k \), so that

\[
\hat{R}\rho(u, q, t) = \int [i \tanh k]\rho_k(u, q, t)e^{iku}dk\frac{dk}{2\pi} = \text{P.V.}\int \rho(\hat{u}, q, t) d\hat{u} \equiv \int \rho(u, q, t)e^{-iku}du \quad \text{(9)}
\]

(P.V. means the principal value integral.) A known analytical function \( \mathcal{Z}(\zeta, q) \) determines parametrically the static bottom profile:

\[
X^0[r, q] + iY^0[r, q] = \mathcal{Z}(r, q),
\]

where \( r \) is a real parameter running from \(-\infty\) to \(+\infty\). The profile of the free surface is now given (in the parametric form as well) by the formula

\[
X^i[u, q, t] + iY^i[u, q, t] = \mathcal{Z}(\xi(u, q, t), q).
\]

For equations to be more short, below we do not indicate the arguments \((u, q, t)\) of the functions \( \psi, \xi, \xi \) (the overline denotes the complex conjugate). Also, we introduce the notation \( \mathcal{Z}'(\xi) \equiv \partial_t\mathcal{Z}(\xi, q) \). The Lagrangian of the system in the variables \( \psi, \xi, \) and \( \xi \) can be re-written as follows (compare to \([9]\)):

\[
\mathcal{L} = \int \mathcal{Z}'(\xi)\overline{\mathcal{Z}'(\xi)}\left[\frac{\xi \xi_u - \xi_i \xi_u}{2i}\right] \psi du dq - \mathcal{K}\{\psi, \xi(u, q, t)\}
\]

\[
+ \frac{g}{2} \int \left[\mathcal{Z}'(\xi) - \overline{\mathcal{Z}'(\xi)}\right]^2 \left[\mathcal{Z}'(\xi)\xi_u + \overline{\mathcal{Z}'(\xi)}\xi_u\right] du dq
\]

\[
+ \int \lambda \left[\xi - \xi - \hat{R}\left(\xi + \xi\right)\right] du dq,
\]

where the indefinite real Lagrangian multiplier \( \Lambda(u, q, t) \) has been introduced in order to take into account the relation \((8)\). Equations of motion follow from the variational principle \( \delta\mathcal{A} = 0 \), with the action \( \mathcal{A} \equiv \int L dt \). So, the variation by \( \delta\psi \) gives us the first equation of motion — the kinematic condition on the free surface:

\[
|\mathcal{Z}'(\xi)|^2 \text{Im} (\xi \xi_u) = \frac{\delta\mathcal{K}}{\delta\psi}
\]

Let us divide this equation by \( |\mathcal{Z}'(\xi)|^2|\xi_u|^2 \) and use the analytical properties of the function \( \xi_u/\xi_u \). As the result, we obtain the time-derivative-resolved equation

\[
\xi_t = \xi_u(T + i) \left[\frac{(\partial\xi_u/\partial\xi) - (\mathcal{Z}'(\xi)|\xi_u|^2}{|\mathcal{Z'}(\xi)|^2|\xi_u|^2}\right],
\]

where the linear operator \( T = \hat{R}^{-1} = -i\cot\hat{k} \) has been introduced. Further, the variation of the action \( A \) by \( \delta\xi \) gives us the second equation of motion:

\[
\left[\psi_u \xi_t - \psi_t \xi_u\right] = (\frac{\delta\mathcal{K}}{\delta\xi} Z'(\xi)\]

\[
+ \frac{g}{2i} \text{Im} \left[\mathcal{Z}(\xi)\right] Z'(\xi)|\xi_u| - \frac{(1 + i\hat{R})\Lambda}{2i}.
\]

