Fast growing double tearing modes in a tokamak plasma

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Configurations with nearby multiple resonant surfaces have broad spectra of linearly unstable coupled tearing modes with dominant high poloidal mode numbers \( m \). This was recently shown for the case of multiple \( q = 1 \) resonances [Bierwage et al., Phys. Rev. Lett. 94 (6), 65001 (2005)]. In the present work, similar behavior is found for double tearing modes (DTM) on resonant surfaces with \( q \geq 1 \). A detailed analysis of linear instability characteristics of DTM with various mode numbers \( m \) is performed using numerical simulations. The mode structures and dispersion relations for linearly unstable modes are calculated. Comparisons between low- and higher-\( m \) modes are carried out, and the roles of the inter-resonance distance and of the magnetic Reynolds number \( S_{H_p} \) are investigated. High-\( m \) modes are found to be destabilized when the distance between the resonant surfaces is small. They dominate over low-\( m \) modes in a wide range of \( S_{H_p} \), including regimes relevant for tokamak operation. These results may be readily applied to configurations with more than two resonant surfaces.

I. INTRODUCTION

Double tearing modes (DTM) are coupled tearing modes on adjacent resonant surfaces which effectively “drive each other” [1, 2, 3]. The resulting instability is stronger than a single tearing mode (STM), whereby modes with high poloidal mode numbers \( m \) may become dominant [4].

DTMs are thought to be involved in a variety of dynamical processes in tokamak plasmas. In the current-ramp-up phase of the tokamak operation they were used to explain the anomalously strong current penetration [4, 5] and short-wavelength MHD activity (e.g., Mirnov oscillations) [3, 5]. Experimental observations of partial and compound sawtooth crashes (internal disruptions) [5, 10, 11, 12, 13] may be explained through kink-tearing modes on double/multiple resonant surfaces [14, 15, 16, 17]. Phenomena associated with low-beta disruptions in the presence of double resonant surfaces with \( q_s > 1 \) may also be understood in terms of DTM activity [14, 15, 18]. In some way related to DTM is the coupling between modes on resonant surfaces with different \( q_s \), which may explain some salient features of major disruptions [5, 10, 11, 21]. A more recent application is related to the formation of internal transport barriers (ITB) in advanced tokamaks. DTM were suggested to be involved in this process due to the fact that ITBs are often observed in the vicinity of resonant surfaces or near \( q_{min} \) [22, 23, 24].

It was found that DTM with any mode number \( m \) become similar to \( m = 1 \) internal kink modes [25] when the distance between the resonant surfaces is sufficiently small [6]. This is reflected by the linear growth rate \( \gamma_{m,n} \) scaling with the magnetic Reynolds number \( S_{H_p} \) as \( \gamma_{m,n} \propto S_{H_p}^{-1/3} \) [2]. When the distance between the resonant surfaces is large, the structure of the instability resembles that of individual STMs localized on each resonant surface. In this case, the scaling law is \( \gamma_{m,n} \propto S_{H_p}^{-3/5} \). Furthermore, it was found that the coupling in a DTM may also destabilize linearly stable tearing modes [15] and speed up the growth of magnetic islands in the nonlinear regime [14, 15, 25, 26]. The instability of DTM may be enhanced by effects like anomalous electron viscosity [6, 29], finite beta and sheared toroidal flows [30]. Poloidal shear flows, on the other hand, may have a stabilizing effect [18, 31, 32], raising the possibility of a dynamical interplay between shear flows and DTM [18, 33]. This idea is supported by recent studies which indicate that MHD activity associated with multiple resonant surfaces may produce a significant amount of sheared poloidal flows [29, 34].

Until now, studies related to DTM in cylindrical (or toroidal) plasmas were focussing mainly on the role of the modes with the lowest poloidal and toroidal mode numbers, \( m \) and \( n \). High-\( m \) tearing modes were associated only with correspondingly high values of \( q_s = m/n \) [1, 4, 5, 34]. Recently it was demonstrated for the case of \( q_s = 1 \) double and triple tearing modes that modes with high \( m \) may dominate over low-\( m \) modes in a cylindrical plasma even on low-\( q \) resonant surfaces [3]. There it was found that in configurations where the distance between neighboring resonant surfaces is sufficiently small, broad spectra of unstable modes exist and that the dominant modes can have \( m \sim O(10) \).

Motivated by the results of Ref. [4], in the present work we investigate the conditions under which broad spectra of multiple tearing modes with dominant higher-\( m \) modes may arise. For this purpose numerical simulations based on the reduced resistive magnetohydrodynamic model in...
cylindrical geometry are used. The dependence of the linear growth rate on parameters such as the inter-resonance distance, the dissipation coefficients as well as the magnetic shear will be addressed.

The main results of the present work are the following. It is shown that broad spectra of unstable DTMs are also found in configurations with \( q_s > 1 \), thus generalizing the results of Ref. [4]. The instability of modes with \( m > 1 \) depends strongly on the distance between adjacent resonant surfaces. For the resistivity dependence, a scaling law \( \gamma_{\text{lin}} \propto S_{Hp}^{-\alpha} \) is found with \( 1/3 \leq \alpha \leq 3/5 \), which is valid in a range of high values of \( S_{Hp} \). This is in agreement with former studies [2, 28]. For intermediate values \( a/R_0 \leq 3 \), this regime where high-

\[ \frac{1}{2} \left( \partial_r f \partial g - \partial_r g \partial f \right) \]

and the Laplacian is approximated by \( \nabla^2 \equiv \nabla \times \nabla \). In this ordering, toroidal topology is preserved by retaining periodicity in the axial coordinate \( z \) and the poloidal angle \( \theta \).

