Conformal Field Theory, 
(2+1)-Dimensional Gravity, and the BTZ Black Hole

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Abstract
In three spacetime dimensions, general relativity becomes a topological field theory, whose dynamics can be largely described holographically by a two-dimensional conformal field theory at the “boundary” of spacetime. I review what is known about this reduction—mainly within the context of pure (2+1)-dimensional gravity—and discuss its implications for our understanding of the statistical mechanics and quantum mechanics of black holes.

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It has been thirty years since Hawking first showed that black holes were thermodynamic objects, with characteristic temperatures and entropies [1]. For most of that time—and, indeed, even before Hawking’s work [2]—it has been assumed that these thermodynamic properties reflect the statistical mechanics of underlying quantum gravitational states. But the detailed nature of these states has remained a mystery. The recent proliferation of state-counting methods, in string theory [3], AdS/CFT [4], loop quantum gravity [5], and induced gravity [6], has, if anything, deepened the mystery: we must now also explain the “universality” of black hole entropy [7], the fact that so many distinct and seemingly orthogonal approaches reach the same conclusion.

The problem of black hole statistical mechanics is especially stark in (2+1)-dimensional spacetime. In three spacetime dimensions, general relativity becomes a topological field theory [8–11] with only a few, nonpropagating degrees of freedom; there seems to be little room for quantum states that might account for black hole thermodynamics. The (2+1)-dimensional BTZ black hole of Bañados, Teitelboim, and Zanelli [12, 13], however, can have an arbitrarily high entropy. If we can explain this entropy in such a simple setting, it may take us a long way towards understanding the general problem. This lower-dimensional model becomes even more interesting when one notes that the near-extremal black holes whose entropy can be computed in string theory almost all have a near-horizon geometry containing the BTZ solution, and that the corresponding entropies can be determined from this dimensionally reduced geometry.

In this review, I will summarize the current—highly incomplete—understanding of the microscopic statistical mechanics of the BTZ black hole. The key will be that much of (2+1)-dimensional gravity can be described “holographically” by a two-dimensional conformal field theory. This conformal field theory is, unfortunately, of a type that is still poorly understood, so many questions remain. But considerable progress has now been made, and some directions for further research are clear.

The literature in this field is by now enormous, and my treatment will necessarily be rather incomplete. This paper should be read as a personal overview, not as a comprehensive review. In particular, I will have relatively little to say about string theory, and will only touch briefly on the extension to supergravity. Several sections of this work are based on a previous paper, Ref. [14].

1 (2+1)-Dimensional Gravity

I will begin with a very brief summary of some key aspects of general relativity in three spacetime dimensions. Much more extensive discussions can be found in [15–17]. The idea that (2+1)-dimensional gravity might be a useful arena for investigating more general questions dates back to at least 1966 [18], but the power of the model only became clear with the work of Deser, Jackiw, and ’t Hooft [19–22]. For us, the first essential feature is that (2+1)-dimensional vacuum gravity has no local degrees of freedom [18, 23]. This can be shown by a simple counting argument: the phase space consists of a spatial metric (three degrees of freedom per point) and its canonical momentum (another three degrees of freedom per point); but the theory also has three constraints that restrict initial data and three arbitrary coordinate choices, leaving no unconstrained, non-“gauge” degrees of freedom.
freedom.

Alternatively, one can note that the curvature tensor in 2+1 dimensions is algebraically determined by the Ricci tensor:

$$G_{\mu\nu} = -\frac{1}{4} \epsilon^{\mu\pi\rho} \epsilon_{\nu\sigma\tau} R_{\pi\rho\sigma\tau}. \quad (1.0.1)$$

In particular, a vacuum solution ($R_{\mu\nu} = 0$) necessarily has a vanishing curvature, and can therefore be constructed by “gluing together” flat pieces of Minkowski space. For a topologically nontrivial manifold, this gluing need not be unique, and one can have a variety of inequivalent “geometric structures” [17, 24]. But these are labeled by a finite number of global parameters that describe the gluing, and do not involve any propagating degrees of freedom. Similarly, any solution of the vacuum field equations with a negative cosmological constant has constant negative curvature, and can be constructed by gluing patches of anti-de Sitter space.

For spacetimes with boundaries or asymptotic regions, this picture becomes a bit more complex. For the action principle to hold—that is, for the Einstein-Hilbert action to have any extrema at all—one must typically introduce boundary conditions on the fields and add boundary terms to the action. These generically break the gauge and diffeomorphism symmetries of the theory: configurations that are gauge equivalent in the absence of boundaries may not be connected by transformations that behave properly at the boundaries. Moreover, the remaining transformations at the boundary are properly viewed as symmetries, not gauge invariances [25, 26]: as explained below in section 2 physical states need not be invariant, but can transform under representations of the group of boundary transformations. As a consequence, new “would-be pure gauge” degrees of freedom appear at the boundary [7].

At first sight, this treatment of boundary gauge transformations may seem somewhat arbitrary. But in certain cases, including Chern-Simons theory—which, as we shall see below, is closely linked to (2+1)-dimensional gravity—the “would-be pure gauge” degrees of freedom are needed to provide a complete intermediate set of quantum states [27]. In particular, consider a Chern-Simons path integral on a manifold $M_1$ with boundary $\Sigma$. The partition function $Z_{(M_1, \Sigma)}[A, g]$ depends on the boundary value of the gauge field $A$ and on a gauge parameter $g$ that acts as a propagating field on $\Sigma$. If $M_2$ is another manifold with a diffeomorphic boundary $\Sigma$, one can “glue” $M_1$ and $M_2$ along $\Sigma$ to obtain a compact manifold $M = M_1 \cup_{\Sigma} M_2$. The path integral equivalent of summing over intermediate states is to set the boundary values of the fields equal and integrate over these boundary fields:

$$Z_M = \int [dA] [dg] Z_{(M_1, \Sigma)}[A, g] Z_{(M_2, \Sigma)}[A, g]. \quad (1.0.2)$$

Witten shows in [27] that this process gives the correct Chern-Simons partition function for $M$ only if the “gauge” degrees of freedom $g$ are included, with the proper Wess-Zumino-Witten action described below in section 4.1. Thus, at least for Chern-Simons theory, the “would-be gauge” degrees of freedom are necessary for a consistent quantum theory.
1.1 Gravity and Chern-Simons Theory

As first noted by Achucarro and Townsend [9] and subsequently extensively developed by Witten [10, 11], vacuum Einstein gravity in three spacetime dimensions is equivalent to a Chern-Simons gauge theory. We will be interested in the case of a negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$. Then the coframe (or “triad” or “dreibein”) $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^a = \frac{1}{2} \epsilon^{abc} \omega^c_{\mu} dx^\mu$ can be combined into two SL($2, \mathbb{R}$) connection one-forms

$$A^{(\pm)} = \omega^a \pm \frac{1}{\ell} e^a.$$  

(1.1.1)

It is straightforward to show that up to possible boundary terms, the first-order form of the usual Einstein-Hilbert action can be written as

$$I = \frac{1}{8\pi G} \int_M \left\{ e^a \wedge \left( d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \right) + \frac{\Lambda}{6} \epsilon_{abc} e^a \wedge e^b \wedge e^c \right\} = I_{CS}[A^{(+)}) - I_{CS}[A^{(-)}]$$  

(1.1.2)

where $A^{(\pm)} = A^{(\pm) a} T_a$ are SL($2, \mathbb{R}$)-valued gauge potentials (see Appendix A for conventions for the generators $T_a$), and the Chern-Simons action $I_{CS}$ is

$$I_{CS} = \frac{k}{4\pi} \int_M \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\},$$  

(1.1.3)

with

$$k = \frac{\ell^2}{4G}.$$  

(1.1.4)

Similarly, the Chern-Simons field equations

$$F^{(\pm)} = dA^{(\pm)} + A^{(\pm)} \wedge A^{(\pm)} = 0$$  

(1.1.5)

are easily seen to be equivalent to the requirement that the connection be torsion-free and that the metric have constant negative curvature, as required by the vacuum Einstein field equations.

This formulation has the enormous advantage that gravity becomes an ordinary gauge theory. In particular, diffeomorphisms, which create endless complications in standard approaches to quantum gravity, are now equivalent on shell to ordinary gauge transformations. Indeed, the Lie derivative of the connection is

$$\mathcal{L}_\xi A = d(\xi \cdot A) + \xi \cdot dA = \xi \cdot F + D_A (\xi \cdot A)$$  

(1.1.6)

where $D_A$ is the gauge-covariant exterior derivative. It is evident that on shell—that is, when $F = 0$—eqn. (1.1.6) is simply an ordinary infinitesimal gauge transformation with gauge parameter $\lambda^a = \xi^\mu A^{\mu a}$.

1.2 The BTZ Black Hole

When the cosmological constant is zero, a vacuum solution of (2+1)-dimensional gravity is necessarily flat, and it can be shown that there are no black hole solutions [28]. It therefore
came as an enormous surprise when Bañados, Teitelboim, and Zanelli showed that vacuum (2+1)-dimensional gravity with $\Lambda < 0$ admitted a black hole solution [12]. A review of this solution is given in [29]; here I will just touch on a few of the most relevant features.

The BTZ black hole in “Schwarzschild” coordinates is given by the metric

$$ds^2 = (N^\perp)^2 dt^2 - f^{-2} dr^2 - r^2 \left( d\phi + N^\phi dt \right)^2$$

(1.2.1)

with lapse and shift functions and radial metric*

$$N^\perp = f = \left( -8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2} \right)^{1/2}, \quad N^\phi = -\frac{4GJ}{r^2} \quad (|J| \leq M\ell).$$

(1.2.2)

The metric (1.2.1) is stationary and axially symmetric, with Killing vectors $\partial_t$ and $\partial_\phi$, and generically has no other symmetries. Although it describes a spacetime of constant negative curvature, it is a true black hole: it has a genuine event horizon at $r_+$ and, when $J \neq 0$, an inner Cauchy horizon at $r_-$, where

$$r^2_{\pm} = 4GM\ell^2 \left\{ 1 \pm \left[ 1 - \left( \frac{J}{M\ell} \right)^{2} \right]^{1/2} \right\},$$

(1.2.3)

i.e.,

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}, \quad J = \frac{r_+r_-}{4G\ell}.$$

(1.2.4)

Kruskal coordinates are discussed in [13]; the Penrose diagram is essentially identical to that of an asymptotically anti-de Sitter black hole in 3+1 dimensions. Another useful coordinate system is based on proper radial distance $\rho$ and two light-cone-like coordinates $u, v = t/\ell \pm \phi$ [30]; the metric then takes the form

$$ds^2 = 4G\ell \left( L^+ du^2 + L^- dv^2 \right) - \ell^2 d\rho^2 + \left( \ell^2 e^{2\rho} + 16G^2L^+L^- e^{-2\rho} \right) dudv$$

(1.2.5)

with

$$L^\pm = \left( \frac{r_+ \pm r_-}{16G\ell} \right)^2.$$

(1.2.6)

In these coordinates, the Chern-Simons connections take the simple form

$$A^{(+)} = \begin{pmatrix} \frac{1}{2} d\rho & -\frac{4G}{\ell}L^+ e^{-\rho} du \\ -e^\rho du & -\frac{1}{2} d\rho \end{pmatrix}, \quad A^{(-)} = \begin{pmatrix} -\frac{1}{2} d\rho & -e^\rho dv \\ -4G L^- e^{-\rho} dv & \frac{1}{2} d\rho \end{pmatrix}$$

(1.2.7)

It is easy to check that these connections satisfy the equations of motion (1.1.5). This solution may be generalized: the Einstein field equations are still satisfied if one allows $L^+$ to be an arbitrary function of $u$ and $L^-$ to be an arbitrary function of $v$. This dependence can be removed by a suitable diffeomorphism, but as we shall see later, such a diffeomorphism does not satisfy appropriate boundary conditions at infinity. $L^+$ and $L^-$ are thus examples of the “would-be pure gauge” degrees of freedom discussed above.

*Many papers use the conventions of [12], in which units are chosen such that $8G = 1$. 

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As a constant curvature spacetime, the BTZ black hole is locally isometric to anti-de Sitter space. In fact, it is globally a quotient space of $\text{AdS}_3$ by a discrete group. We can identify $\text{AdS}_3$ with the universal covering space of the group $\text{SL}(2, \mathbb{R})$; the BTZ black hole is then obtained by the identification \( g \sim \rho^- g \rho^+ \), \( \rho^\pm = \begin{pmatrix} e^{\pi(r_+ \pm r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ \pm r_-)/\ell} \end{pmatrix} \).

Up to a gauge transformation, the group elements $\rho^\pm$ can be identified with the holonomies of the $\text{SL}(2, \mathbb{R})$ connections (1.2.7).

For our purposes, the most important feature of the BTZ black hole is that it has thermodynamic properties closely analogous to those of realistic (3+1)-dimensional black holes: it radiates at a Hawking temperature of

\[
T = \frac{\hbar \kappa}{2\pi} = \frac{\hbar (r_+^2 - r_-^2)}{2\pi \ell^2 r_+},
\]

where $\kappa$ is the surface gravity, and has an entropy

\[
S = \frac{2\pi r_+}{4\hbar G}
\]

equal to a quarter of its area. These features can be obtained in all the usual ways: from quantum field theory in a BTZ background [12, 31, 32]; from Euclidean path integration [33]; from the Brown-York microcanonical path integral [34]; from Wald’s Noether charge approach [29, 35]; and from tunneling arguments [36, 37]. There is even a powerful new method available [38]: one can consider quantum gravitational perturbations induced by a classical scalar source, and then use detailed balance arguments to obtain thermodynamic properties. Together, these results strongly suggest that many of the mysteries of black hole statistical mechanics in higher dimensions can be investigated in this simpler setting as well.

2 Gauge Invariances and Symmetries

I argued in section 1 that most of the degrees of freedom for (2+1)-dimensional anti-de Sitter gravity represent excitations that would naively be considered “pure gauge,” but that become physical at the conformal boundary. As a first step in obtaining these degrees of freedom, one must understand a rather subtle distinction between gauge invariances and symmetries on manifolds with timelike boundaries. The difference between “proper” and “improper” gauge transformations was first, I believe, studied in detail by Benguria et al. [26], although it was to some extent implicit in [25, 39]. This separation of gauge transformations and symmetries has had some interesting applications in gauge theories [40], but while the distinction is well known among experts, it is rarely explained clearly.

