First Order Actions: a New View

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Abstract: We analyse systems described by first order actions using the Hamilton-Jacobi (HJ) formalism for singular systems. In this study we verify that generalized brackets appear in a natural way in HJ approach, showing us the existence of a symplectic structure in the phase spaces of this formalism.

1 Introduction

Systems described by first order actions, \( i.e. \) Lagrangians linear in the velocities \([1, 2]\) appear in many branches in Physics. In 1928 Dirac proposed a system with first order action \([3]\) which has been used since then to describe fermion fields. Later in the 1930’s, a Lagrangian with a linear kinematic term was also proposed in order to describe bosonic fields (DKP theory) \([4, 5, 6]\). In gravitation, this type of Lagrangian appeared for the first time in 1919 with Palatini’s work \([7]\), which came to be the basis of a new method of variation. First order actions were also present in Schwinger’s development of quantum theory \([8, 9]\).

A significant feature of these systems is that they always have a null Hessian matrix, \( i.e. \) they are singular (constrained) systems, and hence they must be properly treated in order to accomplish a hamiltonian formulation.

In the particular case of first-order actions, different approaches can be applied. The most usual one is the formalism developed in 1950 by Dirac \([10, 11, 12]\) to treat general constrained systems, in which the hamiltonian structure is employed \([13, 14, 15, 16, 17]\). One of the main features of the Dirac formalism is the fact that it allows one to introduce generalized brackets which conduct to a consistent quantization of the system. The application of this formalism to first-order action can be found in reference \([18]\).
Another approach to deal with Lagrangians with linear velocities was developed by Faddeev and Jackiw \cite{19} in 1988, where generalized brackets framed on the symplectic structure of phase space were also introduced, leading to a consistent quantization at least to purely bosonic variables.

A third approach that can also be applied to study such systems is the Hamilton-Jacobi (HJ) formalism, which is based on the Carathéodory Equivalent Lagrangian method \cite{20}. This method, developed by Carathéodory to treat regular systems with first derivatives, is an alternative way to obtain the Hamilton-Jacobi equation starting from lagrangian formalism. In 1992 Güler generalized Carathéodory’s method to treat singular systems \cite{21} \cite{22} and more recently Pimentel and Teixeira \cite{23} \cite{24}, worked with Lagrangians with higher order derivatives. In 1998 Pimentel, Teixeira and Tomazelli made an extension to deal with Berezinian singular systems \cite{25}. Important applications of this method can be found in literature \cite{26} \cite{27} \cite{28} \cite{29} \cite{30}, including an application to Lagrangians linear in the velocities \cite{31}, where no generalized brackets are introduced.

In this work we intend to study systems with first-order actions via Hamilton-Jacobi formalism and show how generalized brackets and a symplectic structure appear in a natural way. Accordingly, in the two next sections we will make a review of first-order actions and HJ formalism, respectively. Afterwards we will apply the HJ structure to the Lagrangians of interest and see how generalized brackets are introduced. Then we will show some examples and at last some concluding remarks will be made.

2 First Order Actions

We shall consider here a system whose dynamical evolution is described by some variational principle from the action integral

$$A[z_A] = \int_{t_i}^{t_f} dt L (z_A, \dot{z}_A), \quad A = 1, \ldots, N. \quad (1)$$

In this expression $z_A$ are coordinates and $\dot{z}_A$ are the time derivative of $z_A$. We will assume that the Lagrangian function has linear dependence on the velocities $\dot{z}_A$, i.e.,

$$L (z_A, \dot{z}_A) = \dot{z}_A K^A (z_B) - V (z_B), \quad (2)$$

where $K^A$ and $V$ are arbitrary functions of the coordinates $z_B$.

One can immediately verify that the variational problem remains the same if we consider, instead of $L$, another Lagrangian function $\bar{L} (z_A, \dot{z}_A, t)$, which differs from the first by a total time derivative:

$$\bar{L} (z_A, \dot{z}_A, t) = L (z_A, \dot{z}_A) + \frac{d}{dt} Y (z_A, t) =$$

$$= L + \frac{\partial Y}{\partial t} + \dot{z}_A \frac{\partial Y}{\partial z_A}. \quad (3)$$
An interesting property of this transformation in $L$ is that we can preserve the structure of (2) for $\tilde{L}$, $\tilde{L}(z, \dot{z}, t) = \dot{z}_A \tilde{K}^A(z, t) - \tilde{V}(z, t)$, if $K^A$ and $V$ transform respectively as

$$
\tilde{K}^A(z) = K^A(z) + \frac{\partial Y}{\partial z_A}(z, t), \\
\tilde{V}(z) = V(z) - \frac{\partial Y}{\partial t}.
$$

(4)

Since the variational problem is unchanged by the transformations above, we can expect that the equations of motion depend on quantities that are invariant by these transformations. One example of a quantity that has such a property is the “curl” of $K^A$:

$$
M^{AB} \equiv \frac{\partial K_B}{\partial z_A} - \frac{\partial K_A}{\partial z_B} = -M^{BA} = \tilde{M}^{AB},
$$

(5)

and as it will be seen in the next section, it is related to equations of motions.