Normalizing the time by the poloidal Alfvén time \( \tau_{Alf} = \sqrt{\eta_0 \mu_0 m^2 / B_0 (a/R_0)} \), the radial coordinate by \( a \), and introducing the angular coordinate \( \zeta \equiv z/R_0 q(a) \), the normalized variables are \( \psi = \Psi/aB_0 (a) \) and \( \phi = \Phi/(a^2/\tau_{Alf}) \). Note that \( \psi \) is related to the toroidal angle \( \varphi \) via \( \zeta = -\varphi/q(a) \), with \( q(a) = aB_0/R_0 B_0 (a) \) being the safety factor at \( r = a \) [c.f., Eq. [4]]. The normalization for the source term is \( E_{\zeta} = E_{\psi}/[\eta_0 B_0 (a)/\mu_0 a] \). With these normalizations the Eqs. [1] and [4] become

\[
\begin{align*}
\partial_t \psi &= \left[ \psi, \phi \right] - \partial_r \phi - S_{Hp}^{-1} (\hat{\eta} \psi - E_{\zeta}), \\
\partial_t u &= \left[ u, \phi \right] + \left[ j, \psi \right] + \partial \zeta \jmath + Re_{Hp}^{-1} \nabla^2 \psi, \\
\end{align*}
\]

(3)

where the current density \( j \) and the vorticity \( u \) are related to \( \psi \) and \( \phi \) through \( j = -\nabla^2 \psi \) and \( u = \nabla^2 \phi \), respectively. The magnetic Reynolds number \( S_{Hp} \) (also called Lundquist number) and the kinematic Reynolds number \( Re_{Hp} \) are defined as \( S_{Hp} \equiv \tau_n / \tau_{Alf} \) and \( Re_{Hp} = a^2 / (\nu_m \tau_{Alf} / \rho_m) \), respectively. Here, \( \tau_n = a^2 \mu_0 / \eta \) is the resistive diffusion time. The resistivity profile is given by \( \hat{\eta}(r) \), normalised such that \( \hat{\eta}(r = 0) = 1 \). The relative strengths of kinetic viscosity and (resistive) diffusion are characterized by the Prandtl number \( Pr = S_{Hp} / Re_{Hp} \propto \nu/\eta \), with \( \nu = \nu_m / \rho_m \) being the specific ion viscosity.

### II. MODEL

#### A. Reduced MHD equations

The reduced set of magnetohydrodynamic (RMHD) equations in cylindrical geometry [38] is used. It is obtained from the full MHD model through a high-aspect ratio and low-\( \beta \) expansion: \( \epsilon = a/R_0 \ll 1 \), \( \beta \ll 1 \) (e.g., [30]). Here, \( a \) and \( R_0 \) are, respectively, the minor and major radius of the torus, \( \epsilon \) is the inverse aspect ratio, and the plasma beta \( \beta = p/(B_0^2 / 2 \mu_0) \) is the ratio of thermodynamic pressure to magnetic pressure.

The high-aspect-ratio ordering and the presence of a strong axial magnetic field \( B_0 \) allow to express the magnetic field in terms of a magnetic flux function \( \Psi \) as \( B = B_0 \hat{z} + \nabla \Psi \times \hat{z} \). Here, \( \hat{z} \) is the unit vector in the axial direction and \( B_0 \) is taken to be constant. In the MHD ordering, the single-fluid velocity \( V \) is approximated by the \( \mathbf{E} \times \mathbf{B} \) drift velocity \( V_E = \nabla \Phi \times \hat{z} / B_0 \), where \( \Phi \) is the electrostatic potential (or stream function). In the limit of zero \( \beta \) the pressure equation is decoupled and the RMHD equations take form of a two-field model:

\[
\begin{align*}
\partial_t \Psi &= -B \cdot \nabla \Psi + \frac{\eta}{\mu_0} \nabla^2 \Psi + E_z, \\
\rho_m \partial_t \nabla^2 \Phi &= -\frac{1}{\mu_0} \left( B \cdot \nabla^2 \Psi + \nu_m \nabla^2 \nabla \Phi \right).
\end{align*}
\]

(1)

(2)

Here \( \rho_m \) is the mass density (constant due to the assumption of incompressibility), \( \eta \) the plasma resistivity and \( \nu_m \) is the kinematic viscosity. The time-independent electric field \( E_z \) satisfies \( \nabla E_z \times \hat{z} = 0 \). The convective derivative is defined as \( \partial_t = \partial_r + \mathbf{V}_E \cdot \nabla \).

Cylindrical geometry is chosen, which gives the right-handed set of coordinates \( (r, \vartheta, z) \) where \( r \in [0, a] \) is the radius, \( \vartheta \in [0, 2\pi] \) the poloidal angle and \( z \in [0, 2\pi R_0] \) the axial coordinate (related to the toroidal angle \( \varphi \) via \( z = -R_0 \varphi \)). The Poisson bracket is defined as \( [f, g] = \frac{1}{2} (\partial_r f \partial g - \partial_r g \partial f) \), and the Laplacian is approximated by \( \nabla^2 \equiv \nabla \times \nabla \). In this ordering, toroidal topology is preserved by retaining periodicity in the axial coordinate \( z \) and the poloidal angle \( \theta \).