Let us begin with perhaps the simplest example [41]. Consider the Chern-Simons action (1.1.3) on a manifold with the topology $\mathbb{R} \times \Sigma$, where $\Sigma$ is a two-manifold with boundary $\partial \Sigma$. In canonical form, the action becomes

\[
I_{CS} = \frac{k}{4\pi} \int dt \int_{\Sigma} d^2x \epsilon^{ij} \text{Tr} \left( \dot{A}_i A_j + A_0 F_{ij} \right),
\]

(2.0.1)
from which we can read off the Poisson brackets
\[
\left\{ A^a(x), A^b(x') \right\} = \frac{2\pi}{k} \epsilon_{ij} \hat{g}^{ab} \delta^2(x - x')
\] (2.0.2)

where \( \hat{g}^{ab} = \text{Tr}(T^a T^b) \) is the Cartan-Killing metric on the gauge group. It is apparent from (2.0.1) that \( A_0 \) is a Lagrange multiplier. The corresponding first class constraints

\[
G_a^{(0)} = \frac{k}{4\pi} \hat{g}^{ab} \epsilon_{ij} F_{ij}
\] (2.0.3)

generate gauge transformations; that is, if \( \Sigma \) is closed, the smeared generators

\[
G^{(0)}[\eta] = \int_{\Sigma} d^2 x \eta^a G_a^{(0)}
\] (2.0.4)

have brackets

\[
\left\{ G^{(0)}[\eta], A^a_k \right\} = D_k \eta^a = \delta \eta A^a_k,
\] (2.0.5)

where \( D_k \) is the gauge-covariant derivative. Further, the generators satisfy a Poisson algebra isomorphic to the gauge algebra,

\[
\left\{ G^{(0)}[\eta], G^{(0)}[\xi] \right\} = G^{(0)}[\zeta], \quad \zeta^c = f^c_{ab} \eta^a \xi^b,
\] (2.0.6)

where \( f^c_{ab} \) are the structure constants of the gauge group.

We now make the crucial observation that on a manifold with boundary, the generators \( G_a \) are not “differentiable” [25]: that is, the functional derivative of a smeared generator \( G^{(0)}[\eta] \) involves an ill-defined surface term. Indeed, a simple calculation shows that

\[
\delta G^{(0)}[\eta] = \frac{k}{2\pi} \int_{\Sigma} d^2 x \epsilon^{ij} \eta_a D_i \delta A^a_j = -\frac{k}{2\pi} \int_{\Sigma} d^2 x \epsilon^{ij} D_i \eta_a \delta A^a_j + \frac{k}{2\pi} \int_{\partial \Sigma} \eta_a \delta A^a_k d^k x,
\] (2.0.7)

and if \( \eta \neq 0 \) on the boundary, the last term ruins the Poisson algebra (2.0.6). To restore the algebra, one must add a boundary term \( Q[\eta] \) to \( G^{(0)}[\eta] \), with a variation

\[
\delta Q[\eta] = -\frac{k}{2\pi} \int_{\partial \Sigma} \eta_a \delta A^a_k d^k x.
\] (2.0.8)

The full generator \( G[\eta] = G^{(0)}[\eta] + Q[\eta] \) then has a well-defined functional derivative, and a straightforward computation [42] yields the Poisson algebra

\[
\{ G[\eta], G[\xi] \} = G[\zeta(\eta, \xi)] + \frac{k}{2\pi} \int_{\partial \Sigma} \eta_a d\xi^a.
\] (2.0.9)

I have assumed here that the gauge parameters \( \eta \) and \( \xi \) are independent of the fields, and therefore have vanishing Poisson brackets with the generators \( G \); see [42, 43] for a useful generalization to field-dependent parameters.

Equation (2.0.9) can be recognized as a central extension of the original algebra of gauge transformations. We shall return to this point below. But let us first consider the implications for the symmetries of our Chern-Simons theory.
The quantity $G^{(0)}[\eta]$ vanishes by virtue of the field equations, and its Poisson bracket with any physical observable $\mathcal{O}$ must therefore also vanish: $\{G^{(0)}[\eta], \mathcal{O}\} = 0$. In the quantum theory, the Poisson brackets become commutators, and the corresponding statement is that matrix elements of $[G^{(0)}[\eta], \mathcal{O}]$ between physical states must vanish. On a compact manifold, this is simply the statement that physical observables must be gauge-invariant, and that objects that are “pure gauge” can have no physical meaning. Similarly, the vanishing of $G^{(0)}[\eta]$ normally implies that

$$G^{(0)}[\eta]|_{\text{phys}} = 0,$$

(2.0.10)

although this condition can be weakened slightly, for example by requiring it to hold only for the positive frequency components of $G^{(0)}[\eta]$.

If $\Sigma$ has a boundary, on the other hand, the generator of gauge transformations is not $G^{(0)}[\eta]$, but rather $G[\eta]$. In general, the boundary contribution $Q[\eta]$ to $G[\eta]$ need not vanish. Indeed, if $\eta \neq 0$ at $\partial \Sigma$, setting $G[\eta]$ to zero would be inconsistent with (2.0.9). Hence physical observables need not be invariant under gauge transformations at the boundary; it is enough that they transform under some representation of the algebra (2.0.9). Equivalently, if we look at the algebra of the original constraints $G^{(0)}[\eta]$ in the presence of a boundary, we find [44, 45]

$$\left\{ G^{(0)}[\eta], G^{(0)}[\xi] \right\} = G^{(0)}[f^{c}_{\ ab}\eta^{b}\xi^{c}] + \delta\text{-function boundary terms}. \quad (2.0.11)$$

This means that the constraints $G^{(0)}[\eta]$ become second class at the boundary, and we can no longer consistently impose condition (2.0.10).

Gauge transformations are thus very different in the bulk and at a boundary: in the bulk they are true invariances, but at a boundary they are only symmetries. While I have shown this in detail for Chern-Simons theory, a similar argument holds for any gauge or gauge-like theory. This, as we shall see, has very important implications for the physics of (2+1)-dimensional gravity.

3 Asymptotic Symmetries, AdS/CFT, and State-Counting

The first hint that boundary degrees of freedom can account for the entropy of the BTZ black hole comes from a symmetry argument that does not require knowledge of the detailed dynamics [46,47]. The boundary of an asymptotically anti-de Sitter space is a cylinder, both topologically and metrically, and it is not surprising that the asymptotic diffeomorphisms are related to the diffeomorphisms of a cylinder. What is somewhat surprising is that the resulting Virasoro algebra contains a central term [48], which can be determined by standard methods within the framework of classical general relativity. One can then appeal to the remarkable result, due to Cardy [50, 51], that the asymptotic density of states in a two-dimensional conformal field theory is fixed by a few features of the symmetry algebra, independent of any details of the dynamics.\footnote{As far as I know, the idea of using the Cardy formula to count BTZ black hole states was first suggested in [33], but the authors of that paper were unaware of the computation of the central charge in [48], and therefore gave only qualitative arguments.} While such a symmetry argument does not
explain the underlying quantum mechanical degrees of freedom, it does suggest a solution to the problem of universality [7] discussed in the Introduction.

3.1 Asymptotic Symmetries

The asymptotic symmetries of (2+1)-dimensional asymptotically anti-de Sitter space were first investigated by Brown and Henneaux [48, 49]. The analysis involves two key steps. First, one must find the diffeomorphisms that preserve the asymptotic structure of the anti-de Sitter metric (1.2.1). It is straightforward to show that these are generated by vector fields of the form

\[ \xi^{(+)}(+) = \ell T^+ + \frac{\ell^3}{2r^2} \partial_u T^+ + O\left(\frac{1}{r^4}\right) \]
\[ \xi^{(-)}(+) = \ell T^+ + \frac{\ell^3}{2r^2} \partial_v T^+ + O\left(\frac{1}{r^4}\right) \]
\[ \xi^{(+)}(\pm) = -r \partial_u T^+ + O\left(\frac{1}{r}\right) \]
\[ \xi^{(-)}(\pm) = -r \partial_v T^+ + O\left(\frac{1}{r}\right) \]

(3.1.1)

where, as before, \( u, v = t/\ell \pm \phi \) and \( T^\pm \) are functions of \( u \) and \( v \), respectively. It may be checked that the algebra of such vector fields is closed under commutation: \( \xi^{(\pm)} \) and \( \xi^{(\mp)} \) commute, while the commutators \([\xi^{(\pm)}_1, \xi^{(\pm)}_2] = \xi^{(\pm)} [1, 2]\) yield new vectors of the form (3.1.1) with

\[ T^{+}_{[1, 2]} = 2 ( T^+_1 \partial_u T^+_2 - T^+_2 \partial_u T^+_1 ) , \]
\[ T^{-}_{[1, 2]} = 2 ( T^-_1 \partial_v T^-_2 - T^-_2 \partial_v T^-_1 ) . \]

(3.1.2)

Eqn. (3.1.2) may be recognized as a pair of Virasoro algebras, each with vanishing central charge [52].

The second step is to realize these symmetries as canonical transformations. As in (3+1)-dimensional gravity, the gauge transformations are generated by the Hamiltonian and momentum constraints, whose algebra is (up to some subtleties [53]) the algebra of diffeomorphisms. For a noncompact manifold, however, there is an added complication: as in section 2, boundary terms must be added to the constraints to make them differentiable [25]. The canonical generators thus take the general form

\[ H[\xi] = \int d^2 x \, \xi^\mu \mathcal{H}_\mu + J[\xi] \]  

(3.1.3)

where the boundary terms \( J[\xi] \) are chosen in such a way that the functional derivatives \( \delta H/\delta g_{ab} \) and \( \delta H/\delta \pi^{ab} \), and thus the Poisson brackets of the generators, are well-defined. The presence of such boundary terms can alter the Poisson brackets: in general, a central term appears [48, 54],

\[ \{ H[\xi], H[\eta] \} = H[\{\xi, \eta\}] + K[\xi, \eta] . \]

(3.1.4)
For asymptotically anti-de Sitter space, and in particular for the BTZ black hole, the net effect is that the Virasoro algebras (3.1.2) acquire a central charge:

\[
\{ L^\pm_m, L^\pm_n \} = i(m - n)L^\pm_{m+n} + \frac{ic}{12}m(m^2 - 1)\delta_{m+n,0}
\]

\[
\{ L^+_m, L^-_n \} = 0,
\]

(3.1.5)

where \(L^\pm_n = H_{\xi_n^{(\pm)}}\) are the canonical generators of the asymptotic diffeomorphisms (3.1.1) with \(T^+_n = e^{-inu}\) and \(T^-_n = e^{-in\nu}\), and where

\[
c = \frac{3\ell}{2G}
\]

(3.1.6)

is the central charge. The asymptotic conserved charges \(L^\pm_0\) coming from constant \(T^\pm\) are linear combinations of the mass (associated with a constant time translation \(\xi^t\)) and angular momentum (associated with a constant rotation \(\xi^\phi\)), and a direct computation yields\(^\dagger\)

\[
L^\pm = \frac{(r_+ \pm r_-)^2}{16G\ell},
\]

(3.1.7)

in agreement with the notation of (1.2.6).

Terashima has shown that the same central charge can be obtained from a path integral by way of the Ward-Takahashi identities [55, 56]. A similar, and in some ways simpler, derivation also exists in the Chern-Simons formalism [42, 57]. Recall from (1.1.6) that a diffeomorphism is represented in Chern-Simons theory as a gauge transformation with parameter \(\eta^a = \xi^\mu^a_A\). From (2.0.8), the corresponding boundary term in the generator of gauge transformations is

\[
\delta Q[\eta] = -\frac{k}{2\pi} \int_{\partial\Sigma} \left( \xi^\rho \hat{g}_{ab} A^a_\rho \delta A^b_\phi + \xi^\phi \hat{g}_{ab} A^a_\phi \delta A^b_\phi \right) d\phi.
\]

(3.1.8)

From (3.1.1), \(\xi^{(\pm)\rho} = -\frac{1}{k} \partial_\phi \xi^{(\pm)\phi}\), while from (1.2.7), \(A^\rho_\phi^{(\pm)} = \alpha^{(\pm)}\) should be fixed at infinity. We can thus integrate (3.1.8) to obtain

\[
Q^{(\pm)}[\xi] = -\frac{k}{4\pi} \int \left( \xi^{(\pm)\phi} \hat{g}_{ab} A^{(\pm)a}_\phi A^{(\pm)b}_\phi + 2\xi^{(\pm)\phi} \hat{g}_{ab} \alpha^{(\pm)a} \partial_\phi A^{(\pm)b}_\phi + \xi^{(\pm)\phi} \hat{g}_{ab} \alpha^{(\pm)a} \alpha^{(\pm)b} \right) d\phi,
\]

(3.1.9)

where the last term is an integration constant. Given these generators and the Poisson brackets (2.0.2), it is straightforward to verify the algebra (3.1.5) with central charges

\[
c^{(\pm)} = \frac{3\ell}{G} \alpha^{(\pm)a}. \quad (3.1.10)
\]

\(^\dagger\)The Brown-Henneaux paper [48] actually appeared before the discovery of the BTZ black hole, and discussed conical singularities in asymptotically anti-de Sitter space rather than black holes. But the boundary conditions were deliberately chosen to accommodate more general solutions, and the generalization to the BTZ metric is immediate.
In particular, for the connections (1.2.7), one can read off the values $\alpha^{(\pm)}(\pm) = 1/2$; (3.1.10) then reproduces the central charge (3.1.6). As discussed in section 4.2, the Chern-Simons formulation can be further reduced to Liouville theory [58]; the resulting central charge again agrees with (3.1.6).

A further confirmation of this value of the central charge comes from considering the symmetries of the metric (1.2.5), or, more simply, the Chern-Simons connections (1.2.7). The most general infinitesimal transformation that preserves the form of $A^{(+)}$ is parametrized by a function $\epsilon(u)$, and one finds that [30, 59, 60]

$$\delta L^+ = \epsilon \partial_u L^+ + 2(\partial_u \epsilon)L^+ + \frac{\ell}{8G} \partial^3_u \epsilon.$$  \hfill (3.1.11)

This may be recognized as the transformation law for the holomorphic part of a stress-energy tensor in a conformal field theory with central charge (3.1.6) [52]. This derivation has been extended to supergravity in [61], where it is shown that one can obtain a super-Virasoro algebra. Note that while the transformation (3.1.11) respects our asymptotically anti-de Sitter boundary conditions, it certainly acts nontrivially on boundary values of the fields. As such, it must be considered a symmetry rather than a gauge transformation; that is, as discussed in section 2, the asymptotic fields should transform under some representation of the algebra (3.1.5), but need not be invariant.