3 The Hamilton-Jacobi Formalism for an Arbitrary Lagrangian

The Hamilton-Jacobi (HJ) equation is usually obtained from the hamiltonian approach when a specific canonical transformation is considered. An alternative path to reach HJ formalism was developed by Carathéodory and starts from the lagrangian approach without mentioning the hamiltonian one. According to Carathéodory Equivalent Lagrangian method, a minimum of the action $A$ can be found when we consider a set of functions $\beta_A(z_B, t)$ such that

$$
\tilde{L}(z_A, \dot{z}_A = \beta_A(z_B, t)) = L(z_A, \dot{z}_A) + \frac{\partial}{\partial t} Y(z_A, t) + \frac{\partial Y(z_A, t)}{\partial z_A} \dot{z}_A = 0,
$$

(6)

and in a neighbourhood of $\dot{z}_A = \beta_A(z_B, t)$ the condition $\tilde{L}(z_A, \dot{z}_A) > 0$ is satisfied. From these conditions it follows that

$$
p^A \equiv \left. \frac{\partial L}{\partial \dot{z}_A} \right|_{\dot{z}_A = \beta_B} = -\left. \frac{\partial Y}{\partial z_A} \right|_{\dot{z}_A = \beta_B}.
$$

(7)

If $L$ is a singular Lagrangian then the Hessian matrix, $H^{AB} = \frac{\partial^2 L}{\partial z_A \partial z_B}$, has a null determinant, $\det H^{AB} = 0$; and if this matrix has rank $P = N - R$, then we can find a $P \times P$ submatrix such that

$$
\det H^{ab} = \det \left. \frac{\partial p^a}{\partial z_b} \right|_{\dot{z}_b = \beta_B} \neq 0, \quad a, b = R + 1, \ldots, N.
$$

(8)

For this case we can verify that $R$ momenta $p^\alpha (\alpha = 1, \ldots, R)$ have no dependence on any velocity, which means that $R$ velocities cannot be written as
functions of \( z \) and \( p \), as it happens to \( \dot{z}_b = f_b(z_A, p^\alpha) \). We conclude that \( R \) relations of the type
\[
p^\alpha = -H^\alpha(t, z_\beta \equiv t_\beta, z_\alpha, p^\alpha)
\]
must be satisfied.

Moreover if we define \( H_0 \equiv p^Az_A - L \) it follows from (9) that
\[
p^0 + H^0(t, t_\alpha, z_\alpha, p^\alpha) = 0,
\]
where \( p^0 \equiv \frac{\partial S}{\partial t} \).

We see that equations (9) and (10) lead us to define \( R + 1 \) conditions
\[
\phi'^\alpha \equiv p'^\alpha + H'^\alpha(t_\beta, z_\alpha, p^\alpha) = 0, \quad \alpha', \beta' = 0, 1, ..., R, \tag{11}
\]
where \( t_0 \equiv t \). These conditions \( \phi'^\alpha = 0 \) are usually called constraints, and they constitute a set of first order partial differential equations, called Hamilton-Jacobi Partial Differential Equations (HJPDE).

### 3.1 Integrability Conditions

In order to integrate the HJPDE (11) we can use the method of characteristics[20], which conducts us to total differential equation
\[
\begin{align*}
d\eta^I &= E^{IJ}\frac{\delta \phi^\alpha}{\delta x_\alpha} dt_\alpha = \{\eta^I, \phi^\alpha\} dt_\alpha, \\
dS &= \frac{\partial S}{\partial z_\alpha} dz_\alpha = p^\alpha \frac{\partial \phi^\alpha}{\partial p^\alpha} dt_\alpha,
\end{align*}
\]
\[
I, J = (\zeta, A'); \quad \alpha = 0, ..., R; \quad A' = 0, ..., N; \tag{12}
\]
where \( \{\eta^{1A'}\} = \{z_A\} \) and \( \{\eta^{2A'}\} = \{p^{A'}\} \), \( E^{IJ} = \delta_\beta^\alpha \left[ \delta_\gamma^\beta \delta_\sigma^\alpha - \delta_\gamma^\sigma \delta_\beta^\alpha \right] \), \( I = (\zeta, A'), \quad J = (\sigma, B') \). In this expression we use the definition of Poisson Brackets \( \{F, G\} = \frac{\partial F}{\partial x_\alpha} E^{IJ} \frac{\partial G}{\partial p^\beta} \). According to this method, if the characteristic equations are integrable then the HJPDE will have a unique solution (determined by initial conditions). To obtain (12) we assume that the momenta and coordinates are independent quantities, and we can observe that if \( d\eta^I \) (whose equations will be called equations of motions) constitute an integrable system, then \( dS \) will be integrable as a consequence.

To assure the integrability of the equations of motion we must recall from the theory of differential equations that, associated with a set of total equations, \( dx_I = b_I^J(x_J) dt_\alpha \), there are linear operators \( X^\alpha \) such that
\[
X^\alpha F(x^I) = b_I^J \frac{\partial F}{\partial x_I} = 0. \tag{13}
\]
From this result it is obvious that
\[
\left[X^\alpha, X^\beta\right] F = (X^\alpha X^\beta - X^\beta X^\alpha) F = 0,
\]
if \( F \) is at least twice differentiable.