#### B. RMHD equilibrium and resonant modes

The equilibrium state is defined as \( \partial_t \psi = \partial_t \phi = 0 \). Since the effect of the plasma pressure is neglected, the structure of the equilibrium magnetic field \( \mathbf{B}_{eq} \) is given in terms of the tokamak safety factor \( q \), defined as

\[
q = \frac{\mathbf{B}_{eq} \cdot \nabla \zeta}{\mathbf{B}_{eq} \cdot \nabla \vartheta}.
\]

(5)

The safety factor measures the field line pitch by counting how many times a magnetic field line goes the long way around the torus \( (2\pi R_0) \) after one turn the short way around \( (2\pi r) \). Here, \( q = q(r) \), so that the equilibrium magnetic flux surfaces, where \( \psi = \text{const.} \), are uniquely defined by the radius \( r \). Each field variable \( f \) is written in terms of a time-independent equilibrium component \( f \) and a time-dependent perturbation \( \tilde{f} \) as

\[
f(r, \vartheta, \zeta, t) = \overline{f}(r, \vartheta, \zeta) + \tilde{f}(r, \vartheta, \zeta, t).
\]

(6)
It is assumed that the equilibrium state is free of flows,
\[ \tilde{\phi} = \pi = 0. \] (7)

In general, a tokamak plasma has magnetic surfaces where \( q_s = q(r_s) = m/n \), with integers \( m \) and \( n \). These are called rational or resonant surfaces. The radius \( r_s \) is called resonant radius, because an infinitesimally small helical magnetic perturbation with helicity \( h = m/n \) is resonant with the magnetic field structure in the vicinity of \( r_s \). Such a resonant perturbation does not bend field lines, so there is no restoring force. If the perturbation leads the system to a state of lower energy, the amplitude of the resonant perturbation will grow and the mode is said to be unstable.

C. Fourier representation

In order to study the properties of such resonant modes it is useful to apply a Fourier transform with respect to the periodic coordinates: \( (\vartheta, \zeta) \rightarrow (m,n) \). This representation also gives an efficient and accurate numerical model. Substituting for each field variable \( f \) in Eqs. (3) and (4) the Fourier expansion

\[ f(r, \vartheta, \zeta, t) = \frac{1}{2} \sum_{m,n} \int_{m,n}(r,t) e^{i(m\vartheta - n\zeta)} + c.c., \] (8)

one obtains equations for the individual Fourier modes,

\[ \partial_t \psi_{m,n} = [\psi, \phi]_{m,n} + i n \phi_{m,n} \] (9)

\[ - S^{-1}_{H \psi} \eta j_{m,n} - E_{m,n}, \]

\[ \partial_t u_{m,n} = [u, \phi]_{m,n} + [j, \psi]_{m,n} - i n j_{m,n} \] (10)

\[ + R e^{-1}_{H \psi} \nabla^2_{m,n} u_{m,n}, \]

where \( \nabla^2_{m,n} = \frac{1}{2} \partial_r r \partial_r - m^2/r^2 \) and the nonlinear terms \([f, g]_{m,n}\) have acquired the form of convolutions. In this model, if a perturbation is applied only to modes of a given helicity \( h = m/n \), modes with different helicities will not be excited and the problem is effectively reduced to a two-dimensional one. In the following, we will exclusively refer to individual Fourier components of the field variables and usually omit the \((m,n)\) subscripts for convenience. Note that the equilibrium fields \( \bar{f} = \bar{f}(r) \) have only \((m,n) = (0,0)\) components.

D. Equilibrium model

In order to study configurations with two resonant surfaces, the following model formula for the equilibrium \( q \) profile is used:

\[ q(r) = q_0 F_1 (r). \left\{ 1 + (r/r_0)^{2\mu(r)} \right\}^{1/\mu(r)}, \] (11)

FIG. 1: (Color online) Safety factor profiles with two resonant surfaces used in this study. The Case (Ia), plotted in (a), has two \( q_s = 1 \) resonances, located at \( r = r_{s1} \) and \( r_{s2} \), a relatively small distance \( D_{12} = 0.06 \) apart. In (b), variants of this profile, with different inter-resonance distances \( D_{12} \) and different \( q_s \) are shown. The parameter values required to reproduce Case (Ia) are given in Table I and the geometric characteristics of all cases are listed in Table II.

where

\[ r_0 = r_A \left| \frac{m/(nq_0)}{\mu(r_A)} - 1 \right|^{-1/2\mu(r_A)}, \]

\[ \mu(r) = \mu_0 + \mu_1 r^2, \]

\[ F_1 (r) = 1 + f_1 \exp \left\{ - [(r - r_{s1})/r_{12}]^2 \right\}. \]

The parameter set \( \{q_0, r_A, \mu_0, \mu_1, m, n\} \) is used to design the underlying monotonic profile. The parameters \( \{f_1, r_{s1}, r_{s2}\} \) describe the Gaussian bump which is used to create non-monotonic profiles with two (or three) \( q_s = m/n \) resonant surfaces. The DTM configurations used in this study are shown in Fig. I. The corresponding model parameters are listed in Table II and the geometric characteristics of all cases are given in Table III.

With \( q \) given by Eq. (11), \( \bar{\psi} \) and \( \bar{f} \) were calculated using the relations

\[ q^{-1} = \frac{1}{r} \frac{d\bar{\psi}}{d\vartheta} \quad \text{and} \quad \bar{f} = - \nabla^2 \bar{\psi}. \] (12)

The resonant surfaces where \( q = q_s \) are labeled with \( r_{s1} \) and \( r_{s2} \), and the inter-resonance distance is given by \( D_{12} = |r_{s2} - r_{s1}| \). The magnetic shear profile is defined as

\[ s = \frac{r dq}{q dr} = \frac{d(ln q)}{d(ln r)}, \] (13)

and the local magnetic shear at the resonant radius \( r_{s1} \) is \( s_1 = s(r_{s1}) \).