Yet another derivation of the central charge (3.1.6) comes from considering the quasilocal stress-energy tensor at the conformal boundary of asymptotically anti-de Sitter space [62]. For a region $U$ with timelike boundary $\partial U$, with induced metric $\gamma_{ij}$ on $\partial U$, the Brown-York quasilocal stress-energy tensor [63] is defined as

$$T^{ij} = \frac{2}{\sqrt{|\gamma|}} \frac{\delta I_{EH}}{\delta \gamma_{ij}}$$ \hfill (3.1.12)

where $I_{EH}$ is the Einstein-Hilbert action for $U$ with an appropriate boundary term added to ensure that the variational principle is well defined for fixed $\gamma_{ij}$ [64, 65]: in three dimensions,

$$I_{EH} = \frac{1}{16\pi G} \int_U d^3x \left( R + \frac{2}{\ell^2} \right) + \frac{1}{8\pi G} \int_{\partial U} d^2x \sqrt{|\gamma|} |K|,$$ \hfill (3.1.13)

where $K$ is the extrinsic curvature of $\partial U$. For asymptotically AdS spacetimes, though, the stress-energy tensor (3.1.12) diverges as the boundary approaches conformal infinity. This divergence may be cured by adding a local counterterm $I_{ct}$ [62, 66]; in 2+1 dimensions the appropriate term is

$$I_{ct} = -\frac{1}{8\pi G \ell} \int_{\partial U} d^2x \sqrt{|\gamma|}.$$ \hfill (3.1.14)

With this choice, the trace of the Brown-York stress-energy tensor becomes

$$T = -\frac{\ell}{16\pi G} (2)^{(2)} R,$$ \hfill (3.1.15)

where $(2)^{(2)} R$ is the curvature scalar for the boundary metric $\gamma_{ij}$. But in a general two-dimensional conformal field theory, the conformal anomaly is [52]

$$T = -\frac{c}{24\pi} (2)^{(2)} R,$$ \hfill (3.1.16)
agreeing with (3.1.15) if \( c = 3\ell/2G \). This derivation is closely related to the Fefferman-Graham construction of Liouville theory that will be discussed below in section 4.3; indeed, as we shall see, the anomaly (3.1.15) can be computed directly from an expansion of the metric and action near infinity [66].

Relationships among some of these results have been discussed in [67]. While the derivation of the Virasoro algebra (3.1.5) and central charge (3.1.6) is evidently quite robust, it is worth pointing out that the value of the central charge depends on the choice of boundary conditions. As noted in [14], for example, a boundary condition that fixes \( A_\rho \) at some finite radius \( r \) rather than at infinity will lead to a “blue-shifted” central charge. I shall return to this issue in section 6.

3.2 The Cardy Formula

When the central charge in the symmetry algebra of (2+1)-dimensional asymptotically AdS gravity was first discovered, it was considered to be mainly a mathematical curiosity. This changed when Strominger [46] and Birmingham, Sachs, and Sen [47] independently pointed out that this result could be used to compute the asymptotic density of states. The key to this computation is the Cardy formula [50, 51].

Consider a two-dimensional conformal field theory, whose symmetries are described by a Virasoro algebra with central charge \( c \). Let \( \Delta_0 \) be the smallest eigenvalue of \( L_0 \) in the spectrum, and define an effective central charge

\[
c_{\text{eff}} = c - 24\Delta_0. \tag{3.2.1}
\]

Then for large \( \Delta \), the density of states with eigenvalue \( \Delta \) of \( L_0 \) is\(^\dagger\)

\[
\rho(\Delta) \approx \exp \left\{ 2\pi \sqrt{\frac{c_{\text{eff}}\Delta}{6}} \right\} \rho(\Delta_0). \tag{3.2.2}
\]

\(\dagger\)In conformal field theory parlance, \( \Delta \) is a “conformal weight.”

A careful proof of this result using the method of steepest descents is given in [14]. One can derive the logarithmic corrections to the entropy by the same methods [68]; and indeed, by using results from the theory of modular forms, one can obtain even higher order corrections [69, 70].

Although the mathematical derivation of the Cardy formula is relatively straightforward, I do not know of a good, intuitive physical explanation for (3.2.2). The derivation relies on a duality between high and low temperatures, which arises from modular invariance: by interchanging cycles on a torus, one can trade a system on a circle of circumference \( L \) with inverse temperature \( \beta \) for a system on a circle of circumference \( \beta \) with inverse temperature \( L \). But it would be helpful to have a more direct understanding of why the density of states is identical for systems with very different physical degrees of freedom, but with the same values of \( c \) and \( \Delta_0 \).

For a few cases, the behavior (3.2.2) does have a more immediate explanation. A free scalar field, for example, has creation operators \( a_{-n} \), with \([L_0, a_{-n}] = na_{-n}\). If we choose
a standard vacuum, for which \( a_n |0\rangle = 0 \) for \( n \geq 0 \), then excited states created by applying a chain of creation operators will satisfy
\[
L_0 (a_{-n_1} a_{-n_2} \ldots a_{-n_m}) |0\rangle = (n_1 + n_2 + \cdots + n_m) (a_{-n_1} a_{-n_2} \ldots a_{-n_m}) |0\rangle.
\]
(3.2.3)

The number of states for which \( L_0 \) has eigenvalue \( \Delta \) is thus simply the number of distinct ways of writing \( \Delta \) as a sum of integers. This is the famous partition function \( p(\Delta) \) of number theory, whose asymptotic behavior is \[\ln p(\Delta) \sim 2\pi \sqrt{\Delta/6},\]
(3.2.4)
matching the prediction of the the Cardy formula for \( c = 1 \). More generally, combinatoric methods can be applied if we start with a set of bosonic “creation operators” \( \phi_n^{(M_n)} \), with conformal dimensions
\[
[L_0, \phi_n^{(M_n)}] = \beta n \phi_n^{(M_n)},
\]
(3.2.5)
where \( \beta \) is a constant and the index \( M_n \) distinguishes fields with identical dimensions. Let \( \gamma(n) \) denote the degeneracy at conformal dimension \( \beta n \), i.e., \( M_n = 1, \ldots, \gamma(n) \). We allow \( \gamma(n) \) to be zero for some values of \( n \)—the conformal dimensions need not be equally spaced. Then if the asymptotic behavior of the sum of degeneracies is of the form
\[
\sum_{n \leq x} \gamma(n) \sim K x^u
\]
(3.2.6)
for large \( x \), it can be shown that the number of states with \( L_0 = \Delta \) grows as \[\ln \rho(\Delta) \sim \frac{1}{u} [u + 1]^{u/(u+1)} [K u \Gamma(u + 2) \zeta(u + 1)]^{1/(u+1)} [\Delta/\beta]^{u/(u+1)}.
\]
(3.2.7)

We can now apply the Cardy formula to the BTZ black hole. To do so, we shall assume that \( \Delta_0 = 0 \), so \( c_{\text{eff}} \) is given, up to quantum corrections, by (3.1.6). This assumption can certainly be questioned \[\text{[75]},\] and I shall return to it in section 5. Given such a central charge, though, the Cardy formula for the classical charges (3.1.7) yields and entropy
\[
S = \ln \rho(\Delta^+) + \ln \rho(\Delta^-) = 2\pi \left( \sqrt{\frac{cL^+}{6}} + \sqrt{\frac{cL^-}{6}} \right) = \frac{2\pi r^+}{4G},
\]
(3.2.8)
agreeing precisely with the Bekenstein-Hawking entropy (1.2.10).

Note that the key to this result is the existence of a classical central charge. Indeed, if one is careful about factors of \( \hbar \), the classical Poisson brackets \( \{L, L\} \sim L + c \) become quantum commutators \( [L/\hbar, L/\hbar] \sim L/\hbar + c/\hbar \), giving the factor of \( \hbar \) in the usual Bekenstein-Hawking entropy (see, for instance, \[\text{[76]})\]. A quantum mechanical central charge, on the other hand, will typically be of order 1, as will a quantum correction to the classical value of \( L_0 \). Thus if one wishes to obtain \( c \) and \( L_0 \) strictly as quantum corrections to a symmetry algebra with no classical central charge, one must generate enormously large values of \( c \) and \( L_0 \). This may be possible in certain models—in Sakharov-style induced gravity, for instance, one can obtain an effective Liouville theory whose “classical” central charge is entirely due to quantum effects for a very large number of heavy constituent fields \[\text{[77]}.\]
While the state-counting argument described here was originally applied to the BTZ black hole, the same method correctly counts states in a variety of other asymptotically anti-de Sitter solutions, including black holes coupled to scalar fields \[78–80\] and black holes in gravitational theories with higher-order curvature terms \[81\]. On the other hand, since the method depends only on the asymptotic behavior of the metric, it also gives an entropy \[3.2.8\] for a “star” \[82, 83\], a circularly symmetric, horizonless lump of matter whose exterior is described by the BTZ metric. I shall return to this issue in section 6.

3.3 The Effective Central Charge

It will be important later that the central charge occurring in the Cardy formula is the effective central charge \(c_{\text{eff}}\). Before exploring the significance of this fact, a few subtleties in notation need to be clarified.

The Virasoro algebra \(3.1.5\) is the algebra of holomorphic diffeomorphisms of a conformal field theory on the complex plane, with

\[
L_n = i \int dz z^{n+1} T_{zz}
\]

in the conventions of \[52\]. One can transform to the cylinder with a mapping \(z = e^{r+it}\) and define generators \(L_n^{\text{cyl}}\) as the Fourier components of \(T_{\phi \phi}\); because of the anomalous transformation properties of the stress-energy tensor, one finds that

\[
L_n^{\text{cyl}} = L_n - \frac{c}{24} \delta_{n0},
\]

and the algebra \(3.1.5\) becomes

\[
[L_m^{\text{cyl}}, L_n^{\text{cyl}}] = (m - n)L_{m+n}^{\text{cyl}} + \frac{c}{12} n^3 \delta_{m+n,0}.
\]

The Virasoro eigenvalue \(\Delta_0\) appearing in the definition of \(c_{\text{eff}}\) is the lowest eigenvalue of \(\Delta_0\) on the plane; the corresponding eigenvalue \(3.3.2\) on the cylinder is shifted by the Casimir energy of the fields on a compact space.

For a simple illustration of an “effective central charge,” now consider a standard affine Lie algebra

\[
J^a = \sum_n J_n^a e^{in\phi}, \quad [J_m^a, J_n^b] = if_{abc} J_{m+n}^c - kmg^{ab}\delta_{m+n,0},
\]

with corresponding Virasoro generators given by the Sugawara construction \[52\],

\[
L_n = \frac{1}{2(k + h)} \sum_{m=-\infty}^{\infty} g_{ab} J_{m}^a J_{n-m}^b.
\]

\[\text{This algebra has appeared in equation } 2.0.9\] as the algebra of constraints of a Chern-Simons theory on a manifold with boundary.
where, as in section 3.1, $\hat{g}_{ab}$ is the Cartan-Killing metric, and $h$ is the dual Coxeter number of $G$,

$$f^{ab} f^{de} h = -2 h \hat{g}^{ad}.$$  (3.3.6)

It is straightforward to check that these $L_n$ satisfy the algebra (3.1.5), with a central charge determined by the group, and that the asymptotic density of states is given by the Cardy formula. Now, however, consider the deformed Virasoro algebra [73, 74] generated by

$$\tilde{L}_n = L_n + i n \alpha_a J_n^a + \frac{k}{2} \alpha_a \alpha^a \delta_{n0}.$$  (3.3.7)

It is easy to check that the $\tilde{L}_n$ again satisfy the Virasoro algebra (3.1.5), but with a new central charge

$$\tilde{c} = c + 12 k \alpha_a \alpha^a.$$  (3.3.8)

But the redefinition (3.3.7) has not changed the Hilbert space, so the asymptotic behavior of the density of states should not be affected.

In fact, it is not. Under the deformation (3.3.7), $L_0$ has shifted by a constant, and its lowest eigenvalue is now

$$\tilde{\Delta}_0 = \Delta_0 + \frac{k}{2} \alpha_a \alpha^a.$$  (3.3.9)

The effective central charge, and hence the density of states predicted by the Cardy formula, is thus invariant.

We shall see in section 4.1 that the boundary value $A_\mu^a$ of a Chern-Simons gauge field can be identified with an affine current having an algebra of the form (3.3.4). As a consequence, the Virasoro generators (3.1.9) can be understood as deformed generators of precisely the form (3.3.7). This suggests that we should treat the Cardy formula derivation of the BTZ black hole entropy with caution [75]—it is not at all obvious that the classical central charge of section 3.1 is the correct effective central charge. On the other hand, if the effective central charge is substantially different from the classical value (3.1.6), the success of the counting arguments of the preceding section would become an extraordinary coincidence, crying out for a deeper explanation.

4 Reducing Gravity to Conformal Field Theory

The analysis of the preceding section suggests that quantum general relativity can give the correct counting of microscopic states needed to explain the entropy of the BTZ black hole. It does not, however, tell us what those states are. The main virtue of the Cardy formula, its indifference to the details of the states being counted, is also its main weakness—we can count states without a full quantum theory of gravity, but the actual states remain disguised.

Whether the states of the BTZ black hole can be obtained purely within the framework of gravity has been a hotly debated question. Martinec, for example, has argued that general relativity must be considered an effective field theory, which cannot distinguish among different conformal field theory states with the same expectation values of the stress-energy tensor; only a more complete microscopic theory (string theory or a dual gauge theory)
can describe the true underlying degrees of freedom \[75\]. On the other hand, a number of authors have attempted—with varying degrees of success—to obtain the BTZ black hole entropy by counting states in particular conformal field theories that are, arguably, induced from pure \((2+1)\)-dimensional gravity. To evaluate these attempts, one must first understand how to obtain such conformal field theories.

4.1 From Chern-Simons to Wess-Zumino-Witten

The equations of motion for a Chern-Simons theory are that the field strength \(F_{\mu\nu}\) vanishes. On a topologically trivial manifold—say, \(M = \mathbb{R} \times D^2\)—this implies that the potential \(A\) is “pure gauge,”

\[
A = g^{-1}dg.
\]  

As one might expect from section 2 though, the gauge parameter \(g\) can have nontrivial dynamics on the boundary.