The partial differential equations \( X_\alpha F = 0 \) are said to be complete if
\[
\left[X^\alpha, X^\beta\right] F = C^\alpha_\gamma^\beta X^\gamma F.
\]

4
If this condition is not satisfied we can define a new operator $X$ such that $XF = 0$, and it must be added to previous set $X_\alpha$, and we must verify if this new set is complete. This procedure must be repeated until a complete set is obtained. The total differential equations will be integrable when the associated partial equations constitute a complete set.

We must notice that when we define a new operator $X$ we are imposing a restriction to phase space. In fact we are searching a subspace of the original phase space where the equations of motions can be integrated.

Considering now the specific case of the equations of motion we have $X^\alpha F = \{ F, \phi^\alpha \}$, and using Jacobi identity for Poisson Brackets it follows $[X^\alpha, X^\beta] F = -\{ F, \{ \phi^\alpha, \phi^\beta \} \}$. The equations of motion will be integrable if

$$\{ \phi^\alpha, \phi^\beta \} = C^\alpha_\gamma \phi^\gamma = 0,$$  \hspace{1cm} (15)

or considering the independence of $t_\alpha$,

$$d\phi^\alpha = \{ \phi^\alpha, \phi^\beta \} dt_\beta = 0.$$

If these conditions are not satisfied we must restrict our phase space with new relations $\phi = 0$ until a complete set of partial differential equations is obtained.

4 The HJ Formalism for First Order Actions

Let us now consider the specific case of section. According to the previous section when the condition $\tilde{L} = 0$ and the action is a minimum, the momenta canonically conjugated to $z_A$ and $z_0 = t$ are respectively

$$p^A = -\frac{\partial Y}{\partial z_A}, \hspace{1cm} A = 1, ..., N,$$

$$p^0 = -\frac{\partial Y}{\partial t}.$$  \hspace{1cm} (17)

However when $\tilde{L} = 0$, we have

$$\tilde{L} = \dot{z}_A \left( K^A (z) - p^A \right) - \dot{z}_0 \left( V (z) + p^0 \right) = \dot{z}_A \tilde{K}^A (z, t) = 0, \hspace{1cm} A' = 0, 1, ..., N,$$

where

$$\tilde{K}^0 \equiv - \left( V (z) - \frac{\partial Y}{\partial t} (z, t) \right).$$  \hspace{1cm} (19)

If we consider $\dot{z}_A'$ as independent quantities then

$$\tilde{K}^A' (z, t) = 0 \Rightarrow \left\{ \begin{array}{l} K^A (z) - p^A = 0, \\ V (z) + p^0 = 0. \end{array} \right.$$  \hspace{1cm} (20)

This is the set of HJPDE, which leads us to define the constraints

$$\phi^A' \equiv p^{A'} - K^A' (z) = 0,$$  \hspace{1cm} (21)
with $K^0 \equiv -V(z)$. From this result we see that all the coordinates have the status of parameters and then, to be consistent with the notation of the previous section, we define

$$t_A \equiv z_A,$$
$$t_0 \equiv z_0 \equiv t.$$

### 4.1 Integrability Conditions

To test the integrability conditions we must calculate the total differential of the constraints (21):

$$d\phi^A' = \left\{\phi^A', \phi^B'\right\} dt_B' = \left\{\phi^A', \phi^0\right\} dt_0 + \left\{\phi^A', \phi^B\right\} dt_B,$$

or explicitly

$$d\phi^0 = \left\{\phi^0, \phi^B\right\} dt_B,$$
$$d\phi^A = \left\{\phi^A, \phi^0\right\} dt_0 + \left\{\phi^A, \phi^B\right\} dt_B = \left\{\phi^A, \phi^0\right\} dt_0 + M^{AB} dt_B.$$  

If we consider the independence of the parameters $dt_B'$ then we see that the equations of motion are integrable only if $\left\{\phi^A, \phi^0\right\} = M^{AB} = 0$. If this is not the case we have some problems, because all the coordinates are already parameters and no further restriction can be done, i.e. we cannot define a new constraint $\phi = 0$. And the obvious conclusion is: the system is not integrable.

Of course this analysis is valid when we consider $dt_B'$ as independent quantities. The question naturally arises: can we find a subspace of the parameters space where the system becomes integrable? To answer this question we admit that $M^{AB}$ is not null and that such construction can be done. Hence, in this subspace we have

$$d\phi^0 = 0,$$
$$d\phi^A = 0.$$

The last expression shows

$$d\phi^A = \left\{\phi^A, \phi^0\right\} dt_0 + M^{AB} dt_B = 0 \Rightarrow M^{AB} dt_B = - \left\{\phi^A, \phi^0\right\} dt_0,$$  

and the dependence among $dt_B$ and $dt_0$ becomes clear.

#### 4.1.1 The $M^{AB}$ Regular Case

Let us now consider the case when $M^{AB}$ is a regular matrix. Hence $\det (M^{AB}) \neq 0$ and the matrix $M^{-1}_{AB}$ does exists. In this case it is straightforward to verify that

$$dt_B = -M^{-1}_{BA} \left\{\phi^A, \phi^0\right\} dt_0.$$  

(24)
If we substitute this result in (22) it follows
$$d\phi^0 = -\{\phi^0, \phi^B\} M_{BA}^{-1} \{\phi^A, \phi^0\} dt_0 = \{\phi^B, \phi^0\} M_{BA}^{-1} \{\phi^A, \phi^0\} dt_0,$$
and since $M_{BA}^{-1} = -M_{AB}^{-1}$ it is immediate that $d\phi^0 = 0$, because $\{\phi^B, \phi^0\} \{\phi^A, \phi^0\} = \{\phi^A, \phi^0\} \{\phi^B, \phi^0\}$.