The time-independent source term \( E_z \) compensates the resistive dissipation of the equilibrium current profile, i.e., \( E_z = \eta \bar{J} \) (diffusive equilibrium). For numerical simulations where the temporal evolution of the resistivity profile \( \eta \) is neglected one often assumes \( \eta(r) = \bar{J}(r) = 0 \). This will generally lead to different values of the resistivity at different \( r_{s1} \). In order to simplify the comparison between growth rates of modes associated with different resonant surfaces, a homogeneous resistivity profile, \( \eta(r) = 1 \), is used in this study. Numerical tests indicate that the details of \( \eta \) have no significant effect on any of the qualitative characteristics discussed in this paper.

E. Linearized equations

When the amplitudes and gradients of the perturbed fields are sufficiently small the nonlinear terms in Eqs. (3)
and may be neglected and one obtains the linearized RMHD equations. In the linear system, the time dependence of a perturbed field variable \( \tilde{f} \) takes the form

\[
\tilde{f}(r,t) = f(r) \exp(\lambda t),
\]

where \( \lambda \) is a complex number. Using Eq. (14), the system of equations (3) and (4) becomes

\[
\lambda \tilde{\psi} = i \left( n - \frac{m}{q} \right) \tilde{\phi} + \frac{1}{S_{\text{HP}}} \left( \frac{1}{r} \partial_r r \partial_r - \frac{m^2}{r^2} \right) \tilde{\psi},
\]

\[
\lambda \tilde{\mu} = i \left( n - \frac{m}{q} \right) \left( \frac{1}{r} \partial_r r \partial_r - \frac{m^2}{r^2} \right) \tilde{\psi} + \frac{im}{r} \left( \frac{s(s-2)}{rq} - \frac{\partial_r s}{q} \right) \tilde{\psi} + \frac{1}{Re_{\text{HP}}} \left( \frac{1}{r} \partial_r r \partial_r - \frac{m^2}{r^2} \right) \tilde{\mu}.
\]

Equations (15) and (16) are obtained by applying Eqs. (3) and (4) and expressing the equilibrium fields in terms of the safety factor \( q = q(r) \) and the magnetic shear \( s = s(r) \).

Modes for which the linear growth rate is positive, \( \gamma_{\text{lin}}(m) > 0 \), are said to be linearly unstable. Their amplitudes grow exponentially in time, e.g.,

\[
\delta \tilde{\psi}_{m,n}(r,t) = \delta \tilde{\psi}_{m,n}(r) \exp(\gamma_{\text{lin}} t)
\]

(in RMHD, \( \Re\{\lambda\} = 0 \)). The linear growth rate \( \gamma_{\text{lin}} \) is a function of the mode numbers,

\[
\gamma_{\text{lin}} = \gamma_{\text{lin}}(m,n)
\]

(spectrum of growth rates), and written this way it is generally referred to as the dispersion relation. Since this study is restricted to modes with unique helicity, \( h \), it is sufficient to specify \( m \) for a given \( q_s = h \) (so \( n = m/q_s \)). Along with the radial structure of the eigenmodes, \( \psi(r) \) and \( \phi(r) \), the linear growth rates are our main tool for characterizing the linear instability of DTMs under various conditions.

### F. Numerical method

After discretizing Eqs. (15) and (16) with respect to the radial coordinate, the linearized RMHD model may be written as a generalized eigenvalue problem,

\[
B \mathbf{x} = A \mathbf{x},
\]

where \( \mathbf{x}^T = [\psi, \phi] \), \( A \) is a diagonal matrix containing the eigenvalues \( \lambda \), and \( A \) and \( B \) are coefficient matrices, which also include the finite-difference operators. All eigenmodes \( x_i \) with corresponding eigenvalues \( \lambda_i \) are obtained using an eigenvalue problem (EVP) solver. The accuracy of the results is checked by running tests with different numbers of radial grid points and through comparison with results obtained by solving the linearized version of Eqs. (1) and (2) as an initial value problem (IVP). The numerical convergence and the agreement between the EVP and IVP results are demonstrated in Fig. 2. In Table III the radial resolution used for different values of \( S_{\text{HP}} \) is specified. Due to numerical constraints, the regime \( S_{\text{HP}} > 10^{10} \) was not accessible with sufficient accuracy.
The radial structure of an eigenmode determines which part of the plasma is affected by the instability, since $\phi(r)$ is related to the radial and poloidal components of the plasma displacement velocity,

$$v_r = -\frac{i m}{r} \phi \quad \text{and} \quad v_\theta = \partial_r \phi. \quad (20)$$

In general, for each $(m, n)$ there may be as many unstable eigenmodes as there are resonant surfaces $q_s = m/n$. The radial mode structures of unstable $q_s = 1$ DTM eigenmodes are shown in Fig. 6. The configuration used is Case (Ia), where $D_{12} = 0.06$. For $m = 1$ [Fig. 6(a) and (c)] there are two unstable eigenmodes. For $m > 1$ [Fig. 6(b) and (d)] only one eigenmode is unstable (the $m = 8$ mode structure shown is representative for other modes with $m > 1$). In our notation, the eigenmode $M^{(1)}$ is associated with the innermost resonant surface $r_{s1}$. For $m = 1$ it has a finite amplitude in the region $0 < r < r_{s1}$. The eigenmode $M^{(2)}$ is active in the region $0 < r < r_{s2}$ for $m = 1$, and mainly in the region $r_{s1} < 0 < r_{s2}$ for $m > 1$.