As Witten first suggested \[11\], this dynamics can be described by a Wess-Zumino-Novikov-Witten (WZ[N]W) model \[84\]. Perhaps the simplest way to understand this relation \[85,86\] is to begin with the canonical formalism of section 2 and substitute \(A_i = g^{-1}\partial_i g\) into the action \(2.0.1\). A straightforward computation shows that the resulting action for \(g\) is a WZW action on the boundary \(\partial M = \mathbb{R} \times S^1\).

A closely related but slightly more general approach \[87,88\] starts with the observation \[89\] that on a general three-manifold with boundary, the Chern-Simons action is not quite gauge invariant: under a gauge transformation

\[
A = g^{-1}dg + g^{-1}\bar{A}g,
\]  

the action \(1.1.3\) transforms as

\[
I_{CS}[A] = I_{CS}[\bar{A}] - \frac{k}{4\pi} \int_{\partial M} \text{Tr}\left((dg^{-1}) \wedge \bar{A}\right) - \frac{k}{12\pi} \int_M \text{Tr}\left(g^{-1}dg\right)^3.
\]  

For a closed manifold, the last term in \(4.1.3\) is proportional to a topological invariant, a winding number \[84\]; for \(k\) an integer, \(I_{CS}\) shifts by \(2\pi kN\), so despite first appearances, \(\exp\{iI_{CS}\}\) is invariant. For a manifold with boundary, however, this term cannot, in general, be discarded.

A further gauge dependence appears because one must add a surface term to the action when \(M\) is not compact. For a manifold with boundary, the Chern-Simons action has no extrema: a variation of \(A\) gives

\[
\delta I_{CS}[A] = \frac{k}{2\pi} \int_M \text{Tr}\left[\delta A \left(dA + A \wedge A\right)\right] - \frac{k}{4\pi} \int_{\partial M} \text{Tr}\left(A \wedge \delta A\right),
\]  

and as in section 2 one must add a boundary contribution to the action to cancel the boundary term in \(4.1.4\). The form of this new term will depend on our choice of boundary conditions. Good boundary conditions generally require that we fix half the canonical data—positions but not momenta, for instance—but from the Poisson brackets \(2.0.2\), the
gauge potentials $A$ are both positions and momenta. We thus need additional information to separate out “half the data” to be held constant at $\partial M$.

Typically, this added information comes in the form of a choice of complex structure on $\partial M$. If we choose such a complex structure and prescribe a fixed boundary value for, say, $A_z$, the appropriate boundary term in the action is easily seen to be

$$I_{\text{bdry}}[A] = \frac{k}{4\pi} \int_{\partial M} \text{Tr} A_z A_\bar{z}, \quad (4.1.5)$$

which transforms as

$$I_{\text{bdry}}[A] = I_{\text{bdry}}[\bar{A}] + \frac{k}{4\pi} \int_{\partial M} \text{Tr} \left( \partial_z g^{-1} g^{-1} \partial_{\bar{z}} g^{-1} + \partial_{\bar{z}} g^{-1} \partial_z g^{-1} \right). \quad (4.1.6)$$

Combining (4.1.5) and (4.1.6), we see that

$$(I_{\text{CS}} + I_{\text{bdry}})[A] = (I_{\text{CS}} + I_{\text{bdry}})[\bar{A}] + k I^+_{\text{WZW}}[g^{-1}, \bar{A}], \quad (4.1.7)$$

where

$$I^+_{\text{WZW}}[g^{-1}, \bar{A}] = \frac{1}{4\pi} \int_{\partial M} \text{Tr} \left( g^{-1} \partial_z g^{-1} \partial_{\bar{z}} g^{-1} - 2g^{-1} \partial_{\bar{z}} g \partial_z \bar{A} \right) + \frac{1}{12\pi} \int_M \text{Tr} \left( g^{-1} dg \right)^3 \quad (4.1.8)$$

is the chiral WZW action for $g$ coupled to a background field $\bar{A}_z$.

As anticipated, the gauge parameter $g$ has become dynamical at the boundary $\partial M$. In particular, in a path integral evaluation of the partition function or correlators, one can perform the usual Faddeev-Popov trick of splitting the integral into an integral over $\bar{A}$ and one over $g$, but since the action depends on $g$, the latter may no longer simply be divided out. A similar phenomenon occurs in anomalous gauge theories [90,91]; the difference here is that the extra $g$-dependent piece appears only at the boundary.

Note that the WZW current

$$J_z = -k \partial_z g^{-1} = -kg A_z g^{-1} + k \bar{A}_z \quad (4.1.9)$$

is essentially the same as the gauge field $A_z$. From the perspective of conformal field theory, the role of the background field $\bar{A}_z$ is to permit this current to have a fixed, nontrivial holonomy. For the BTZ black hole, in particular, this holonomy is given by (1.2.8). In some references, the background field is absorbed into $J_z$ by redefining $g$; but if the holonomy is nontrivial, such a redefinition requires that $g$ be multivalued.

In a related derivation of the WZW action, Fjelstad and Hwang start with the Chern-Simons action in the canonical form of section 2 and note that the boundary term in (2.0.7) makes the constraints second class at the boundary [45]. There is a standard procedure for restoring the full symmetry to a system with second class constraints: one can add new degrees of freedom that convert the second class constraints to first class constraints [92]. Reference [45] shows that the new degrees of freedom are precisely those of a WZW theory, and that different gauge choices lead either to a pure chiral WZW theory or a pure Chern-Simons theory. This derivation is essentially the converse of the one described above; instead
of isolating “would-be gauge” degrees of freedom \( g \), Fjelstad and Hwang show that one can add in new degrees of freedom \( g \) to restore full invariance of the action.

A key test of these results comes from looking at the “gluing” of Chern-Simons theories. As described above in section 1, one can begin with a compact manifold \( M \) and split it along a surface \( \Sigma \) to obtain two manifolds \( M_1 \) and \( M_2 \), each with a boundary diffeomorphic to \( \Sigma \). In general, the partition functions for \( M_1 \) and \( M_2 \) will depend on the boundary values of \( A \), and one should be able to recover the partition function for \( M \) by integrating over these values. But as Witten has shown \([27]\), one obtains the correct composition law \((1.0.2)\) only by including chiral WZW actions in the partition functions \( Z(M_1, \Sigma) \) and \( Z(M_2, \Sigma) \) from the start.

### 4.2 From Wess-Zumino-Witten to Liouville

Given the Chern-Simons form \((1.1.2)\) of the \((2+1)\)-dimensional gravitational action, the arguments of section 4.1 allow us to obtain a chiral \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) WZW action at the boundary. In particular, for the BTZ black hole, the spacetime manifold has the topology \( \mathbb{R} \times D^2 \), and the WZW action describes the dynamics at the boundary \( \mathbb{R} \times S^1 \) at conformal infinity. This appearance of a conformal field theory at the conformal boundary of an asymptotically anti-de Sitter space is perhaps the simplest example of the famous AdS/CFT correspondence of string theory \([46]\).

But as Coussaert, Henneaux, and van Driel first pointed out \([58]\), we have not yet exhausted the full set of boundary conditions implied by the asymptotic behavior \((1.2.7)\) of the connection. The additional boundary conditions further simplify the boundary theory, eventually reducing it to Liouville theory.

As a first step in this reduction, note that for the Chern-Simons connection \((1.2.7)\) describing the BTZ black hole, \( A_u^{(\pm)} = 0 \). This is exactly the type of boundary condition discussed in section 4.1 with \( u \leftrightarrow z \) and \( v \leftrightarrow \bar{z} \). We thus obtain a sum of two chiral WZW models with opposite chiralities,

\[
I = kI_{WZW}^+[g_1^{-1}, \bar{A}_u = 0] - kI_{WZW}^-[g_2^{-1}, \bar{A}_v = 0]. \tag{4.2.1}
\]

But by the Polyakov-Wiegmann formula \([93]\), a pair of chiral WZW actions combines naturally to form a single nonchiral WZW action

\[
I = kI_{WZW}[g = g_1g_2^{-1}]. \tag{4.2.2}
\]

As a second step in the reduction, we can impose the additional conditions—which also follow from \((1.2.7)\)—that \( A_u^{(\pm)} \sim const. T^- \) and \( A_v^{(\mp)} \sim const. T^+ \) at constant \( \rho \). These conditions translate into constancy of certain \( SL(2, \mathbb{R}) \) components of two currents: after a \( \rho \)-dependent gauge transformation,

\[
(\partial_v g^{-1})^- = 1, \quad (g^{-1} \partial_u g)^+ = 1, \tag{4.2.3}
\]

where \( X^\pm = \text{Tr}(XT^\pm) \). Constraints of this kind were first treated classically by Forgacs et al. \([94]\), and, in a slightly different form, by Alekseev and Shatashvili \([95]\); later the full quantum theory was analyzed by a number of authors \([96–99]\). The effect of the constraints
is to eliminate two degrees of freedom, one directly and one because the conditions, treated as first class constraints, generate gauge transformations that can be factored out. The classical derivation is straightforward: if we decompose the SL(2, \mathbb{R}) group element \( g \) as

\[
 g = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\varphi/2} & 0 \\ 0 & e^{-\varphi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix},
\]

(4.2.4)

the WZW action (4.1.8) is simple to compute, yielding

\[
k I_{\text{SL}(2, \mathbb{R})}^+ = \frac{k}{4\pi} \int dudv \left[ \frac{1}{2} \partial_u \varphi \partial_v \varphi + 2e^{-\varphi} \partial_u X \partial_v Y \right],
\]

(4.2.5)

while the constraints (4.2.3) become

\[
 (\partial_v g^{-1})^- = e^{-\varphi} \partial_v Y = 1, \quad (g^{-1} \partial_u g)^+ = e^{-\varphi} \partial_u X = 1.
\]

(4.2.6)

Inserting these constraints into (4.2.5), we see that the action reduces to a Liouville action,

\[
 I = I_{\text{Liou}} = \frac{k}{8\pi} \int d^2 x \sqrt{g} \left( \frac{1}{2} g^{ab} \partial_a \varphi \partial_b \varphi + \frac{1}{2} \varphi R + \lambda e^\varphi \right).
\]

(4.2.7)

In the original derivation of Coussaert et al., the metric in (4.2.7) was a flat metric on the cylinder at infinity, but a slight generalization of the asymptotic conditions [100, 101] permits a general curved metric and nontrivial holonomies.

Note that with fields normalized as in (4.2.7), the coefficient in front of the Liouville action is \( c/48\pi \), where \( c \) is the classical central charge of the Virasoro algebra of the Liouville theory [102]. Thus, using (1.1.4),

\[
 c = 48\pi \cdot \frac{k}{8\pi} = 6k = \frac{3\ell}{2G},
\]

(4.2.8)

in agreement with the Brown-Henneaux central charge (3.1.6) obtained from the algebra of asymptotic symmetries.

Similar constructions exist for (2+1)-dimensional supergravity [61, 103]. There are also hints that a discretized boundary Liouville theory can be obtained from discretized (2+1)-dimensional gravity [104].

The Liouville action (4.2.7) depends on a background metric \( g_{ab} \) at the conformal boundary. In both the Chern-Simons approach of [100, 101] and the Fefferman-Graham construction described below in section 4.3 a conformal structure—that is, a conformal equivalence class of metrics—is prescribed as part of the boundary data. One can make the dependence on the metric more explicit by starting with a fixed canonical metric and performing a quasiconformal deformation,

\[
 z \rightarrow f(z, \bar{z}), \quad \bar{z} \rightarrow \bar{f}(z, \bar{z}),
\]

(4.2.9)

which can change the conformal structure. Such transformations have been considered in [105, 106]. They are parametrized by a Beltrami differential \( \mu \), such that

\[
 \partial_{\bar{z}} f - \mu \partial_z f = 0.
\]

(4.2.10)
The effect of the transformation (4.2.9) is to change the Liouville action to a more general “Beltrami-Liouville action,” which includes Polyakov’s light cone action for $f$ [107] and which makes the dependence on the boundary conformal structure manifest.

One useful check on this derivation comes from looking at Ward identities. It has been known for some time that the Virasoro Ward identities for Liouville theory (in its nonlocal Polyakov form [107–109]) can be derived from an $SL(2, \mathbb{R})$ Chern-Simons theory [110], and thus, perhaps, from (2+1)-dimensional gravity [88, 110, 111]. Bañados and Caro have recently shown how the full, nonchiral Liouville Ward identities can be obtained from asymptotically anti-de Sitter (2+1)-dimensional gravity [112].

### 4.3 Asymptotic AdS and the Fefferman-Graham Construction

The preceding section relied heavily on the Chern-Simons formulation of (2+1)-dimensional gravity, and thus on special features peculiar to three dimensions. It is naturally of interest to see whether a similar reduction to a boundary conformal field theory can be found in the standard metric formalism, perhaps allowing easier generalizations to more than 2+1 dimensions.

The key to such a reduction comes from a theorem of Fefferman and Graham [113, 114], who show that given a conformal metric at the boundary of an asymptotically anti-de Sitter spacetime, there exists a (formal) asymptotic expansion of the metric that solves the vacuum Einstein field equations. In particular, in three spacetime dimensions, one can find coordinates such that

$$ds^2 = \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} g_{ij}(r, x) dx^i dx^j$$

with

$$g_{ij}(r, x) = g^{(0)}_{ij}(x) + \frac{\ell^2}{r^2} g^{(2)}_{ij}(x) + \frac{\ell^4}{r^4} g^{(4)}_{ij}(x),$$

where indices $i, j$ run from 1 to 2.\(^\star\) The Einstein field equations then become [66, 100, 115–117]

$$\nabla^i g^{(2)}_{jk} - \nabla^j g^{(2)}_{ik} = 0,$$

where indices are raised and lowered and covariant derivatives defined in terms of the conformal boundary metric $g^{(0)}_{ij}$. Note that these equations can be obtained directly from the Hamiltonian and momentum constraints of (2+1)-dimensional gravity, without using the full “bulk” field equations.