Now considering the dependence established by (25), the differential of any function $E = E (z, p)$ is given by
$$dE = [[E, \phi^0] - \{E, \phi^B\} M_{BA}^{-1} \{\phi^A, \phi^0\}] dt_0.$$

We can now introduce new Brackets
$$\{F, G\}_* = \{F, G\} - \{F, \phi^B\} M_{BA}^{-1} \{\phi^A, G\},$$
(26)
such that
$$dE = \{E, \phi^0\}_* dt_0.$$ (27)

In particular if we consider in (26) functions $F = F(z_A)$ and $G = G(z_B)$, then
$$\{F, G\}_* = \frac{\partial F}{\partial z_A} M_{AB}^{-1} \frac{\partial G}{\partial z_B},$$ (28)
and if $F = z_A$ and $G = z_B$,
$$\{z_A, z_B\}_* = M_{BA}^{-1}.$$ (29)

Equations (28) and (29) show there is a symplectic structure in phase space in HJ approach.

We can still verify the consistence of this construction by taking $E = z_C$ in (27) and see if (25) is obtained:
$$dz_C = \{z_C, \phi^0\}_* dt_0 = [[z_C, \phi^0] - \{z_C, \phi^B\} M_{BA}^{-1} \{\phi^A, \phi^0\}] dt_0 = -\delta_{BC} M_{BA}^{-1} \{\phi^A, \phi^0\} dt_0 = -M_{CA}^{-1} \{\phi^A, \phi^0\} dt_0.$$
If we consider that $z_C = t_C$ then the verification is straightforward.

At last, expliciting $\{\phi^A, \phi^0\}$, we see
$$dz_C = M_{CA}^{-1} \frac{\partial V}{\partial z_A} dt_0.$$ (30)

This result is in agreement to that one presented in [18], where the lagrangian and hamiltonian approach are considered.

4.1.2 The $M^{AB}$ Singular Case

In what follows we will consider the case when $M^{AB}$ is a singular matrix (i.e. $\det (M^{AB}) = 0$) of rank $P = N - R$. Even in this case the expression (24) holds,
and from this result it is quite simple to verify that if $\lambda^{(\alpha)}$ are $R$ eigenvectors of $M^{AB}$, $M^{AB}\lambda^{(\alpha)}_A = 0$ then

$$\frac{\partial V}{\partial z_A} \lambda^{(\alpha)}_A \ dt_0 = 0.$$ 

Since $M^{AB}$ has rank $P = N - R$ then there is a submatrix $P \times P$ of $M^{AB}$ such that

$$\det \left( M^{ab} \right) \neq 0, \quad a, b = 1, \ldots, P,$$

which implies in the existence of $M^{-1}_{ab}$. Hence we can rewrite (24) as (considering $a = 1, \ldots, P$; $\alpha = P + 1, \ldots, N$)

$$- \{ \phi^A, \phi^0 \} \ dt_0 = M^{AB} \ dt_B = M^{Ab} \ dt_b + M^{A\beta} \ dt_\beta.$$

Taking $A = a$ we see some $dt_b$ can be expressed as a linear combinations of $dt_\beta$

and $dt$:

$$dt_B = -M^{-1}_{ba} \left[ \{ \phi^a, \phi^0 \} \ dt_0 + M^{a\beta} \ dt_\beta \right] \Rightarrow dt_b = -M^{-1}_{ba} \left\{ \phi^a, \phi^\alpha \right\} \ dt_\beta', \quad \alpha' = \{0, \alpha\}. \quad (31)$$

If we consider now the case $A = \alpha$ it follows

$$M^{ab} \ dt_b = -M^{\alpha\beta} \ dt_\beta - \{ \phi^\alpha, \phi^0 \} \ dt,$$

and if we use (32) and consider $dt_\beta$ and $dt_0$ as independent parameters, then

$$\left\{ \left\{ \phi^\alpha, \phi^0 \right\} = \left\{ \phi^\alpha, \phi^b \right\} M^{-1}_{ba} \left\{ \phi^a, \phi^0 \right\} \right\} \left\{ \left\{ \phi^\alpha, \phi^\beta \right\} = \left\{ \phi^\alpha, \phi^b \right\} M^{-1}_{ba} \left\{ \phi^a, \phi^\beta \right\} \right\},$$

which tell us that, if (32) is satisfied, $\{ \phi^\alpha, \phi^0 \}$ and the elements $\{ \phi^\alpha, \phi^\beta \}$ of the matrix $M^{AB} = \{ \phi^A, \phi^B \}$ must be related to $\{ \phi^a, \phi^0 \}, \{ \phi^a, \phi^\beta \}, \{ \phi^\alpha, \phi^b \}$. The second expression is in agreement with the fact that $M^{AB}$ is singular, while the first one must be faced as conditions that actually fix the subspace of the parameters where the system can be integrable. It becomes clear from the results above that the case $A = a$ brings information about the dependence among the $P$ parameters $dt_b$ and the $R$ parameters $dt_\beta$, while the case $A = \alpha$ establishes $R$ conditions that determine the subspace of integrability.