After multiplying Eq.(15) by \(-2i\xi_u\) we have

\[
\left\{[\psi_t + g \text{Im} \mathcal{Z}(\xi) |\xi_u|^2 - \psi_u \xi_t \xi_u\} |\mathcal{Z}'(\xi)|^2
\]

\[
= (1 + i\hat{R})\Lambda - 2i \left[\frac{(\partial\xi_u/\partial\xi)}{\delta\xi}\right] Z'(\xi)\xi_u,
\]

where \( \Lambda \) is another real function. Taking the imaginary part of Eq.(16) and using Eq.(13), we find \( \Lambda \):

\[
\tilde{\Lambda} = \tilde{T} \left[\psi_u \frac{\delta\mathcal{K}}{\delta\psi}\right] + 2\tilde{T} \text{Re} \left[\frac{(\partial\xi_u/\partial\xi)}{\delta\xi} Z'(\xi)\xi_u\right].
\]

After that, the real part of Eq.(16) gives us the Bernoulli equation in a general form:

\[
\psi_t + g \text{Im} \mathcal{Z}(\xi) = \psi_u \tilde{T} \left[\frac{(\delta\mathcal{K}/\delta\psi)}{Z'(\xi)|\xi_u|^2\right] + \tilde{T} \left[\psi_u (\delta\mathcal{K}/\delta\psi)\right]
\]

\[
+ 2 \text{Re} \left(\tilde{T} - i\right)[(\delta\xi_u/\partial\xi)Z'(\xi)|\xi_u|^2] \left|Z'(\xi)|^2|\xi_u|^2\right],
\]
Equations (14) and (18) completely determine the evolution of the system, provided the kinetic energy functional $K\{\psi, Z, \overline{Z}\}$ is explicitly given. It should be emphasized that in our description a general expression for $K$ remains unknown. However, under the conditions $|\varphi_q| \ll 1$, the potential $\varphi(u, v, q, t)$ is efficiently expanded into a series on the powers of the small parameter $\epsilon$:

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \ldots, \quad \varphi^{(n)} \sim \epsilon^n, \quad (19)$$

where $\varphi^{(n+1)}$ can be calculated from $\varphi^{(n)}$, and the zeroth-order term $\varphi^{(0)} = \text{Re} \phi(w, q, t)$ is the real part of an easily represented (in integral form) analytical function with the boundary conditions $\text{Re} \phi|_{t=1} = \psi(u, q, t)$, $\text{Im} \phi|_{v=0} = 0$. Correspondingly, the kinetic energy functional will be written in the form

$$K = K^{(0)} + K^{(1)} + K^{(2)} + \ldots, \quad K^{(n)} \sim \epsilon^n, \quad (20)$$

where $K^{(0)}\{\psi\}$ is the kinetic energy of a purely 2D flow,

$$K^{(0)}\{\psi\} = \frac{1}{2} \int \left[ (\varphi_u^2 + \varphi_v^2) + J(Q \cdot \nabla \varphi)^2 \right] du \, dv \, dq$$

(21)

and other terms are corrections due to gradients along $q$. Now we are going to calculate the first-order correction $K^{(1)}$.

### III. THE FIRST-ORDER CORRECTIONS

As the result of the conformal change of two variables, the kinetic energy functional is determined by the expression

$$K = \frac{1}{2} \int \left[ (\varphi_u^2 + \varphi_v^2 + J(Q \cdot \nabla \varphi)^2 \right] du \, dv \, dq,$$

(22)

where the conditions $x_u = y_v, x_v = -y_u$ have been taken into account, and the following notations are used:

$$J \equiv |z_u|^2, \quad (Q \cdot \nabla \varphi) \equiv a \varphi_u + b \varphi_v + \varphi_q,$$

$$a = \frac{x_v y_q - x_q y_v}{J} \sim \epsilon^{1/2}, \quad b = \frac{y_u x_q - y_q x_u}{J} \sim \epsilon^{1/2}.$$  

Consequently, the Laplace equation in the new coordinates takes the form

$$\varphi_{uu} + \varphi_{vv} + \nabla \cdot (Q \cdot \nabla \varphi) = 0,$$

(23)

with the boundary conditions

$$\varphi|_{t=1} = \psi(u, q, t), \quad [\varphi_v + bJ(\varphi_q + a \varphi_u + b \varphi_v)]|_{v=0} = 0. \quad (24)$$