As a first step, we can now reproduce the conformal anomaly described at the end of section 3.1 [66, 117]. The coordinate transformations that preserve the form (4.3.1) of the metric are [117, 118]

$$\delta r = -r \delta \sigma(x)$$

$$\delta x^i = \int_0^r \frac{\ell^2}{r^2} g^{ij}(x, r') \partial_j \sigma dr',$$

\(^\star\)In general, one obtains an infinite series for $g_{ij}$, which includes logarithmic terms as well; but in three dimensions, the series terminates [115], as is evident in the general solution (1.2.5).
and it is straightforward to check that the corresponding change in the metric is
\[
\delta g^{(0)}_{ij} = -2\delta\sigma g^{(0)}_{ij}
\]
\[
\delta g^{(2)}_{ij} = -\ell^2 \nabla_i \nabla_j \delta\sigma.
\]
(4.3.4)
The boundary term in the Einstein-Hilbert action (3.1.13) is chosen in such a way that the boundary variation vanishes when the metric is held fixed at the boundary. We therefore know from general principles—and also, of course, from explicit computation—that
\[
\delta I_{EH} = -\int_{\partial U} d^2x \pi_{ij} \delta\gamma^{ij},
\]
where in three dimensions the canonical momentum \(\pi_{ij}\) is \([15]\)
\[
\pi_{ij} = \frac{1}{16\pi G} \sqrt{\gamma} (K_{ij} - \gamma_{ij} K).
\]
(4.3.5) The counterterm (3.1.14) also varies in an easily calculable fashion. Then using (4.3.2) and (4.3.4), it is straightforward to show that
\[
\delta I = \frac{1}{8\pi G\ell} \int_{\partial U} d^2x \sqrt{-\ell^2 \nabla_i \nabla_j \delta\sigma (\ell^2 \nabla_i \nabla_j \phi + \ell^2 (0) g_{ij} (\lambda e^{2\varphi} - \frac{1}{2} \ell^2 (0) g_{kl} \nabla_k \phi \nabla_l \phi))},
\]
(4.3.7) which can be recognized as the standard expression for the conformal anomaly of a two-dimensional field theory with central charge \(c = 3\ell/2G\) \([52]\). More generally, the variation can be used to extract the “holographic stress tensor” (4.3.9) at the boundary \([119–122]\).

We can now obtain Liouville theory in several ways. The most direct \([116, 117]\) is to simply postulate that \(g^{(2)}_{ij}\), which has one independent degree of freedom, can be written in terms of a scalar field \(\varphi\) such that \(e^{\varphi}\) has conformal weight \(-1\) (that is, under a Weyl transformation \(\delta\varphi = -\sigma\)). It is not hard to show that the combination that has the transformation property (4.3.4) is
\[
g^{(2)}_{ij} = \ell^2 \left[-\nabla_i \nabla_j \varphi + \nabla_i \varphi \nabla_j \varphi + g_{ij} \left(\lambda e^{2\varphi} - \frac{1}{2} g^{(0)}_{kl} \nabla_k \varphi \nabla_l \varphi\right)\right],
\]
(4.3.8) where \(\lambda\) is an arbitrary constant. This expression is closely related to the stress-energy tensor of Liouville theory. Indeed, the Einstein equations (4.3.2) imply that
\[
(0) \nabla_i T^i_j = 0 \quad \text{with} \quad T^i_j = \frac{1}{8\pi G\ell} \left(g^{(2)}_{ij} + \ell^2 (0) \delta^i_j R\right),
\]
(4.3.9) and with the identification (4.3.8), \(T^i_j\) is precisely the stress-energy tensor obtained from the Liouville action (4.2.7).

A roughly equivalent procedure \([115]\) is to use a Liouville field as an auxiliary field in order to directly integrate the Einstein equations (4.3.2) or (4.3.9). Since these are
differential equations, \((g_{ij})^{(2)}\) will be a nonlocal function of the prescribed data \((g_{ij})^{(0)}\). One can, however, restore locality by introducing an auxiliary field \(\varphi\). The expression (4.3.8) can then be viewed as simply giving an integral of (4.3.9); the nonlocality is now hidden in the fact that \(\varphi\) must itself obey the Liouville equation of motion, and thus depends nonlocally on \((g_{ij})^{(0)}\). Alternatively, one can “integrate the anomaly”; that is, one can look directly for an action depending on the boundary metric whose conformal variation is given by (4.3.7) [115,123,124]. The result has a unique nonlocal piece, the Polyakov action [107,108]

\[
I_{Pol} = \int d^2x \sqrt{\gamma} R \Box^{-1} R,
\]

which is essentially equivalent to the Liouville action [109].

Such nonlocality should not be surprising, in view of our picture of the dynamical degrees of freedom as “would-be gauge” excitations. The great strength of the Chern-Simons formalism is that the gauge transformations are local. In the metric formalism, on the other hand, gauge transformations—diffeomorphisms—are not local: after all, a diffeomorphism moves points. The surprise is not that the dynamical description of boundary diffeomorphisms is nonlocal, but rather that it is “local enough” to allow an easy description.

We have not yet directly related our Liouville field to the asymptotic diffeomorphisms of the black hole metric. This can be done [125–127], essentially by considering the finite version of the transformation (4.3.3). Let \(\rho = \ln r\). Under a diffeomorphism\(^\dagger\)

\[
\rho \to \rho + \frac{1}{2} \varphi(x) + e^{-2\rho} f(x) + \ldots
\]

\[
x^i \to x^i + e^{-2\rho} h^i(x) + \ldots,
\]

the demand that the metric remain in the form (4.3.8) leads to the relations [129]

\[
(h^{(2)}_i = -\frac{\ell^2}{4} e^{-\varphi} \partial_i \varphi, \quad f^{(2)} = -\frac{\ell^2}{16} e^{-\varphi} (g^{ij})^{(0)} \partial_i \varphi \partial_j \varphi.
\]

It is then easy to show that the spatial metric transforms to (4.3.8) with \(\lambda = 0\). But the field \(\varphi\) is no longer an auxiliary field introduced to integrate the field equations; rather, it is an explicit parametrization of the asymptotic diffeomorphisms (4.3.11).

One can take this argument a step further, and compute an action for \(\varphi\). In the original Einstein-Hilbert action (3.1.13) with the counterterm (3.1.14), we are instructed to integrate out to a constant value \(\rho = \bar{\rho}\), sum the terms in the action, and only then take the limit \(\bar{\rho} \to \infty\). With the coordinate transformation (4.3.11), though, we should place the boundary at a location at which the new radial coordinate is constant; that is, in the original coordinate system,

\[
\rho = \bar{\rho} + \frac{1}{2} \varphi + e^{-2\rho} f + \ldots
\]

\(^\dagger\)The exact version of this transformation, to valid all orders in \(e^{-2\rho}\), may be found in [125,127].
A straightforward calculation then shows that the Einstein-Hilbert action reduces to the Liouville action \((4.2.7)\) at the boundary \([126,128]\). Moreover, by comparing the BTZ metric \((1.2.5)\) with the expansion \((4.3.8)\) of the metric, we can confirm that the constants \(L^\pm\) are indeed the classical values of the Virasoro generators \(L\) and \(\bar{L}\) of the Liouville theory. We thus confirm the picture of the boundary degrees of freedom as “would-be diffeomorphisms” that become dynamical at the conformal boundary.\(^4\)

It is worth noting that the metric derivations of Liouville theory typically lead to an action in which the constant \(\lambda\) either vanishes or is put in by hand. The Chern-Simons derivation, on the other hand, naturally leads to a nonzero value of \(\lambda\). The difference arises from the difference in boundary conditions. Rather than fixing the metric at the boundary, the Chern-Simons derivation fixes components of the connections \(A^{(\pm)}\). This change has two consequences \([130]\): the coefficient of the extrinsic curvature term in the action \((3.1.13)\) is altered, and the extrinsic curvature is itself replaced by a first-order form,

\[
\int d^2x \sqrt{\gamma} \bar{K} = \int \omega_a \wedge e^a. \tag{4.3.14}
\]

One can always find a local Lorentz frame in which \(\bar{K} = K\). But \(\bar{K}\) is not invariant under local Lorentz transformations, and new degrees of freedom—“would-be local Lorentz transformations”—appear. The Liouville potential term in the Chern-Simons derivation can be traced directly to these extra degrees of freedom \([126]\).

This difference may not be as large as it first appears, however. As noted in a different context in \([124]\), the vanishing of \(\lambda\) in the metric formalism comes from an exact cancellation between the divergence of the Einstein-Hilbert action and the boundary counterterm \((3.1.14)\). If this term is interpreted as in conventional renormalization, we might instead expect a finite remainder, which, it has been argued, can give precisely the missing potential term in the Liouville action.

### 4.4 Euclidean Gravity and Liouville Theory

If we allow ourselves to analytically continue from Lorentzian to Riemannian metrics, there is another more manifestly geometrical way of obtaining Liouville theory from the asymptotic behavior of three-dimensional gravity. This Euclidean theory has been developed in depth by Krasnov \([127, 131–136]\); here I will merely summarize some key features.

Any solution of three-dimensional (Euclidean) gravity with a negative cosmological constant is a constant negative curvature space, and can be expressed as a quotient of hyperbolic three-space \(\mathbb{H}^3\) by discrete groups of isometries, \(M = \mathbb{H}^3/\Gamma\). Equivalently, one can view \(M\) as a piece of \(\mathbb{H}^3\) with boundaries “glued” together. The Euclidean continuation of the BTZ black hole, for example, is a quotient of \(\mathbb{H}^3\) by a group generated by a single element \([33]\); in the upper half-space model of \(\mathbb{H}^3\),

\[
\frac{dz^2}{z^2} = \ell^2 \left( dx^2 + dy^2 + dz^2 \right), \quad z > 0, \tag{4.4.1}
\]

\(^4\)See also \([115]\) for the case of a static boundary metric; that paper also discusses some important subtleties involving the lower range of the integration over \(\rho\) in the bulk action.
the identification is just a dilatation combined with a rotation around the $z$ axis, with parameters determined by the mass and angular momentum of the black hole. By using more complicated isometries, this picture can be extended to more elaborate configurations; for example, there are Euclidean versions of multiple-horizon black holes, and of black holes combined with point particles.

One can now compute the Einstein-Hilbert action for such configurations. As in the preceding section, the result is divergent, with both an area and a logarithmic divergence that must be regulated. The delicate issue is how to choose a boundary analogous to “constant $\rho$” upon which to perform the regularization, in such a way that the identifications $\Gamma$ act nicely. Specifically, one would like a boundary whose metric approaches the standard upper half-space metric of $\mathbb{H}^2$ as the regulator is removed. The answer is given by classical geometry: one should choose a surface \[ z(x, y) = \epsilon e^{-\phi(x, y)} \] (4.4.2) where $\phi(x, y)$ is a solution of the Liouville equation. Given such a choice of boundary, it can be shown that the regulated Einstein-Hilbert action is precisely the Liouville action (4.2.7), with additional boundary terms [137] related to the action of the identifications $\Gamma$ on the boundary.

It is clear that this result should be related to the “would-be diffeomorphism” description [126] discussed at the end of the preceding section. In particular, (4.4.2) is probably at least roughly the Euclidean equivalent of the asymptotic diffeomorphism (4.3.11), although details have not yet been worked out. These results may also offer further insight into the missing “potential term” discussed at the end of that section. The Liouville field in (4.4.2) comes from uniformization of the boundary surface, that is, from a Weyl transformation to the canonical metric of constant curvature at the boundary. For a single BTZ black hole, the Euclidean boundary is a torus, and the canonical metric is flat; as a result, the classical Liouville equation has $\lambda = 0$. For more complicated topologies—multiple-horizon solutions, for example—the Euclidean boundary is a surface of genus $g > 1$, the canonical metric is one of constant negative curvature, and the classical Liouville equation has a nonvanishing $\lambda$. This suggests that an extension of [126] to multiple-horizon black holes might automatically introduce a nontrivial value of $\lambda$.

The boundary Liouville action described here is relatively new, but it has already lead to a number of interesting applications. In particular, the formalism has been used to compute the thermodynamics of multiple-horizon black holes [136] (a Cardy formula interpretation of these results would be very interesting) and to analyze the quantum production of black holes by point particle collisions [135]. The formalism also gives a Liouville theory expression for the probability of point particle emission by a BTZ black hole [134] that appears to be at least qualitatively correct (see also [138]).

4.5 Projective Structures

The asymptotic dynamics of (2+1)-dimensional anti-de Sitter gravity certainly appears to contain Liouville theory, but there are some hints that additional structure may be necessary as well. The general form (1.2.5) of the asymptotic metric incorporates both a
complex structure\(^5\) and a pair of functions \(L^\pm\). As noted in [59,140], the latter combine to form a holomorphic quadratic differential, essentially a transverse traceless rank two tensor. Such data, in turn, determine a projective structure [141] at infinity, that is, a collection of complex coordinate patches whose transition functions are fractional linear transformations. The space of projective structures is closely related to the cotangent bundle of the space of conformal structures, or Teichmüller space; a short introduction may be found in Appendix C of [127].

The derivations of Liouville theory described above have treated the \(L^\pm\) as functions of the Liouville field \(\varphi\), thus reducing the dynamics to that of a single field with a single background metric. But this may not be sufficiently general. Krasnov has argued in [127] that the full space of projective structures is the correct moduli space of classical multi-black-hole solutions, at least in the Euclidean setting, and that the inclusion of projective structures in the path integral leads to good holomorphic factorization properties for the Euclidean partition function. The space of projective structures is twice as large as the space of conformal structures, and one may argue that this is the “right” size. A single \(\text{SL}(2,\mathbb{R})\) Chern-Simons theory gives a partition function \(Z[\mu, \bar{\mu}]\) that depends on a Beltrami differential [121] and can be interpreted as a “quantization of Teichmüller space” [110]; it is not unreasonable to expect the partition function of an \(\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})\) Chern-Simons theory to be, in some sense, twice as big. Indeed, the derivation of Ward identities in [112] seems to require such a treatment of the \(L^\pm\).

In the Fefferman-Graham approach of section 4.3 it is possible to see where this extra structure might be hidden. There, the metric coefficients \(L^\pm\) were determined as functions of the Liouville field by integrating the asymptotic Einstein equation (4.3.9). But the solution (4.3.8) is not unique: as noted in [115,142], one can add to \(T_{ij}\) any solution \(\tilde{T}_{ij}\) of

\[
\nabla_i \tilde{T}^i_j = 0, \quad \tilde{T}^i_i = 0,
\]

that is, any holomorphic quadratic differential. It has been pointed out in [124] that a shift arises naturally when one considers asymptotically nontrivial diffeomorphisms: because of the anomalous transformation properties (3.1.11), a holomorphic diffeomorphism can shift \(L^+(u)\) by an arbitrary function of \(u\) and \(L^-(v)\) by an arbitrary function of \(v\).

Despite these intriguing results, though, the role of projective structures in (2+1)-dimensional asymptotically anti-de Sitter gravity remains fairly mysterious. To the best of my knowledge, no one has yet systematically studied their role, and there has been no work on quantization in this context.