With (32) the differential of $E = E(z, p)$ becomes

$$dE = \left[ \{ E, \phi^\alpha \} - \{ E, \phi^b \} M^{-1}_{ba} \left\{ \phi^a, \phi^\alpha \right\} \right] \ dt_\beta' \Rightarrow dE = \left\{ E, \phi^\alpha \right\} \ dt_\beta'.$$ 

\[ (33) \]
where the Brackets \{F, G\}_*, are introduced
\[
\{F, G\}_* \equiv \{F, G\} - \{F, \phi^b\} M^{-1}_{ab} \{\phi^a, G\}.
\] (34)

And if we consider \(F = F(z_A)\) and \(G = G(z_B)\),
\[
\{F, G\}_* = \frac{\partial F}{\partial z_a} M^{-1}_{ab} \frac{\partial G}{\partial z_b},
\]
and for \(F = z_A\) and \(G = z_B\)
\[
\{z_A, z_B\}_* = \delta_a M^{-1}_{ab} \delta_B 
\]
These two last results show the existence of a reduced symplectic structure in phase space of a singular \(M^{AB}\).

Taking \(E = z_C\) in (33) it follows
\[
dz_C = \left\{ F, \phi^{\beta'} \right\}_* \ dt_{\beta'} = \left[ \left\{ z_C, \phi^{\beta'} \right\} - \left\{ z_C, \phi^b \right\} M^{-1}_{ba} \left\{ \phi^a, \phi^{\beta'} \right\} \right] \ dt_{\beta'} =
\]
\[
= \left[ \delta^{\beta'}_{\beta} - \delta^b_{\beta} M^{-1}_{ba} \left\{ \phi^a, \phi^{\beta'} \right\} \right] \ dt_{\beta'} =
\]
and for \(C = \beta'\):
\[
dz_{\beta'} = dt_{\beta'},
\]
that shows \(z_{\beta'}\) remain arbitrary parameters in this construction. For \(C = b\),
\[
dz_b = -M^{-1}_{ba} \left\{ \phi^a, \phi^{\beta'} \right\} \ dt_{\beta'},
\]
which is consistent with (32) since \(z_b = t_b\). This expression can still be written as
\[
dz_b = M^{-1}_{ba} \left[ \frac{\partial K^a}{\partial z_{\beta'}} - \frac{\partial K^{\beta'}}{\partial z_a} \right] \ dt_{\beta'},
\]
when we explicit the Poisson Bracket \(\{\phi^A, \phi^{\beta'}\}\).

5 Examples

In order to illustrate how the method works, let us consider the two examples studied in [31], where the HJ method was also applied.

1- Starting with the Lagrangian
\[
L = (z_2 + z_3) \dot{z}_1 + z_4 \dot{z}_3 + W(z_2, z_3, z_4),
\]
\[
W(z_2, z_3, z_4) = \frac{1}{2} \left[ (z_4)^2 - 2z_2z_3 - (z_3)^2 \right],
\]
in a four dimensional (coordinate) space, it is immediate to identify $K^A$ and the constraints $\phi^A$:

$$
\begin{align*}
K^1 &= z_2 + z_3, \\
K^2 &= 0, \\
K^3 &= z_4, \\
K^4 &= 0, \\
V &= -W,
\end{align*}
\Rightarrow
\begin{align*}
\phi^1 &= p^1 - K^1 = p^1 - (z_2 + z_3) = 0, \\
\phi^2 &= p^2 - K^2 = p^2 = 0, \\
\phi^3 &= p^3 - K^3 = p^3 - z_4 = 0, \\
\phi^4 &= p^4 - K^4 = p^4 = 0, \\
\phi^0 &= p^0 + V = p^0 - \frac{1}{2} \left[ (z_4)^2 - 2z_2z_3 - (z_3)^2 \right] = 0.
\end{align*}
$$

From here, the matrix $M^{AB}$ is given by

$$
M^{AB} = \frac{\partial K^B}{\partial z_A} - \frac{\partial K^A}{\partial z_B} = \{\phi^A, \phi^B\} \Rightarrow (M^{AB}) = \begin{pmatrix}
0 & -1 & -1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0
\end{pmatrix},
$$

which is regular and whose inverse is

$$
(M^{-1}_{AB}) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0
\end{pmatrix}.
$$

Now we can find the equations of motion by constructing the generalized brackets and using (27), or using (30) in a straightforward way:

$$
\begin{align*}
\begin{cases}
\dot{z}_1 &= z_3 dt, \\
\dot{z}_2 &= z_4 dt, \\
\dot{z}_3 &= -z_4 dt, \\
\dot{z}_4 &= -z_2 dt,
\end{cases}
\Rightarrow
\begin{cases}
\dot{z}_1 &= z_3, \\
\dot{z}_2 &= z_4, \\
\dot{z}_3 &= -z_4, \\
\dot{z}_4 &= -z_2.
\end{cases}
\end{align*}
$$

Manipulating the second and fourth equations we see that

$$
\dot{z}_2 + z_2 = 0 \Rightarrow z_2 = -A \cos t + B \sin t,
$$

and by direct substitution into the second equation it follows

$$
z_4 = A \sin t + B \cos t.
$$

By integration it is verified that

$$
z_3 = A \cos t - B \sin t + C, \\
z_1 = A \sin t + B \cos t + Ct + D.
$$

Now, substituting these results in the constraints we can obtain the momenta:

$$
\begin{align*}
p^1 &= C, \\
p^2 &= 0, \\
p^3 &= A \sin t + B \cos t, \\
p^4 &= 0,
\end{align*}
$$
and the problem is completely solved since we know all phase space variables. Comparing these results with those obtained in [31] we see some differences, the main one being the linear dependence of \( z_1 \) with \( t \). In fact we verify that the result of [31] is a particular case of the one obtained here.