Under the condition $\epsilon \ll 1$ it is possible to write the solution as the series (19), with the zeroth-order term satisfying the 2D Laplace equation

$$\varphi^{(0)}_{uu} + \varphi^{(0)}_{vv} = 0, \quad [\varphi^{(0)}_v]|_{v=1} = \psi(u, q, t), \quad [\varphi^{(0)}_v]|_{v=0} = 0.$$

Thus, it can be represented as $\varphi^{(0)} = \text{Re} \phi(w, q, t)$, where

$$\phi(w, q, t) = \int \frac{\psi(s, q, t) e^{ikw}}{\cosh k} \frac{dk}{2\pi}, \quad (25)$$

$$\psi_k(q, t) = \int \phi(u, q, t) e^{-iku} du.$$

(26)

On the free surface

$$\phi(u + i, q, t) \equiv \Psi(u, q, t) = (1 + i \hat{k})\psi(u, q, t). \quad (27)$$

For all the other terms in Eq. (19) we have the relations

$$\varphi^{(n+1)}_{uu} + \varphi^{(n+1)}_{vv} + \nabla \cdot (Q \cdot \nabla \varphi^{(n)}) = 0$$

(28)

and the boundary conditions $\varphi^{(n+1)}|_{v=1} = 0$,

$$[\varphi^{(n+1)}_v + bJ(\varphi^{(n)}_q + a \varphi^{(n)}_u + b \varphi^{(n)}_v)]|_{v=0} = 0.$$  

Noting that $\int (\varphi^{(0)}_v \varphi^{(1)}_v + \varphi^{(0)}_u \varphi^{(1)}_u dudv = 0$ (it is easily seen after integration by parts), we have in the first approximation

$$K^{(1)} = \frac{1}{2} \int J(\varphi^{(1)}_v + a \varphi^{(0)}_u + b \varphi^{(0)}_v) dudv$$

(29)

$$\frac{1}{2} \int \left( \psi|_{v=0} - \int \frac{\phi|_{v=0} z_q}{z_u} \right)^2 dudq.$$  

Since $z(w)$ and $\phi(w)$ are represented as $z(u + iv) = e^{k(1-v)}Z(u)$ and $\phi(u + iv) = e^{k(1-v)}\Psi(u)$, we can use for $v$-integration the following formulas:

$$\int du \int_0^1 [e^{k(1-v)}A(u)][e^{k(1-v)}B(u)] dv$$

$$= \frac{\left( e^{2k} - 1 \right)}{2k} A_k \overline{B_k} \frac{dk}{2\pi}$$

$$= -i \frac{1}{2} \int B(u) \partial_u^{-1} A(u) du$$

$$+ i \frac{2}{2} \int B^{[b]}(u) \partial_u^{-1} A^{[b]}(u) du.$$  

(30)

with $A^{[b]}(u) = e^k A(u), B^{[b]}(u) = e^k B(u)$. As the result, we obtain from Eq. (29) the expression of the form

$$K^{(1)} = K^{(1)}_{[s]} - K^{(1)}_{[b]}, \quad \text{where } K^{(1)}_{[s]} = F\{\Psi, \overline{\Psi}, Z, \overline{Z}\}, \quad K^{(1)}_{[b]} = F\{\Psi^{[b]}, \overline{\Psi^{[b]}}, Z^{[b]}, \overline{Z^{[b]}}\}, \quad \text{with } Z = Z^{[s]}, Z^{[b]} = e^k Z, \quad \Psi^{[b]} = e^k \Psi = (\cosh k)^{-1} \Psi.$$  