In Figure 6 the mode structures of $q_s = 2$ DTM eigenmodes with $(m, n) = (2, 1)$ are plotted. Results are shown for two cases with different distances $D_{12}$: Case (IIIa) with $D_{12} = 0.31$ [Fig. 6(a) and (c)], and Case (IIIb) with $D_{12} = 0.06$ [Fig. 6(b) and (d)]. In agreement with previous works on DTMs, it is found that two individual modes are present in the case of larger $D_{12}$: $M^{(1)}$ with even parity around $r_{s1}$ and $M^{(2)}$ with odd parity around $r_{s1}$. In the case of smaller $D_{12}$ only the eigenmode $M^{(2)}$ is found to be unstable.

The $M^{(1)}$ eigenmode is essentially an STM associated with the $r = r_{s1}$ resonant surface (here, a negative-shear surface), since it is practically unaffected by the presence of resonant surfaces beyond $r = r_{s1}$. The actual DTMs (for sufficiently small $D_{12}$) are the $M^{(2)}$ eigenmodes. The radial structure of DTMs with $m > 1$ is very similar for different $q_s$, as may be seen by comparing the profiles (b) and (d) in Figs. 6(b) and (d). Let us note that Cases (II), (IV) and (V) of Table II, where $q_s = 3/2, 5/2$ and 3, respectively, have eigenmode structures very similar to those found for $q_s = 2$ in Fig. 6.

The local resistivity $\eta(r_{s1})$ and the magnetic shear $s(r_{s1})$ at a resonant surface $r_{s1}$, together with the distance between neighboring resonant surfaces $D_{12}$, determine which eigenmode, $M^{(1)}$ or $M^{(2)}$, will be the dominant mode for a given $m$. An exception is the STM-like eigenmode $M^{(1)}$: as its mode structure indicates, it is not affected by $D_{12}$.

## III. RESULTS

### A. Mode structures

Although, it is difficult to obtain reliable values for the ion viscosity in the tokamak core, it is usually thought to be very low. However, it is required in most nonlinear simulations for the purpose of providing a cut-off at short wavelengths in order to be consistent with the finite number of grid points or Fourier modes. As may be seen from Fig. 6 as long as the Prandtl number satisfies $Pr \lesssim 0.1$, the viscosity has practically no effect on the linear growth rates of DTMs. A similar result is also obtained for other values of $\delta_{\ell s}$ as well as for single and triple tearing modes, so it may be considered a generic characteristic of tearing modes. The mode structures in Figs. 6 and 7 and all the following results were obtained in the regime where $Pr < 0.01$, so the viscosity effect will not be discussed further in this paper. Let us note that the stabilizing effect of viscosity for $Pr \gtrsim 1$, evident in Fig. 6 was pointed out previously by Öfman [31], in a study on DTMs in the presence of equilibrium shear flows.

### B. Viscosity effect

Recently it was discovered that configurations with nearby $q_s = 1$ resonant surfaces have broad spectra of linearly unstable modes [3]. In particular, if the distance
surfaces will grow in time, starting from
During this process the distance between the resonant
10 are given for the linear resistive layer width
12 = 1 DTMs. In (a) the q profiles are shown
in Fig. 7 (a), represent the equilibria at successive instants in time during such a process. Similar cases were
previously investigated in Ref. 28 where the effect of
12 on $\gamma_{\text{lin}}(m = 3, n = 1)$ and the corresponding scaling
exponent $\alpha$ (in $\gamma_{\text{lin}} \propto S_{\text{HP}}^{-\alpha}$) was characterized. In
Ref. 28, the role of $q_{\text{min}}$ was studied in the context of
partial and full reconnection.

The dispersion relations for the three cases of Fig. 6(a)
are plotted in Fig. 7(b). Clearly, the width of the spectrum is reduced as $D_{12}$ increases, and the $m = 1$ mode
eventually becomes dominant.

Note that Cases (Ia)–(Ic) all have different magnetic shears $s_1$ and $s_2$ [cf., Table IV]. The effect of varying
only $D_{12}$ is illustrated in Fig. 5. Starting with Case (IIb),
where $D_{12} = 0.21$, the q profile is gradually modified in
such a way that $D_{12}$ decreases down to 0.08, while both
shears, $s_1$ and $s_2$, are held constant. The profiles used
are plotted in Fig. 5(a) and the corresponding dispersion relations in Fig. 5(b). These results show that $D_{12}$
controls the breadth of the spectrum. Modes with higher
$m$ and with higher growth rates than the lower-$m$ modes appear when $D_{12}$ is decreased.

It is known that a tearing mode is stable when the
local magnetic shear at the resonant surface is zero [28].
The results in Fig. 7 indicate that for DTM with $m > 1$ the
destabilizing effect of small $D_{12}$ dominates over the
effect of low local shears $|s_1|$ and $s_2$ become small when
$\Delta q = q_q - q_{\text{min}}$ is reduced [cf., Table IV].