5 Counting States

We have now answered the first question raised in the Introduction: although (2+1)-dimensional gravity on a spatially compact manifold is “topological,” on a noncompact

\(^5\)For a single Lorentzian black hole, the complex structure is almost, but not quite, trivial: it is determined by the relative scale of \(t\) and \(\phi\) in the coordinates \(u, v\) of (1.2.5). For a multi-black hole solution, the choice is generally much more complicated. Note that for an orientable two-manifold, a complex structure and a conformal structure are equivalent; see [139] for a good physicists’ introduction.
manifold the theory acquires a new set of dynamical degrees of freedom, described by an SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) WZW action or a Liouville action. The next question is whether we can count these degrees of freedom to obtain a microscopic explanation of the entropy of the BTZ black hole.

This is, unfortunately, a far more difficult question than one might naively expect. WZW models are well understood, but only for compact gauge groups; noncompact groups such as SL(2, \mathbb{R}) are much harder to handle. Similarly, one sector of Liouville theory, the “normalizable” sector, is reasonably well understood [102], but the degrees of freedom relevant to the BTZ black hole come mainly from the much more poorly understood “non-normalizable” sector. Nevertheless, some progress has been made in counting states, and there is hope that we may soon understand the problem better.

5.1 WZW Approaches: General Considerations

An obvious first step towards counting BTZ black hole states is to try to understand the states of the SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) WZW model induced at the conformal boundary. WZW models have been studied extensively (see, for example, [52]), and in many cases their states and operators are well understood. Unfortunately, though, WZW models with noncompact groups are much more poorly understood; although there has been some progress—see, for example, [143–154]—we remain largely ignorant.

If we wish to explain black hole entropy by counting WZW states, we are immediately confronted with a problem. In general, a WZW model for group $G$ is characterized by an affine current algebra (3.3.4), with Virasoro operators given by (3.3.5). The central charge may be computed directly by evaluating the commutators of the $L_m$; for SL(2, \mathbb{R}), one obtains

$$c = \frac{3k}{k - 2}.$$  \hspace{1cm} (5.1.1)

In particular, in the semiclassical ($k \gg 1$) limit, $c \approx 3$. One can understand this limit by rescaling the currents in (3.3.4) to $\tilde{J} = J/\sqrt{k}$. For large $k$, the term involving the structure constant is suppressed, and one obtains three decoupled U(1) current algebras, each having $\tilde{k} = 1$. But the resulting central charge is then much smaller than the Brown-Henneaux value (3.1.6), and the Cardy formula would seem to give an entropy that is drastically too small.

But the Cardy formula also suggests a possible escape [14]. The central charge appearing in the expression for the density of states is actually the effective central charge (3.2.1). If we can find a model for which the lowest eigenvalue of $L_0$ is

$$\Delta_0 = -\frac{k}{4},$$  \hspace{1cm} (5.1.2)

then from (3.2.1) and (1.1.4), we would obtain $c_{\text{eff}} \approx 6k = 3\ell/2G$, matching the central charge of the asymptotic algebra. Of course, a negative value of $\Delta_0$ would ordinarily imply a nonunitary theory, and while this is not necessarily fatal [155, 156], it may seem unnatural. On the other hand, SL(2, \mathbb{R}) WZW models already have problems related to unitarity [143, 144], so this might not be so terrible.
In fact, the choice (5.1.2) has some justification, both from gravity and from conformal field theory. On the gravity side, if one normalizes the Virasoro algebra so that the zero-mass BTZ black hole has $L_0 = 0$, then there exists a large class of point particle states for which the classical value of $L_0$ is negative [48], with values ranging from 0 down to precisely $-k/4$. The lowest eigenvalue (5.1.2) corresponds to empty anti-de Sitter space. The relevance of such point particle states receives some support from their apparent relation to the quasinormal modes of the BTZ black hole [157], and we shall see below in section 5.4 that closely related states may be quite important in understanding quantum Liouville theory. Classically, point particles with $L_0 < 0$ can collide to form black holes [158], and Krasnov has suggested that perhaps the entropy of the BTZ black hole can be viewed as counting possible point particle constituents [132].

It is also worth noting that in supergravity, the massless black hole is the vacuum state of the Ramond sector of the theory, while anti-de Sitter space is the vacuum state of the Neveu-Schwarz sector [159]. The $L_0$ eigenvalues of these two vacua differ by $c/24 = k/4$, in agreement with the comments above. Unfortunately, this argument does not fix an absolute value of $\Delta_0$, since the superconformal algebra may be easily deformed in a way that shifts both $\Delta_0$ and $c$ [14].

The value (5.1.2) also has a (somewhat weaker) rationale from the conformal field theory side. The Virasoro generator $L_0$ of equation (3.3.4) involves an important zero-mode term:

$$L_0 = \frac{1}{k-2} \left( (J_0^0)^2 - (J_1^0)^2 - (J_2^0)^2 \right) + N,$$

where

$$N = \frac{2}{k-2} \sum_{m=1}^{\infty} \left( J_{-m}^0 J_m^0 - J_{-m}^1 J_m^1 - J_{-m}^2 J_m^2 \right)$$

is essentially the sum of three number operators. From the representation theory* of $SL(2, \mathbb{R})$, we learn that for the principal discrete series,

$$L_0 = -\frac{j(j+1)}{k-2} + N,$$

where $j$ is a negative integer or half-integer. In particular, for $j = -k/2$,

$$L_0 = -\frac{k}{4} + N.$$

Now, in an SU(2) Chern-Simons theory, $j = k/2$ is the highest admissible value (the highest integrable representation), and Hwang has argued that $j = -k/2$ may play an equivalent role for $SL(2, \mathbb{R})$ [163–165]. Similarly, in the Euclidean partition function approach of [166], $|j| = |k/2|$ is the maximal value appearing in the partition function. While certainly none of this is conclusive in the absence of a carefully worked out quantization of the $SL(2, \mathbb{R})$ WZW model, these results are at least suggestive.

*For a description of the representations of $SL(2, \mathbb{R})$, see, for example, [160–162]; a brief summary appears below in section 5.3.
An alternative approach to the “low central charge problem,” proposed independently and in slightly different forms by Bañados [167, 168] and Fjelstad and Hwang [169], is to allow some or all of the currents \((3.3.4)\) to have fractional modes,

\[
J^a = \sum_n \sum_{p=0}^{N-1} J^a_{n+p/N} \exp \left\{ i \left( n + \frac{p}{N} \right) \phi \right\}.
\]

(5.1.7)

In general, this changes the physics; currents with fractional modes correspond to metrics with certain conical singularities. This may be reasonable from the point of view of Liouville theory; Krasnov [132] and Chen [170] have both argued for an important role for point particles in understanding the states of the BTZ black hole.

Once one has decided to admit such conical singularities, though, it is not obvious how—or whether—one should pick a particular value of \(N\), that is, why one should still demand periodicity under transformations

\[
\phi \to \phi + 2\pi N.
\]

(5.1.8)

Bañados has argued that such a periodicity for the Virasoro generators, with \(N = c\), can come from demanding unitarity of the Virasoro algebra [167,168]. Given a Virasoro algebra with central charge \(c\), one may check that a transformation

\[
L_n \to \tilde{L}_n = NL_{n/N}
\]

(5.1.9)

gives a new Virasoro algebra with central charge \(\tilde{c} = c/N\). Requiring that \(N = c\) ensures that this new algebra is unitary; apart from a few exceptional values of \(N\) and \(c\), this will not be the case if \(N > c\) [52]. Alternatively, looking at the full \(\text{SL}(2,\mathbb{R})\) current algebra, Fjelstad and Hwang have argued that a similar periodicity, with \(N = k/2\), can come from the condition that the current zero-mode \(J^0_0\) have integer eigenvalues [169]. Fractional periodicity of the currents \(J^a\) implies a fractional “spectrally flowed sector” similar to those described below in section 5.3; the Fjelstad-Hwang condition follows from (5.3.5). In either case, plausible combinatorial arguments then lead to the correct BTZ black hole entropy.

Yet another speculative approach to nonintegral moding has appeared in [14]. Instead of only allowing states built from the vacuum by currents \(J^a_{-n}\) as in (5.3.3), one might also consider states formed by acting on a suitable vacuum by the group-valued WZW field \(g\).

The conformal weight of \(g\) is not integral: for a discrete representation of spin \(j\),

\[
\Delta_j(g) = \frac{j(j - 1)}{k - 2},
\]

(5.1.10)

offering a new source for fractional conformal weights. Indeed, if we assume that all spins appear and that representations with spin \(j\) occur with a multiplicity \(2j+1\), as suggested by the Plancherel measure [164], the combinatorial formula (3.2.7) yields a density of states [14]

\[
\ln \rho(\Delta) \sim 2\pi \sqrt{\frac{k\Delta}{3}}.
\]

(5.1.11)

This is the right order of magnitude, but differs from the correct answer (3.2.8) by a factor of \(1/\sqrt{3}\). The missing factor may reflect the “bimodular” properties of the WZW
model [171]—$g$ transforms on one side under the standard SL(2, $\mathbb{R}$) Lie algebra and on the other side under a quantum group, and one may speculate that the “extra” quantum group transformation leads to a further degeneracy in the number of states within a given representation of SL(2, $\mathbb{R}$).

5.2 Counting WZW States

Despite the absence of a complete quantization of the SL(2, $\mathbb{R}$) WZW model, a number of attempts have been made to count WZW states. The early efforts circumvented the difficulties arising from the noncompactness of SL(2, $\mathbb{R}$) by, implicitly or explicitly, looking at analytic continuations to models with more compact-group-like behavior.

The first attempt at such a counting of states, Ref. [172], looked at states that are exactly diffeomorphism-invariant, and in particular obey the condition $L_0|\text{phys}⟩ = 0$. This equality is achieved by balancing the negative contribution of the zero-modes in (5.1.3) with positive non-zero-mode “oscillator” contributions, with the implicit assumption that each component of $g_{ab}J^a_mJ^b_{-m}$ in (3.3.5) makes a positive contribution, despite the indefiniteness of the metric. Unlike most later papers, [172] looked at the WZW model induced at the horizon rather than at infinity. This changes the natural choice of boundary data, and one must perform a functional Legendre transformation to recover the usual partition function and density of states. Details of this transformation may be found in Appendix B of [14], while [173] contains a more careful derivation of the boundary conditions. Given these assumptions, though, one obtains precisely the correct Bekenstein-Hawking entropy (1.2.10). A similar computation gives the correct entropy for (2+1)-dimensional de Sitter space [174]. In retrospect, the success of this approach is a bit surprising, since a crucial step depends on a cancellation between quantum corrections in the two SL(2, $\mathbb{R}$) factors of the gauge group.

A second paper, Ref. [166], made an explicit analytic continuation from SL(2, $\mathbb{R}$) × SL(2, $\mathbb{R}$) to a “Euclidean” SL(2, $\mathbb{C}$) Chern-Simons theory, whose partition function is fairly well understood [146, 147]:

$$Z_{\text{SL(2, C)}}[\tilde{A}^+, \tilde{A}^-] = |Z_{\text{SU(2)}}[\tilde{A}]|^{-2},$$

where the SU(2) partition function $Z_{\text{SU(2)}}[\tilde{A}]$ on a torus of modulus $\tau$ coupled to a background field $\tilde{A}_z$ can be written explicitly in terms of Weyl-Kac characters for affine SU(2). The density of states can be extracted from the partition function (5.2.1) by a contour integral, since

$$Z_{\text{SL(2, C)}}(\tau)[\tilde{A}^+, \tilde{A}^-] = \text{Tr} \left\{ e^{2\pi i L_0} e^{-2\pi i \tilde{L}_0} \right\} = \sum \rho(\Delta, \bar{\Delta}) q_1^{\Delta - \bar{\Delta}} q_2^{\Delta + \bar{\Delta}},$$

where $q_1 = e^{2\pi i \gamma_1}$, $q_2 = e^{-2\pi i \gamma_2}$, and $\rho(\Delta, \bar{\Delta})$ is the number of states for which the Virasoro generators $L_0$ and $\tilde{L}_0$ have eigenvalues $\Delta$ and $\bar{\Delta}$. The zero-modes again play a crucial role: the partition function $Z_{\text{SU(2)}}[\tilde{A}]$ is dominated by a zero-mode contribution coming from the coupling to the fixed boundary data $\tilde{A}_z$ in (4.1.7) and (4.1.8). This prefactor gives the leading contribution to the density of states $\rho(\Delta, \bar{\Delta})$, which again reproduces the
Bekenstein-Hawking entropy (1.2.10). A similar argument has appeared in [43], and an extension to (2+1)-dimensional de Sitter space has also been found in [175]. Unfortunately, like most Euclidean path integral methods, the computation of [166] gives us the correct entropy without actually telling us much about the microscopic states responsible for that entropy. There are some hints—for example, the continuation requires that we flip the sign of the coupling constant $k$, a transformation that also shows up in some analyses of the $SL(2, \mathbb{R})$ WZW partition function [176]. To go further, though, we can no longer avoid the task of quantizing the $SL(2, \mathbb{R})$ WZW model.

5.3 Toward a Quantum $SL(2, \mathbb{R})$ WZW Model

I believe it is fair to say that no one yet fully understands how to quantize the $SL(2, \mathbb{R})$ WZW model. There has been a good deal of work on this problem in the past several years, though, particularly in the context of string theory and the AdS/CFT correspondence. My treatment here will be sketchy, but I will try to summarize the major progress and open questions.

Quantization of the algebra (3.3.4) requires two steps: we must choose a “vacuum” representation for the algebra of the zero-modes $J^0$, and must then build up a representation of the rest of the $J^a$, which should be ghost-free (that is, states should have positive norms) and should allow the construction of a modular invariant partition function. The first step is classical: the representation theory of $SL(2, \mathbb{R})$ was analyzed by Bargmann in 1947 [177], and is discussed in, for example, [160–162]. As in the better-known case of $SU(2)$, one can find a basis of simultaneous eigenfunctions of $J^0$ and $c^2 = \eta_{ab} J^a_0 J^b_0$, with\footnote{Note that conventions vary; the equations below depend on the choice of signature of the metric.}

\[
J^0_0 |j, m\rangle = m |j, m\rangle, \quad c^2 |j, m\rangle = -j(j + 1)|j, m\rangle.
\]

Unlike the case of $SU(2)$, the unitary representations are all infinite dimensional, and come in a number of classes:

1. the discrete representations $D^\pm_j$, $j < 0$, $-2j \in \mathbb{Z}$;
2. the principal continuous representations $C^\epsilon_j$, $\epsilon = 0, 1/2$, $j = -\frac{1}{2} + is$ with $s$ real;
3. the complementary representations $E_j$, $-1 < j < 0$;
4. the identity representation.