2- Let us now consider the Lagrangian
\[
L = (z_2 + z_3) \dot{z}_1 + k \dot{z}_3 + W(z_2, z_3),
\]
\[
W(z_2, z_3, z_4) = \frac{1}{2} \left[ k^2 - 2z_2z_3 - (z_3)^2 \right],
\]
in a three dimensional (coordinate) space. We identify
\[
\begin{align*}
K_1 &= z_2 + z_3, \\
K_2 &= 0, \\
K_3 &= k, \\
V &= -W,
\end{align*}
\]
and construct \( M^{AB} \):
\[
(M^{AB}) = \begin{pmatrix}
0 & -1 & -1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]
This is a singular matrix and has rank 2, and then we must find an invertible submatrix \( M^{ab} \); this can be done by choosing
\[
(M^{ab}) = \begin{pmatrix}
M^{11} & M^{13} \\
M^{31} & M^{33}
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \Rightarrow (M^{-1}_{ab}) = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]
This choice implies that \( t_3 = z_2 (\phi^a = \phi^2) \) and \( t_b = \{z_1, z_3\} (\phi^a = \{\phi^1, \phi^3\}) \), and the construction of the generalized brackets leads us to the following equations of motion:
\[
\begin{align*}
dz_1 &= (z_2 + z_3) \, dt, \\
dz_3 &= -dz_2.
\end{align*}
\]
By direct integration of the second equation it follows
\[
z_3 = -z_2 + C;
\]
which shows us that
\[
z_1 = Ct + D.
\]
Now we must look for the condition that fixes the subspace where the system is integrable:
\[
\begin{align*}
\{\phi^2, \phi^0\} &= \{\phi^2, \phi^0\} M^{-1}_{a0} \{\phi^a, \phi^0\} \Rightarrow \\
\Rightarrow -z_3 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -z_2 + z_3 \end{pmatrix} \Rightarrow z_2 = 0.
\end{align*}
\]
Hence we see that this system is integrable in the subspace where \( z_2 = 0 \), which leads to conclude that \( z_3 = C \). Substituting these results in the constraints, it follows

\[
\begin{aligned}
p^1 &= C, \\
p^2 &= 0, \\
p^3 &= k.
\end{aligned}
\]

The problem is then completely solved. If we now compare these results with those obtained in [31] we see that they are in agreement if we correctly fix the values of \( C_1 \), \( C_2 \), and \( C_3 \) of reference [31].

3 - Now we will consider a third example of a system of fields known as Proca’s model. The Lagrangian density considered here has the Palatini’s form, where the fields \( A_\mu (x) \) and \( F^{\mu\nu} (x) (\mu, \nu = 0, 1, 2, 3; i, j = 1, 2, 3) \) are considered as independent fields (\( z_A = A_\mu, F^{\mu\nu} \)):

\[
\mathcal{L} = \frac{1}{4} A_\nu \partial_\mu (F^{\mu\nu} - F^{\nu\mu}) - \frac{1}{4} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu
\]

\[
= -\frac{1}{4} (F^{0\nu} - F^{\nu0}) \partial_0 A_\nu + \frac{1}{4} A_\nu \partial_0 F^{0\nu} - \frac{1}{4} A_i \partial_0 F^{0i} - \mathcal{H},
\]

where

\[
\mathcal{H} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{4} (F^{i\nu} - F^{\nu i}) \partial_i A_\nu - \frac{1}{4} A_\nu \partial_i (F^{i\nu} - F^{\nu i}) - \frac{1}{2} m^2 A_\mu A^\mu.
\]

We then identify

\[
\begin{aligned}
K^\nu (x) &= -\frac{1}{4} (F^{0\nu} (x) - F^{\nu0} (x)), \\
K_{00} (x) &= 0, \\
K_{0i} (x) &= \frac{1}{2} A_i (x), \\
K_{i0} (x) &= -\frac{1}{2} A_i (x), \\
K_{ij} (x) &= 0, \\
V (x) &= \mathcal{H} (x),
\end{aligned}
\]

\[
\Rightarrow
\begin{aligned}
\phi^\nu (x) &= \pi^\nu (x) + \frac{1}{4} (F^{0\nu} (x) - F^{\nu0} (x)) = 0, \\
\phi_{00} (x) &= \Pi_{00} (x) = 0, \\
\phi_{0i} (x) &= \Pi_{0i} (x) + \frac{1}{4} A_i (x) = 0, \\
\phi_{i0} (x) &= \Pi_{i0} (x) + \frac{1}{4} A_i (x) = 0, \\
\phi_{ij} (x) &= \Pi_{ij} (x) = 0, \\
\phi^\ell (x) &= p^\ell + \mathcal{H} (x) = 0.
\end{aligned}
\]