The dependence of the DTM growth rates on the parameter $q_{\text{min}}$ (meaning, simultaneous variation of
$D_{12}$, $s_1$ and $s_2$) was studied previously by Ishii et al. 28 for
the $(m, n) = (3, 1)$ mode and a set of q profiles similar to
those in Fig. 6(a). They found that the curve $\gamma_{\text{lin}}(q_{\text{min}})$ is
not monotonic. In Fig. 7(a) it is shown that this is also
true for the dependence of $\gamma_{\text{lin}}$ on $D_{12}$ alone (constant
shears): in the range shown (0.08 $\leq D_{12} \leq 0.21$), the
growth rate of the $M^{(2)}(m = 1)$ mode, $\gamma_{\text{lin}}^{(2)}(m = 1)$, has
a maximum around $D_{12} = 0.14$. A comparison between the results for $S_{\text{HP}} = 10^6$ in Fig. 6(a) and $S_{\text{HP}} = 10^7$
in Fig. 6(b) shows that the value $S_{\text{HP}}$ influences the
location of the peak. This is most likely related to the
distance $D_{12}$ becoming comparable to the linear resistive
layer width $\delta_\eta$, which is estimated after Ref. 2,

$$
\delta_\eta \simeq \left[ \frac{\gamma_{\text{lin}}(m)}{(m/r_{\text{min}})^2 B_s^2 S_{\text{HP}}} \right]^{1/4},
$$

where $B_s = s(r_{\text{si}})/q(r_{\text{si}})$. In the following section, the
role of $S_{\text{HP}}$ will be studied in more detail.

It is noted that the growth rate $\gamma_{\text{lin}}^{(1)}(m = 1)$ does not depend on $D_{12}$, as it is expected from the mode structure
$M^{(1)}(m = 1)$ [Fig. 3(a) and (e)]. Hence, the increase of the
$m = 1$ growth rate in Fig. 7(b) is caused by the increase in the magnetic shear $|s_1|$.
FIG. 10: (Color online) DTM dispersion relations for $S_{\text{Hp}} = 10^6$, $10^7$, and $10^8$ obtained with Case (Ia) where $q_s = 1$. For $m = 1$ there are two unstable eigenmodes, $M^{(1)}$ and $M^{(2)}$. The growth rates of both the dominant ($M^{(1)}$) and the secondary ($M^{(2)}$) $m = 1$ eigenmode are shown for $S_{\text{Hp}} = 10^6$ (they almost coincide for higher $S_{\text{Hp}}$). Vertical dashed lines indicate the locations $m_{\text{peak}}$ of the peaks $\gamma_{\text{peak}} = \max(\gamma_{\text{lin}}(m))$. Similar results are obtained for cases with $q_s > 1$.

E. Role of resistivity

In the previous sections the linear instability of DTMs was investigated at $S_{\text{Hp}} = 10^6$, since this value lies in the regime where nonlinear simulations of MHD instabilities are often performed: $10^4 \lesssim S_{\text{Hp}} \lesssim 10^7$. While large tokamaks typically operate in regimes where $S_{\text{Hp}} \gtrsim 10^8$ (except in the early current-ramp-up phase), it is difficult to access this low-collisional regime with nonlinear simulations. Therefore, the linear study of the $S_{\text{Hp}}$ dependence of DTM growth rates is an important tool for relating nonlinear simulation results to plasma conditions.

In Fig. 11 the dispersion relations of Case (Ia) is shown for $S_{\text{Hp}} = 10^6$, $10^7$ and $10^8$. The growth rates of all modes are found to decrease with increasing $S_{\text{Hp}}$, as it is expected for resistive instabilities. Let us remark that the width of the spectrum is not affected by the variation of $S_{\text{Hp}}$; unstable modes are found in the range $1 \leq m \leq 18$. However, it can be seen that the growth rates of modes with higher $m$ drop more rapidly than growth rates of low-$m$ modes, when $S_{\text{Hp}}$ is increased, which is also reflected by the shift of the peak $\gamma_{\text{peak}}$ to lower $m$. This is important, because the mode number of the fastest-growing mode determines the size of the magnetic islands formed in the early nonlinear regime.

In the remaining part of this section, the $S_{\text{Hp}}$ dependence of DTM growth rates is examined in detail. Comparisons between low- and higher-$m$ modes are made and the role of the distance $D_{12}$ will also be emphasized. First, cases where $q_s = 1$ are considered, whereby a distinction is made between DTMs with $m = 1$ and $m > 1$. The results obtained for $q_s = 1$ are readily applied to cases with $q_s > 1$, as will be shown at the end of this section where $q_s = 2$ DTMs are considered.

**DTMs with $m = 1$**

It is known that DTMs with $m = 1$ behave similarly to $m = 1$ STMs, regardless of the location of the resonant surfaces and their mutual distance. Due to this close relationship between the $m = 1$ STMs and the $m = 1$ DTMs it is possible to decompose the non-monotonic $q$ profile into two monotonic ones — one with a resonance at $r_{s1}$ and negative shear $s_1$, the other with a resonance at $r_{s2}$ and positive shear $s_2$ — and compare the growth rates and $S_{\text{Hp}}$ dependences of the STM eigenmodes with the corresponding DTMs. Our aim is to characterize deviations between STMs and DTMs in certain ranges of $S_{\text{Hp}}$. For clarity, both $M^{(1)}$- and $M^{(2)}$-type eigenmodes in a double-tearing configuration are referred to as “DTMs.”

In Fig. 11 this comparison is performed for Case (Ia), where $D_{12} = 0.06$ [Fig. 11(a)]. The results for Cases (Ib) and (Ic), where $D_{12} = 0.21$ and $D_{13} = 0.31$, respectively, are presented in Fig. 12.

In Case (Ia), $|s_1| \approx s_2$ [cf., Table II], so that the growth rates of the two STMs are almost equal (Fig. 11). Both have $\alpha = 0.32 \approx 1/3$, in agreement with linear theory. It can be observed that the DTM growth rates coincide with the corresponding STM growth rates only at relatively high values of $S_{\text{Hp}}$: $\gamma_{\text{lin}}^{(1)}$ for $S_{\text{Hp}} > 10^7$ and $\gamma_{\text{lin}}^{(2)}$ for $S_{\text{Hp}} > 10^8$. The agreement between $m = 1$ DTMs and STMs for intermediate $S_{\text{Hp}}$ is significantly improved when the distance between the resonant surfaces is increased. This can be seen by comparing Fig. 11 with Fig. 12.