Additional possibilities appear if one considers the universal covering of $SL(2, \mathbb{R})$; in particular, various quantities need no longer be integers. One can also obtain a different set of “hyperbolic” representations by simultaneously diagonalizing $J^0$ and $c^2$ [178]; in contrast to the case of $SU(2)$, the signature of the Cartan-Killing metric now distinguishes $J^0$ from $J^1$ and $J^2$. I will mention these representations briefly at the end of this section.

It is not obvious \textit{a priori} which representations should be relevant to the $SL(2, \mathbb{R})$ WZW model. Only the discrete and continuous representations occur in the Peter-Weyl decomposition of square integrable functions on the group manifold, and it has been argued that
these are therefore the only relevant representations for string theory [152, 165]. If we identify the zero-mode contribution to $L_0$ in (5.1.3) with the asymptotic charge (1.2.6), we see that the principal continuous representations correspond to black holes, while, as noted in section 5.1, the discrete representations correspond to point particles. This picture receives further support from the analysis of the Chern-Simons holonomies of these solutions, which can be related to orbits in $\text{SL}(2, \mathbb{R})$ and thence to particular representations [162].

We must next consider representations of the full affine algebra (3.3.4) of currents $J^a_n$. As in standard conformal field theory [52], one can construct a representation by starting with a “vacuum” $|j,m\rangle$ for which

$$J^a_n |j,m\rangle = 0 \quad \text{for } n > 0 \quad (5.3.2)$$

and building a tower of states by acting with current operators $J^{-n}:$

$$|\psi\rangle = J^a_{n_1} J^a_{n_2} \ldots J^a_{n_\ell} |j,m\rangle, \quad n_1, n_2, \ldots, n_\ell > 0. \quad (5.3.3)$$

The resulting representation spaces are denoted with hats; for example, the space of states of the form (5.3.3) built over a discrete representation $D_{\pm j}$ is $\hat{D}_{\pm j}$. One immediately confronts a serious problem: even if the “vacuum” states all have positive norm, it is generally easy to create negative norm excited states [143, 144]. This in itself is not fatal, since one can still require a physical state condition of the form

$$L_0 |\text{phys}\rangle = \alpha |\text{phys}\rangle. \quad (5.3.4)$$

(Classically, diffeomorphism invariance requires $\alpha = 0$, but as we know from string theory, there may be quantum corrections.) A number of authors have investigated the question of whether one can prove a “no-ghost theorem” for physical states [152, 163–165, 179–183]; the conclusion is that as long as $k > 2$ and one restricts to states $-k/2 \leq j < 0$ in the discrete representations, one can indeed ensure that the Hilbert spaces built over $D_{\pm j}$ and $C_j$ are unitary. Alternatively, it may be possible to use a different free-field representation of the WZW model [149, 185, 187], in which zero-modes are treated quite differently, to achieve a ghost-free spectrum.

We should also demand that when placed on a torus, our WZW model is invariant under “large” diffeomorphisms, diffeomorphisms that cannot be continuously deformed to the identity. This is the demand for a modular invariant partition function [52]. For the representations (5.3.3), it seems likely that no such partition function is possible [181, 184]. But as Henningson et al. first pointed out [152, 164, 176], the $\text{SL}(2, \mathbb{R})$ WZW model has additional “winding sectors”: for any integer $w$, the transformation

$$J^a_n \to \tilde{J}^a_n = J^a_n - \frac{k}{2} w \delta_{n,0}, \quad J^\pm_n \to \tilde{J}^\pm_n = J^\pm_{n \pm w}$$

$$L_n \to \tilde{L}_n = L_n + w J^0_n - \frac{k}{4} w^2 \delta_{n,0}, \quad (5.3.5)$$

where $J^\pm = J^1 \pm i J^2$, preserves the commutation relations (3.3.4) and generates a new representation labeled by $w$. Such a transformation is called “spectral flow” in conformal
field theory. For the SL(2, ℝ) WZW model, it has its origin in the fact that SL(2, ℝ) has
a compact direction; \( w \) corresponds to a winding number around this \( S^1 \). The “flowed”
representations have eigenvalues of \( L_0 \) that are unbounded below, but one can again prove
a “no ghost” theorem for states obeying \( (5.3.4) \) [152].

The new spectrally flowed sectors \( (5.3.5) \) are precisely what is needed to achieve modular
invariance. Henningson et al. show in [176] that if one includes all integral values of \( w \), it
is relatively straightforward to write down a modular invariant partition function. (See
also [185, 186].) Moreover, the result is essentially an analytic continuation of the partition
function for the SU(2) WZW model, providing some further justification for the Euclidean
methods of Ref. [166].

We now have a candidate for our Hilbert space: the direct sum/integral [152, 154]
\[
\bigoplus_{w=-\infty}^{\infty} \bigoplus_{j=-k/2}^{-1/2} \mathcal{D}_j^{\pm,w} \bigoplus_{\epsilon,s} \mathcal{C}_{\epsilon,w}^{s+1/2}
\]
\( (5.3.6) \)

The resulting partition function has been computed in [186]. The question now is whether
one can count states in this Hilbert space to obtain the BTZ black hole entropy. The answer
is not yet known. Troost and Tsuchiya have pointed out two serious problems [162]: the
spectrum of \( L_0 \) is not bounded below for the spectrally flowed states, and the Hilbert space
generally includes negative-norm states. Both of these problems can be solved by imposing
the physical state condition \( (5.3.4) \), but this is presumably not what we want to do; we are
interested in counting states with \( L_{\pm 0} \) given by \( (1.2.6) \).

Part of the problem may be that we are looking at a pure WZW model, rather than a
WZW model coupled to a background field as in \( (4.1.8) \). If we include \( \bar{A}_z \) in the current
as in \( (4.1.9) \), using the BTZ connection \( (1.2.7) \) as our background field, the zero-mode
contribution to \( L_0 \) changes. In fact, we find a shift precisely of the form \( (5.3.5) \), with a
“winding number” (see also [187])
\[
\bar{w} = r_+ \pm r_- \ell.
\]
\( (5.3.7) \)

In general, this \( \bar{w} \) is not an integer—indeed, if it is, then the holonomies \( (1.2.8) \) are trivial—and
the BTZ black hole does not pick out an ordinary spectrally flowed sector. But the background dependence of the WZW action suggests that we should perhaps not allow
arbitrary spectral flow, but should use the asymptotic data to restrict the representations
we are considering. This, of course, means giving up modular invariance for any particular
black hole, but this may not be unreasonable. In particular, it is known that modular
transformations of the classical Euclidean BTZ black hole do not preserve its physical
meaning; rather, one obtains a family of inequivalent solutions, including hot empty anti-
de Sitter space [188].

If we restrict ourselves to the “winding sector” determined by \( \bar{w} \), the physical state
condition \( (5.3.4) \) becomes
\[
L_0|_{\text{phys}} = \left( -\frac{j(j+1)}{k-2} + N \right)|_{\text{phys}} = \left( \alpha - \bar{w}m + \frac{k}{4}\bar{w}^2 \right)|_{\text{phys}}.
\]
\( (5.3.8) \)
This gives a value of $L_0$ that agrees, for large $\bar{w}$, with the classical value (1.2.6). But it leaves us with the dilemma discussed in section 5.1: there do not seem to be enough degrees of freedom in the number operator $N$ to account for the Bekenstein-Hawking entropy.

On the other hand, (5.3.8) is very similar to the diffeomorphism-invariance condition of [172], so it could be that a correct combination of left- and right-moving states might yield the correct entropy. The similarity with [172] also suggests that the conventional boundary conditions might be more easily expressed a different, “hyperbolic” representation of $\text{SL}(2, \mathbb{R})$ [178], in which the noncompact generator $J_0^2$ is diagonalized. This is, indeed, the natural representation if we wish to view the black hole as a quotient space of $\text{AdS}_3$, since the identification (1.2.8) is by hyperbolic elements of $\text{SL}(2, \mathbb{R})$. There has been some work on the $\text{SL}(2, \mathbb{R})$ WZW model in this context [185, 187, 189, 190], but, to the best of my knowledge, no one has yet attempted to count states.

5.4 Liouville Theory

The normal boundary dynamics induced from a Chern-Simons theory is described by a WZW action. But as we saw in section 4.2, slightly stronger anti-de Sitter boundary conditions reduce the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ WZW model of (2+1)-dimensional gravity to Liouville theory. The central charge (4.2.8) of the Liouville theory has the correct value to give the BTZ black hole entropy via the Cardy formula. We might therefore hope that a direct counting of states in Liouville theory could explain this entropy.

The argument is not, unfortunately, quite so straightforward, since the central charge in the Cardy formula is the “effective” central charge (3.2.1), which is not obviously the same as (4.2.8). Quantum states in Liouville theory split into two sectors, “normalizable” states and “nonnormalizable” or “Hartle-Hawking” states. The names come from the Schrödinger picture description of wave functions: if one writes a wave function $\Psi[\varphi]$ and considers its dependence on the zero-mode $\varphi_0$, the “normalizable” states are (delta function) normalizable with respect to the measure $d\varphi_0$ [102]. The $\text{SL}(2)$-invariant vacuum is not a normalizable state, and as a result, the usual operator-state correspondence breaks down: states created by functionally integrating over a disk with a local operator insertion are non-normalizable.

Quantum Liouville theory has been studied extensively—for a review, see [192]—but the emphasis has been largely on the normalizable sector. In this sector, the eigenvalues of $L_0$ are bounded below [102] by

$$\Delta_0 = \frac{c - 1}{24}. \quad (5.4.1)$$

As Kutasov and Seiberg have stressed [191], this corresponds to an effective central charge $c_{\text{eff}} = 1$, that of a single free boson. This clearly does not give enough states to account for the entropy of a BTZ black hole, and has led to the suggestion that Liouville theory is only an effective field theory that does not describe the actual degrees of freedom [75].

The nonnormalizable states, on the other hand, have values of $\Delta_0$ that go down to zero. They are, in fact, closely analogous to the point particle states described in section 5.1 that fill in the gap between the massless black hole and anti-de Sitter space [157, 170]. If we can
formulate a sensible quantum theory that includes these states, the Cardy formula tells us that we ought to be able to obtain the correct black hole entropy.

The most interesting effort I know of to formulate such a theory is due to Chen [170]. From the work of Gervais, we know that the Liouville field can be written in terms of chiral fields with conformal weights [193–196]

\[
\Delta_j = -j - \frac{\tilde{\gamma}^2}{2} j(j + 1), \quad \tilde{\gamma}^2 = \frac{c - 13 - \sqrt{(c - 1)(c - 25)}}{6} \approx \frac{12}{c}, \quad (5.4.2)
\]

whose exchange algebra is that of the quantum group \( U_q(sl_2) \) with \( q = e^{\pi i \tilde{\gamma}^2/2} \approx e^{\pi i/k} \). When \( q \) is a root of unity—classically, when \( 2k \in \mathbb{Z} \)—such a group has only a finite number of irreducible representations. A second set of chiral vertex operators can be formed by the replacement \( \tilde{\gamma} \rightarrow 2/\tilde{\gamma} \); together, these give a quantum group structure of the form \( U_q(sl_2) \circ U_{\tilde{q}}(sl_2) \) [196].

Following Gervais [193–196], Chen observes that the conformal weights (5.4.2) of the Hartle-Hawking states formed by insertions of these operators are of the Kac form [52]; that is, the states

\[
|\psi\rangle = L_{-n_1} L_{-n_2} \ldots L_{-n_\ell} |\Delta_{j,\tilde{j}}\rangle \quad (5.4.3)
\]

become singular at level \((2j + 1)(2\tilde{j} + 1)\). This behavior occurs in the “minimal models” of conformal field theory, where its implications are well understood [52]:

1. the singular states are orthogonal to every other state in the representation;
2. the singular states are null, that is, they have zero norm.

Chen argues that the first property continues to hold for Liouville theory, but that the second property does not—because of the absence of an \( SL(2) \)-invariant vacuum, the Ward identities of Liouville theory differ from those of standard conformal field theories, and the usual proofs break down.

If the singular “decoupling states” do, in fact, have finite norms, then each pair of allowed representations of \( U_q(sl_2) \) and \( U_{\tilde{q}}(sl_2) \) gives a different, decoupled “vacuum” upon which one can build ordinary states by acting with operators \( L_{-n} \). If \( 2k \) is an odd integer, the counting of such “vacua” is straightforward; one finds a total of \( 12/\tilde{\gamma}^2 \approx 6k \) distinct sectors. Since these sectors decouple from each other, the system acts, at least for large \( k \), like a system of \( 6k \) scalar fields, with \( k \) given by \( (1.1.4) \). The Hardy-Ramanujan expression \( (3.2.4) \) for the partition function [71], discussed in section \( 3.2 \), then allows us to directly count states. The result correctly reproduces the Bekenstein-Hawking entropy \( (1.2.10) \) of the BTZ black hole.

Some added support for this picture comes from the close connection between the decoupling states’ conformal weights, the conformal weights of point particles, and the quasi-normal modes of the BTZ black hole [157]. The picture is not at all complete, however. In particular, the proof that the “decoupling states” have finite norm is still sketchy, we do not understand the relationship between these states and the normalizable sector, and

\[\text{Note that these are essentially the same as the } SL(2, \mathbb{R}) \text{ zero-modes} \].
we do not know what happens when the coupling constant $2k$ is not an odd integer. Still, I believe that at this writing, this is probably the best candidate we have for a genuine conformal field theoretic counting of BTZ black hole states.

5.5 Stringy Results

While the work I have discussed so far can be understood in the context of pure (2+1)-dimensional gravity, many of the results were inspired by string theory. As noted in section 1.2, (2+1)-dimensional anti-de Sitter space is naturally isometric to $\text{SL}(2, \mathbb{R})$, so string theory on $\text{AdS}_3$ can be formulated in terms of strings moving on a group manifold. This topic was first investigated by Balog et al. [98], who discovered the problems of nonunitarity and ghosts discussed in section 5.3. The idea that the BTZ black hole could be treated as an exact string theory background was first introduced by Horowitz and Welch [197] and Kaloper [198] in 1993, and this area has become a minor industry in itself.

A full description of stringy approaches is too far afield for this review, but a few highlights deserve mention. Section 5 of Ref. [4] has a much more comprehensive discussion and a good list of references, and I will undoubtedly omit some important results.