The \( M^{AB} \) matrix, now defined as

\[
M^{A_\mu B_\nu} = \frac{\delta K^B (y)}{\delta z_A (x)} - \frac{\delta K^A (x)}{\delta z_B (y)},
\]

is

\[
(M^{A_\mu B_\nu}) = \begin{pmatrix}
0 & 0 & \frac{1}{2} \delta^\mu_\ell \delta (x - y) & -\frac{1}{2} \delta^\mu_\ell \delta (x - y) & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} \delta^\ell_\mu \delta (x - y) & 0 & 0 & 0 & 0 \\
\frac{1}{2} \delta^\mu_\ell \delta (x - y) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

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From this results we see that for any function and it is not inversible. So we must find an inversible submatrix $M^a_{b^v}$, what can be done by choosing $t_3(x) = \{ A_0(x), F^{00}(x), F^{j0}(x), F^{ij}(x) \}$ and $t_b = \{ A_i(x), F^{ij}(x) \}$, such that

$$
(M^a_{b^v}) = \begin{pmatrix}
  0_{3 \times 3} & 1_{3 \times 3} \\
-1_{3 \times 3} & 0_{3 \times 3}
\end{pmatrix} \frac{1}{2} (x - y) \Rightarrow \\
(M^{-1}_{c_{a_s}a}) = \begin{pmatrix}
  0_{3 \times 3} & -1_{3 \times 3} \\
1_{3 \times 3} & 0_{3 \times 3}
\end{pmatrix} 2\delta (z - x)
$$

In order to construct the generalized brackets we must calculate $\{ \phi^{a_s}, \phi^{b_s}_x \}$:

$$
\{ \phi^i(y), \phi^0(x) \} = M^{i\nu}_{\nu} = 0, \\
\{ \phi^0(y), \phi^{00}(x) \} = M^{i\nu}_{\nu} = 0, \\
\{ \phi^i(y), \phi^{j0}(x) \} = M^{i\nu}_{\nu} = -\frac{1}{4} \delta_{ij}\delta (x - y), \\
\{ \phi^i(y), \phi_{mn}(x) \} = M^{i\nu}_{\nu} = 0,
$$

$$
\{ \phi^0(y), \phi^0(x) \} = M^{00}_{\nu} = 0, \\
\{ \phi^0(y), \phi^{00}(x) \} = M^{00}_{\nu} = 0, \\
\{ \phi^0(y), \phi^{j0}(x) \} = M^{00}_{\nu} = 0, \\
\{ \phi^0(y), \phi_{mn}(x) \} = M^{00}_{\nu} = 0,
$$

$$
\{ \phi^i(y), \phi^i(x) \} = \frac{\partial H(x)}{\partial F^{00}(y)} = -\frac{1}{4} [F^{ij}(x) - F^{ji}(x)] \frac{\partial y}{\partial i} \delta (x - y) + \\
+ \frac{1}{4} \left[ \partial_j F^{ij}(x) - \partial_i F^{ij}(x) \right] \delta (x - y) + m^2 A^i (x) \delta (x - y), \\
\{ \phi^0(y), \phi^i(x) \} = -\frac{\partial H(x)}{\partial A_i(y)} = \left[ \frac{1}{2} F_{0i}(x) + \frac{1}{4} \partial_i A_0(x) \right] - \frac{1}{4} A_0(x) \partial_i \delta (x - y).
$$

From this results we see that for any function $G = G(z, p)$

$$
dG = \left\{ G, \phi^{b_s}_x \right\} dt_{\beta^v} + \\
- \left\{ G, \phi^{c_s} \right\} \left(M^{-1}_{c_{a_s}a} \right) \left\{ \phi^{a_s}, \phi^i(x) \right\} dt (x) + \\
- \left\{ G, \phi^{c_s} \right\} \left(M^{-1}_{c_{a_s}a} \right) \left\{ \phi^{a_s}, \phi^{j0}(x) \right\} dF^{j0}(x).
$$

Before obtaining the field equations we will look for the condition that fixes the subspace where the system is integrable,

$$
\{ \phi^{a_w}, \phi^i(x) \} = \{ \phi^{a_w}, \phi^{b_s}_x \} M^{-1}_{b_{a_s}a} \left\{ \phi^{a_s}, \phi^i(x) \right\}.
$$

For $\phi^{a_w} = \phi^0 (w)$,

$$
\{ \phi^0 (w), \phi^i(x) \} = -\frac{1}{4} [F^{j0}(x) - F^{0j}(x)] \delta_i \delta (x - y) + \\
+ \frac{1}{4} \left[ \partial_j F^{j0}(x) - \partial_i F^{0j}(x) \right] \delta (x - y) + \\
m^2 A^0 (x) \delta (x - y),
$$

$$
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$$
and
\[ \{ \phi^0 (w), \phi^\alpha \} M^{-1}_{\alpha \beta} \{ \phi^\beta, \phi^\gamma (x) \} = 0, \]
which lead us to conclude that
\[ \frac{1}{2} \left[ \partial_j F^{0j} (x) - \partial_j F^{0j} (x) \right] + m^2 A^0 (x) = 0. \] (36)