The linear growth rate of a tearing mode plotted as a function of the magnetic Reynolds number, $\gamma_{\text{lin}}(S_{\text{Hp}})$, always has a maximum at a certain value $S_{\text{Hp}} = S_{\text{max}}$. For instance, for $\gamma_{\text{lin}}^{(2)}$ in Fig. 11 one finds $S_{\text{max}} \approx 5 \times 10^3$. In the regime where $S_{\text{Hp}} < S_{\text{max}}$ the width of the linear resistive layer $\delta_\eta$ [Eq. (22)] is comparable to $D_{12}$. Due to the strong resistive diffusion in the regime $S_{\text{Hp}} \lesssim S_{\text{max}}$, the two resonant surfaces are effectively seen as a single $q_s = 1$ “surface.” This leads to the reduction in the DTM growth rate apparent in Fig. 11. For $S_{\text{Hp}} \ll S_{\text{max}}$ a linear dependence $\gamma_{\text{lin}} \propto S_{\text{Hp}}$ (i.e., $\alpha = -1$) is obtained independently of $m$.

In summary, for $m = 1$ DTMs it is found that for sufficiently small $D_{12}$, the growth rate of the $M^{(2)}$ mode exhibits no scaling law in a wide range of $S_{\text{Hp}}$. For $S_{\text{Hp}} \ll S_{\text{max}}$ a linear dependence is observed and for $S_{\text{Hp}} \gg S_{\text{max}}$ the scaling exponent $\alpha = 1/3$ is obtained for both eigenmodes. However, the range of $S_{\text{Hp}}$ in which deviations from the $\alpha = 1/3$ scaling are observed may extend to high $S_{\text{Hp}}$ when $D_{12}$ is small.
FIG. 13: (Color online) $S_{Hp}$ dependence of the linear growth rates of $q_s = 1$ DTMs with $m = 1, 2, 3$ for a small distance $D_{12} = 0.06$ [Case (Ia)]. The dotted lines represent the scaling law $\gamma_{lin} \propto S_{Hp}^{-\alpha}$ with $\alpha = 1/3$ and $3/5$.

FIG. 14: (Color online) $S_{Hp}$ dependence of the linear growth rates of $q_s = 1$ DTMs with $m = 1, 2$ for the distance $D_{12} = 0.21$ [Case (Ib)].

DTMs with $m > 1$

Here the $S_{Hp}$ dependence of $q_s = 1$ DTMs with $m > 1$ is analyzed. First, consider Case (Ia) where the inter-resonance distance is small: $D_{12} = 0.06$. As indicated in Fig. 14, we have selected the growth rates $\gamma_{lin}^{(m)}(m = 2)$ and $\gamma_{lin}^{(m)}(m = 8)$, and plotted them as functions of $S_{Hp}$ in Fig. 13. It is noted that in Case (Ia) only $M^{(2)}$-type modes are unstable for $m > 1$. For comparison, $\gamma_{lin}^{(1)}(m = 1)$ and $\gamma_{lin}^{(2)}(m = 1)$ from Fig. 14 are shown as well. It can be seen that, similarly to the $m = 1$ mode, $\gamma_{lin}^{(2)}(m = 2)$ and $\gamma_{lin}^{(2)}(m = 8)$ have $\alpha = -1$ for $S_{Hp} \ll S_{max}$, where $S_{max}(m = 2) \approx 10^8$ and $S_{max}(m = 8) \approx 2 \times 10^6$. In the range of $S_{Hp}$ shown in Fig. 13 only $\gamma_{lin}^{(2)}(m = 8)$ approaches the scaling $\alpha = 3/5$ in the limit of high $S_{Hp}$. The growth rate of the $m = 2$ mode, $\gamma_{lin}^{(2)}(m = 2)$, has $\alpha$ close to but somewhat larger than $1/3$. Calculations with $m = 4$ and $m = 6$ (not shown here) gave intermediate values $1/3 < \alpha_m < 3/5$ (with $\alpha_m$, being the scaling exponent for the mode number $m$ in the limit of high $S_{Hp}$).

In Fig. 14, the growth rates of the $m = 1$ and $m = 2$ eigenmodes are compared for Case (Ib), where $D_{12} = 0.21$. In contrast to Case (Ia) [Fig. 14, $D_{12} = 0.06$], now there are two unstable $m = 2$ eigenmodes. A scaling exponent $\alpha_m = 3/5$ is obtained for both $m = 2$ eigenmodes [while $\alpha_m = 1/3$ in Case (Ia)].

As a result of the complicated dependence of $\gamma_{lin}$ on the parameter set $\{m, s_1, s_2, D_{12}, S_{Hp}\}$, growth rates $\gamma_{lin}(m > 1)$ may arise above $\gamma_{lin}(m = 1)$ in certain regimes of the parameter space. This can be seen in Fig. 14, $\gamma_{lin}^{(2)}(m = 8) > \gamma_{lin}^{(1)}(m = 1)$ for $4 \times 10^4 \lesssim S_{Hp} \lesssim 2 \times 10^8$, and $\gamma_{lin}^{(2)}(m = 2) > \gamma_{lin}^{(1)}(m = 1)$ for $S_{Hp} \gtrsim 8 \times 10^5$ (upper limit not known). In this regime dispersion relations are found to peak at $m_{peak} > 1$.