The functional $F$ is defined below:

$$F = \frac{i}{8} \int (Z_u \Psi_q - Z_q \Psi_u) \partial_u^{-1} \left[ (Z_u \Psi_q - Z_q \Psi_u)^2 / Z_u \right] \overline{Z}$$

$$- Z \left[ (Z_u \Psi_q - Z_q \Psi_u)^2 / Z_u \right] du.$$  

(31)
From here one can express the variational derivatives \( \delta K^{(1)}/\delta \psi \) and \( \delta K^{(1)}/\delta Z \) by the formulas

\[
\frac{\delta K^{(1)}}{\delta \psi} = \left[ (1 - i \hat{R}) \frac{\delta F}{\delta \psi} + (1 + i \hat{R}) \frac{\delta F}{\delta \bar{\psi}} \right] - \left[ \cosh \hat{k} \right]^{-1} \left( \frac{\delta K^{(1)}}{\delta \psi} + \frac{\delta K^{(1)}}{\delta \bar{\psi}} \right),
\]
\[
\frac{\delta K^{(1)}}{\delta Z} = \frac{\delta F}{\delta Z} - e^{-k} \left( \frac{\delta K^{(1)}}{\delta \psi} + \frac{\delta K^{(1)}}{\delta \bar{\psi}} \right),
\]

The derivatives \( \delta F/\delta \psi \), \( \delta F/\delta Z \), \( \delta K^{(1)}/\delta \psi \), and \( \delta K^{(1)}/\delta Z \) are calculated in a standard manner, for instance,

\[
\frac{\delta F}{\delta \psi} = \frac{i}{8} \left[ \left( Z_u \Psi_q - Z_q \Psi_u \right) + \hat{u}_q \left( \Psi_q - Z_q \Psi_u / Z_u \right) \right] - \frac{i}{8} \left[ \Psi_u \hat{q} \hat{u}^{-1} \left( Z_u \Psi_q - Z_q \Psi_u \right) + \left( \Psi_q - Z_q \Psi_u / Z_u \right) \right].
\]

Now one can substitute \( \delta K/\delta \psi \approx -\hat{R} \psi_u + (\delta K^{(1)}/\delta \psi) \) and \( \delta K/\delta Z \approx (\delta K^{(1)}/\delta Z) \) into the equations of motion (14) and (18), keeping in mind that \( Z = Z(\xi, q) \), \( Z_u = Z'(\xi) \xi_u \), and so on. Thus, the weakly 3D equations of motion are completely derived, and our main goal is achieved.

The answers are more compact in the limit \(|k| \gg 1\), corresponding to the “deep water”, when \( \hat{R} \to \hat{H}, \hat{T} \to -\hat{H} \), with \( \hat{H} \) being the Hilbert operator: \( \hat{H} = i \text{sign} \hat{k} \). In this case \( K^{(1)} |_{[b]} \to 0 \), and therefore

\[
\mathcal{K}_{\text{deep}} \approx -\frac{1}{2} \int \psi \hat{H} \psi_u d\xi dq + F(\Psi, \nabla Z, \bar{Z}).
\]

After appropriate rescaling of the variable \( u \), we can write

\[
Z = u + (i - \hat{H}) Y(u, q, t), \quad Z_u = 1 + (i - \hat{H}) Y_u.
\]

The equations of motion for quasi-plane waves on the deep water look as follows:

\[
\int_Z Z_t = Z_u (\hat{H} - i) \left[ \hat{H} \Psi_u - (\delta F/\delta \psi)/|Z_u|^2 \right],
\]

\[
\psi_t + g Y = \psi_u \hat{H} \left[ \hat{H} \Psi_u - (\delta F/\delta \psi)/|Z_u|^2 \right]
+ \hat{H} \left[ \psi_u \hat{H} \Psi_u - (\delta F/\delta \psi) \right]/|Z_u|^2
- 2 \text{Re} \left( (\hat{H} + i)[Z_u (\delta F/\delta Z)] \right)/|Z_u|^2,
\]

where \( \delta F/\delta \psi = 2 \text{Re} \left[ (1 - i \hat{H}) (\delta F/\delta \Psi) \right] \).

IV. SUMMARY

Thus, now we have nonlinear evolution equations for weakly 3D steep water waves, as for deep water case, as for arbitrary quasi-1D bottom profile. The obtained equations are intended to describe, for example, the sudden formation of giant waves in open sea, as well as overturning waves on a beach. The following step should be development of an efficient numerical method for simulation these equations.