Considering now \( \phi^{\alpha w} = \phi_{00} (w) \),
\[ \left\{ \phi_{00} (w), \phi^i (x) \right\} = \frac{1}{2} F_{00} (x) \delta (w-x), \]
\[ \left\{ \phi_{00} (w), \phi^B \right\} M^{-1}_{B \bar{A}} \left\{ \phi^{\bar{A} s}, \phi^t (x) \right\} = 0, \]
\[ \therefore F_{00} (x) = 0. \] (37)

For \( \phi^{\alpha w} = \phi_{ij} (w) \),
\[ \left\{ \phi_{ij} (w), \phi^i (x) \right\} = \frac{1}{2} \left[ F_{ij} (x) - (\partial_i A_j (x) - \partial_j A_i (x)) \right] \delta (w-x), \]
\[ \left\{ \phi_{ij} (w), \phi^\bar{B} \right\} M^{-1}_{\bar{B} \bar{A}} \left\{ \phi^{\bar{A} s}, \phi^t (x) \right\} = 0, \]
\[ \therefore F_{ij} (x) = \partial_i A_j (x) - \partial_j A_i (x). \] (38)

At last, considering \( \phi^{\alpha w} = \phi_{j0} (w) \),
\[ \left\{ \phi_{j0} (w), \phi^i (x) \right\} = \frac{1}{2} \left[ F_{j0} (x) - \partial_j A_0 (x) \right] \delta (w-x), \]
\[ \left\{ \phi_{j0} (w), \phi^\bar{B} \right\} M^{-1}_{\bar{B} \bar{A}} \left\{ \phi^{\bar{A} s}, \phi^t (x) \right\} = -\frac{1}{2} \left[ F_{j0} (x) - \partial_j A_0 (x) \right] \delta (w-x), \]
\[ \therefore F_{j0} (x) = -F_{0j} (x). \] (39)

Now we are able to obtain the field equations. Taking \( G = A_j (w) \) in (35), we have
\[ dA_j (w) = [F_{0j} (w) + \partial_j A_0 (w)] dt. \] (40)

If we now consider \( G = F^{0j} (w) \),
\[ d \left[ F^{0j} (w) - F^{0j} (w) \right] = \left[ \partial_m F^{jn} (w) - \partial_m F^{mn} (w) - 2m^2 A^k (w) \right] dt. \] (41)

The results above (eqs. 36-41) can be summarized as
\[ F_{\mu \nu} (x) = -F_{\nu \mu} (x) = \partial_{\mu} A_{\nu} (x) - \partial_{\nu} A_{\mu} (x), \]
\[ \partial_{\nu} F^{\mu \nu} (x) + m^2 A^{\nu} (x) = 0. \]

Moreover, when we take the divergence of this last result, as consequence of the antisymmetry property of \( F_{\mu \nu} \), it follows:
\[ \partial_{\nu} A^{\nu} (x) = 0. \]

When we compare these results with those obtained in reference [32], where an analysis of the Proca model is conducted with the usual Hamilton-Jacobi formalism, the agreement is manifest.
6 Final Remarks

In this work we have studied how systems described by Lagrangians with linear velocities can be treated in Hamilton-Jacobi formalism. Initially we observed that all momenta were constrained and therefore all coordinates had the status of parameters. Then, after applying integrability conditions, we saw that if the conditions \( \{ \phi^A, \phi^B \} = 0, M^{AB} = 0 \) are satisfied, then the system of total differential equation is integrable, and all the parameters are independent.

But, if these conditions are not satisfied then the system of differential equation can be integrable only in a subspace of the phase space. Two distinct cases were analysed. Firstly we considered the \( M^{AB} \) regular case, where the system is integrable in the subspace where all the parameters are time-dependent (see (25)). In this subspace we verified that new generalized brackets could be introduced, which allowed us to recognize the symplectic structure of phase space in HJ approach.

Secondly we considered the \( M^{AB} \) singular case, and we saw that integrability is achieved in a subspace where \( t_0 \) and \( t_\beta \) are independent parameters, while the other parameters are dependent of these. We also verified that some extra conditions must be satisfied in order that this subspace could be determined. In this subspace generalized brackets were also introduced and the sympletic structure could also be recognized. Here one interesting feature can be pointed out. In this case we can separate the parameters \( t_B \) in two distinct sets: one is composed by the independent parameters \( t_\beta \), and the other by the dependent ones, \( t_b \). In the same way, the associated constraints can also be separated in two sets composed by \( \phi^b \) (these compose the invesible submatrix \( M^{ab} \)) and \( \phi^\beta \), respectively. This separation is analogous to that one made in Dirac hamiltonian formalism, where the constraints are separated in first and second classes (the constraints \( \phi^b \) would be classified as second class constraints, while \( \phi^\beta \) would be called first class in Dirac formalism). And in the same way that in the hamiltonian formalism the second class constraints lead one to introduce Dirac Brackets, in HJ approach the constraints \( \phi^b \) allows us to define the generalized brackets. We must emphasize that this separation of the constraints in two sets is not necessary in HJ formalism, but we see that it can be made naturally, and that it clarify the equivalence between Dirac and HJ formalisms.

Moreover it is easily seen that the introduction of generalized brackets can be done not only in systems described by first order actions, but also in any system which has a non-null matrix \( M^{AB} = \{ \phi^A, \phi^B \} \) with an invesible submatrix \( M^{ab} \).

At last we must notice that, although we considered only usual variables in this work, the extension to treat berezinian variables is quite immediate.

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