**AdS/CFT Correspondence:** The connection between three-dimensional asymptotically anti-de Sitter gravity and two-dimensional conformal field theory is probably the simplest example of Maldacena’s celebrated conjecture of a duality between string theory in asymptotically AdS spacetimes and conformal field theory one dimension lower [199]. Strictly speaking, the proposed correspondence involves string theory in ten-dimensional asymptotically AdS space, and one should really consider a product space such as $\text{AdS}_3 \times S^3 \times T^4$. Many of the results we have obtained in the context of pure conformal field theory have a simple AdS/CFT interpretation. For instance, the appearance of the Liouville stress-energy tensor in the metric, as described in section 4.3, is exactly what one would expect from the AdS/CFT relation between bulk gravitons and boundary stress-energy tensors [200–202]. The nonlocality of the action (4.3.10) is also natural: the action is the generator of boundary stress-energy tensor correlators, and thus cannot be local. The (2+1)-dimensional model has also provided one of the relatively few known time-dependent tests of the AdS/CFT correspondence, by showing that the response of the boundary conformal field theory to small perturbations is directly related to the behavior of quasinormal excitations of the BTZ black hole in the bulk [203, 204].

The AdS/CFT picture also allows us to extend some of these results beyond three space-time dimensions. In [46], it was shown that for certain higher-dimensional near-extremal black holes with near-horizon geometries that looks like that of the BTZ black hole, the three-dimensional Cardy formula can again be used to count states. This argument has been extended to a wide variety of higher-dimensional stringy black holes in, for example, [205–211]. The connection between quasinormal modes and Liouville states of [157] has been extended to these higher-dimensional black holes as well [212], where such “stringy” properties as D-brane charge fractionization have been reproduced.

**Behind the Horizon:** If the AdS/CFT correspondence is correct, the conformal field
theory at the boundary of AdS$_3$ should contain complete information about the interior, including information about the interior of any event horizons. At first sight, this seems implausible, since the region inside a horizon is causally disconnected from the conformal boundary at infinity. A number of recent papers have shown, however, that various correlation functions at infinity can, in fact, probe the interior of a BTZ horizon [213–217]. Such quantities can be determined in the Lorentzian theory by analytic continuation from Euclidean values, and with a suitable choice of contour, a continuation can probe the region behind the horizon, giving information about the inner horizon and the singularity. One can understand these results as coming from the smoothness of the BTZ metric and the analyticity of field theoretic probes; this analyticity allows us to determine properties of the metric even in region we cannot directly measure. It is noteworthy that while these results cannot be obtained in pure gravity—one needs a probe such as a scalar field—neither do they depend on the full apparatus of string theory.

**Stringy Descriptions of the BTZ Black Hole:** Much of the work on SL(2, R) WZW models described in section 5.3 was motivated by the attempt to understand string propagation on AdS$_3$. Although they can exist without string theory, the WZW models have features whose simplest interpretations come from string theory. For example, the spectrally flowed sectors (5.3.5) in the continuous representation describe “long strings” that wind around the center of AdS$_3$ [152], while the restriction $-k/2 \leq j < 0$ required for the no-ghost theorem reflects a “stringy exclusion principle” [165, 188].

A central feature of the string theoretical description is the existence of two different conformal symmetries, one of the string world sheet and one of the target space. In such a model, the Brown-Henneaux central charge (3.1.6) is the central charge of the latter symmetry. The generators of the spacetime Virasoro algebra can be constructed in perturbative string theory [219–221], in a manner that parallels the “would-be diffeomorphism” description of section 4; in particular, the generators of the algebra correspond to “pure diffeomorphism” vertex operators that receive contributions only from the asymptotic boundary. Further, at least for the string theory defined on AdS$_3 \times S^3 \times T^4$, it can be shown that $\Delta_0 = 0$ in (5.2.1)—at least when $\ell$ is less than the string scale [222]—so we need not worry about the “effective central charge” dilemma of section 3.3. It should be noted, though, that most of the work in this area has been on “Euclidean AdS$_3$” rather than the real Lorentzian sector of the theory. First steps toward an extension from pure AdS$_3$ to a BTZ black hole background have been taken in [162, 187].

Using D-brane technology, one can also obtain a rather detailed string theoretic model of gravity on AdS$_3 \times S^3 \times M^4$, where $M^4$ is either a four-torus or a K3 manifold. The description involves a collection of D1 branes along a noncompact direction and a collection of D5 branes wrapping $M^4$ and sharing the noncompact direction with the D1 branes. These results are reviewed in section 5 of [4]. The dual conformal field theory has calculable central charge, which again matches the value (3.1.6) obtained from pure (2+1)-dimensional general relativity; moreover, a fairly explicit description of the degrees of freedom responsible for the black hole entropy is now possible.

**CFT and Information Loss:** Pure gravity in 2+1 dimensions cannot directly address
the “information loss paradox,” the question of whether the formation and subsequent evaporation of a black hole is unitary. There are, after all, no propagating degrees of freedom in the purely gravitational theory, and thus no way for a black hole to evaporate. If (2+1)-dimensional gravity is part of a larger string theory, though, the AdS/CFT correspondence implies that the process should be unitary, since it can be described entirely in terms of an ordinary conformal field theory.

This leads to an apparent paradox [223]. The relevant boundary field theory lives in a finite volume, and should therefore be subject to Poincaré recurrences. In particular, the correlation functions of small disturbances should be quasiperiodic over a long enough time scale. In the bulk, though, a perturbed black hole relaxes exponentially, decaying through quasinormal modes. The (2+1)-dimensional model, in which both quasinormal modes and boundary correlators are exactly computable [203], offers a simple arena for investigating this issue. Proposals to resolve the contradiction by summing over bulk topologies apparently fail [224,225]; it has been speculated that one may ultimately have to alter the physics near the horizon to eliminate quasinormal excitations [225,226].

6 What States Are We Counting?

Despite some differences among approaches to the boundary dynamics, it seems certain that asymptotically anti-de Sitter gravity in 2+1 dimensions acquires new degrees of freedom at the conformal boundary, and that these are associated with a change in the physical meaning of gauge symmetries at the boundary. Nemanja Kaloper and John Terning have suggested a useful analogy [227]: if one thinks of the boundary as breaking gauge invariance, then the WZW or Liouville fields are essentially the Goldstone bosons of this symmetry-breaking, confined to the boundary because it is only there that the gauge transformations are not exact invariances. We have also seen that it is plausible, though certainly not proven, that we can count the states of these boundary degrees of freedom to obtain the Bekenstein-Hawking entropy of the BTZ black hole. Despite this progress, though, deep questions remain about the physical meaning of these degrees of freedom.

One basic problem may be posed as follows. Suppose one replaces the BTZ black hole by a (2+1)-dimensional “star,” a finite axially symmetric distribution of matter. The exterior metric of such a configuration is still the BTZ metric [82], and thus the asymptotic degrees of freedom are identical to those we have discussed above. Our state-counting arguments would then apparently attribute the same entropy to a star as to a black hole.

A perfect fluid is only a phenomenological model of a star, of course, and it is possible that a realistic quantum field theory of the constituents would alter boundary data and explain the differences in entropy. This, of course, is a basic premise of the AdS/CFT correspondence in string theory [4]. Certainly, new matter fields can lead to modifications to the Fefferman-Graham results of section 4.3 [119,228]. Moreover, as described in section 5.5, there is good evidence that such objects as scalar field correlators at the conformal boundary can probe the interior of an asymptotically BTZ spacetime [213–217], perhaps telling us whether or not a black hole is present (though see [218]).

Nevertheless, it remains unclear whether we can explain the apparent surplus of purely gravitational states in an asymptotically BTZ spacetime containing no black holes. Exact
(2+1)-dimensional solutions can be found for boson stars [229], for example, that have the same asymptotics as the BTZ black hole but, presumably, different entropies. Perhaps worse, one can couple scalar fields to (2+1)-dimensional gravity in such a way that new exact black hole solutions appear [230], having the same asymptotic symmetries as the BTZ black hole, but for which the naive application of the Cardy formula gives the wrong entropy. These solutions are probably unstable against decay into ordinary BTZ black holes [228], and Park has argued that the correct “effective central charge” \[ \tilde{c}(r_0) \] removes the discrepancy [80], but it is clear that we must proceed with caution.

This problem is closely related to the question of where the degrees of freedom of the black hole live. As long as we are considering a single BTZ black hole in otherwise empty space, the answer to this question is irrelevant, since there are no “bulk” degrees of freedom. But if we wish to isolate the states of a black hole in a spacetime in which other matter or fields are present, it would be helpful to be able to move the boundary closer to the event horizon. Similarly, there are multi-black-hole solutions whose asymptotic symmetries are indistinguishable from those of a single BTZ black hole [231]; presumably we would like to be able to attribute separate entropies to the separate horizons.

In a string theory setting, arguments for including conformal field theories at both the conformal boundary and the event horizon have appeared in [232, 233]. For pure (2+1)-dimensional gravity, the question has been investigated in [14], taking advantage of the Chern-Simons derivation of the central charge. Ref. [14] starts with the general expression \[ (3.1.8) \] for the boundary term in the generator of diffeomorphisms and asks the effect of imposing different boundary conditions. For example, we might require that the induced boundary metric \( g_{\phi \phi} \) be fixed—that is, that the Lie derivative \( \mathcal{L}_\xi g_{\phi \phi} = 0 \)—at a boundary \( r = r_0 \). An easy computation [14, 43] shows that this requires

\[
\xi^\rho = \frac{N(\infty)}{N(r_0)} \partial_\phi \xi^\phi, \tag{6.0.1}
\]

where \( N(r) \) is the BTZ lapse function \[ (1.2.2) \]. The resulting central charge can be obtained from \[ (3.3.8) \]: it is

\[
\tilde{c}(r_0) = c + \left( \frac{N(\infty)}{N(r_0)} \right)^2 \frac{3\ell}{2G}, \tag{6.0.2}
\]

significantly different from the conformal boundary case. Fixing the extrinsic curvature rather than the metric at the boundary gives a different shift in \( c \); yet another value comes from fixing the mean curvature of the boundary itself [14]. It would thus appear that the location of the boundary is crucial.

In fact, this is not the case. As stressed in section \[ 3.3 \] the relevant quantity is not the central charge, but the “effective” central charge. In each of the cases I have described, the change in the central charge of the Virasoro algebra is precisely canceled by a shift in the minimum conformal weight \( \Delta_0 \), leaving the effective central charge \( c_{\text{eff}} \) unaltered. Thus if we believe the counting arguments of section \[ 3.2 \] we have a great deal of latitude in the placement of the boundary upon which the “would-be gauge” degrees of freedom live.

For a spacetime containing more than a single black hole, of course, an obvious choice of boundary is the event horizon itself. There has been a fair amount of recent work on
the question of whether the entropy of a black hole in any dimension can be obtained from conformal symmetry or “would-be gauge” degrees of freedom at the horizon—see, for example, [76, 234–245]—but I believe it is fair to say that the results, while intriguing, are still far from conclusive.

7 Open Questions

It is customary to end a review of this sort with a list of remaining questions. In this field, most of the deep questions are still open. We know that (2+1)-dimensional gravity induces a two-dimensional conformal field theory at a boundary, including the conformal boundary of asymptotically anti-de Sitter space. We know that a good way to understand the new degrees of freedom is as “would-be gauge degrees of freedom” that become dynamical because the boundary changes the meaning of the symmetries. And we know what the central charge of the conformal field theory is, and that it (somehow) gives the right state-counting for the BTZ black hole. Beyond that, we have a number of suggestive hints, but we lack any very solid answers.

In particular:

1. Why does the Cardy formula work so well in giving the entropy of the BTZ black hole? How is it that a computation that relies only on classical features—classical “charges” and the Poisson brackets of classical asymptotic symmetry generators—gives a correct enumeration of quantum states? These classical features must imprint themselves on the quantum theory in a very fundamental way, determining basic properties of the Hilbert space, but we do not know how this happens. Indeed, we do not yet have a physically intuitive explanation of the Cardy formula itself.

2. Can we construct an appropriate quantum $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ WZW model and understand it well enough to count states? The Cardy formula, naively applied, hints at trouble. To reproduce the BTZ black hole entropy, we will have to do something new, whether introducing (and understanding) new vacuum or spectrally flowed sectors, building fractional conformal weight states, or, most likely, doing something no one has yet thought of.

3. Can we count states in the nonnormalizable sector of Liouville theory? Here, the Cardy formula looks promising, but despite some progress, we remain rather far from a complete understanding of the relevant quantum theory.

4. Can we describe states at the horizon, or only at infinity? If we can only construct the theory at infinity, how do we distinguish different configurations with the same asymptotic behavior?

5. Can we couple the “would-be gauge degrees of freedom” to matter? While a count of these degrees of freedom may give us the BTZ black hole entropy, Hawking radiation requires something to be radiated, and a quantum theory will be consistent only if that radiation can react back on the gravitational degrees of freedom. Two rather different papers, one looking at the conformal boundary [38] and one at the horizon [246], have
begun to address this question, and the results of [38] suggest an interesting connection to Hawking radiation. But so far we have only very preliminary results.

6. Do the “would-be gauge degrees of freedom” provide the fundamental description of the states of the quantum BTZ black hole, or are they only effective fields that reflect a more fundamental underlying theory? This question may not have a unique answer—there may well be different quantum theories of gravity, which need not agree about the source of black hole statistical mechanics—but we do not even know whether such choices exist.

7. Is the progress we have achieved unique to 2+1 dimensions, or can any of our results be extended to higher-dimensional spacetimes?

We have a lot of work to do. Perhaps a future review article will be able to answer some of these questions.

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Appendix A Conventions

It this appendix, I briefly summarize the conventions used in this paper.

I use the “mostly minuses” or “west coast” metric signature, in which the Minkowski metric is $\eta_{ab} = diag(1, -1, -1)$. My $SL(2, \mathbb{R})$ generators are

$$
T^0 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T^1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

(A.1)

so

$$
[T^a, T^b] = e^{ab}_c T^c
$$

(A.2)

with $\epsilon_{012} = \epsilon^{012} = 1$, and

$$
\hat{g}^{ab} = Tr(T^a T^b) = -\frac{1}{2} \eta^{ab}.
$$

(A.3)

My curvature tensor conventions are

$$
[\nabla_\mu, \nabla_\nu] v^a = R_{\mu\nu}^\rho v^\rho v^b
$$

(A.4)

with

$$
R_{\mu\rho} = R_{\mu\rho}^\nu.
$$

(A.5)
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