Cases with $q_s = 2$

In Fig. 15, the $S_{Hp}$ dependence of the linear growth rates of $q_s = 2$ DTMs is shown. A case with relatively large $D_{12}$ is plotted in Fig. 15(a). Both $m = 2$ eigenmodes follow the scaling law $\gamma_{lin} \propto S_{Hp}^{-3/5}$ rather well. When $D_{12}$ is reduced the growth-rate scalings deviate from this power law in a wide range of $S_{Hp}$, as can be seen in Fig. 15(b). While the higher-$m$ mode still approaches $\alpha_m = 3/5$, the lower-$m$ mode has $\alpha_m$ close to $1/3$. Note the similarity between Fig. 15 and the $q_s = 1$ results shown above.

IV. DISCUSSION AND CONCLUSIONS

Double tearing modes with high poloidal mode numbers are destabilized when the distance between the resonant surfaces is small. For a given inter-resonance distance the mode number of the fastest growing mode was observed to shift to lower $m$ when $S_{Hp}$ is increased (Fig. 10). This is related to the fact that modes with different mode numbers $m$ approach scaling laws $\gamma_{lin} \propto S_{Hp}^{-\alpha_m}$ with different exponents $\alpha = \alpha_m$. Moreover, and in agreement with earlier works [4, 23], it was found that the scaling exponent $\alpha_m$ is a function of the inter-resonance distance $D_{12}$ (e.g., Figs. 13 and 15).

Linear tearing mode theory predicts two characteristic values for the $S_{Hp}$-scaling exponent: $\alpha = 1/3$ for $m = 1$ modes [23] and DTMs on nearby resonant surfaces [2], and $\alpha = 3/5$ for STMs [24]. When the distance $D_{12}$ is increased, DTMs are transformed into STMs and $\alpha_m$ gradually increases from $1/3$ to $3/5$ (except for $m = 1$ modes) [25]. In the present work, it was observed that the transition from $\alpha_m = 1/3$ to $3/5$ also occurs when $m$ is increased (Figs. 13, 14 and 15). However, the variation of $m$ does not change the character of the DTM mode structures, which is in contrast with the above-mentioned effect of increasing $D_{12}$ (Figs. 8 and 9). For practical reasons, the knowledge of the mode number of the dominant mode $m_{peak}$ in the spectrum $\gamma_{lin}(m)$ is important, since it determines the structure of the magnetic islands in the early nonlinear regime [4]. The observations that $m_{peak}$ varies with $S_{Hp}$ and that the growth rates $\gamma_m(m)$ may approach their respective characteristic scaling exponent $\alpha_m$ only at very high values of $S_{Hp}$ [c.f., Figs. 13, 14 and 15(b)] have the following important consequence. Results obtained from nonlinear simulations run in the collisional regime (e.g., $S_{Hp} \sim 10^6$) for a given configuration may not be easily extrapolated to higher $S_{Hp}$, since magnetic islands with different poloidal mode numbers are expected for different $S_{Hp}$. Note, however, that the effective magnetic Reynolds number in the reconnection regions may be smaller due to the action of micro-turbulence (anomalous resistivity) [34, 40]. In summary, the linear instability characteristics of DTMs with high poloidal mode numbers $m$ were studied numerically. High-$m$ tearing modes become unstable
when two or more resonant surfaces \( q(r_{x1}) = q(r_{x2}) = q \) are formed in a tokamak plasma and when the distance between these resonances \( D_{12} = |r_{x2} - r_{x1}| \) is still small. It was shown that despite the low magnetic shear in the vicinity of \( q_{\text{min}} \), modes with high \( m \) have high growth rates due to the destabilizing effect of small \( D_{12} \). The width of the DTM spectrum and mode number of the dominant mode were found to be independent of the \( q \)-value. Broad spectra of unstable DTMs, with dominant modes having \( m > 1 \), were found in a wide range of magnetic Reynolds numbers, including the regimes in which tokamaks operate. Let us note that this result may also be applied to configurations with more than two resonant surfaces and configurations with low magnetic shear.

The findings of this linear study motivate a nonlinear investigation of DTMs. Nonlinear simulations were performed in the past for relatively large inter-resonance distances (e.g., \( q \)). According to our results, during the stage where the inter-resonance distance is still small, fast growing high-\( m \) DTMs may significantly modify the \( q \) profile near \( q_{\text{min}} \) and thereby affect the long-term evolution. This might have important implications for the understanding of the sawtooth crash and other applications of DTM dynamics mentioned in the introduction, such as the anomalous current penetration during the current-ramp up phase, off-axis sawteeth or the formation of ITBs.

An effect that was neglected here but may decouple DTMs in practice is differential rotation. However, its influence decreases with decreasing inter-resonance distance. A factor that is expected to be important in regions with low magnetic shear near resonant surfaces (as in the vicinity of \( q_{\text{min}} \)) is the pressure gradient. This and other extensions are left for future study.

Acknowledgments

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[28] Y. Ishii, M. Azumi, G. Kurita, and T. Tuda, Phys. Plas-
mas 7, 4477 (2000).
(a) Case (Ia)

\[ r_s = 0.99 \]

\[ D_{12} = 0.06 \]

(b) Cases:
- (Ia,b,c)
- (II)
- (III a,b)
- (IV)
- (V)
Case (Ia)

\[ S_{H_p} = 10^8 \]

\[ N_r = 730 \]

\[ \gamma_{lin}(m=1) \]

\[ \gamma_{lin}(m=2) \]

\[ \gamma_{lin}(m=5) \]

\[ \gamma_{lin}(m=8) \]

\[ m=1 \]

\[ m=2 \]

\[ m=5 \]

\[ m=8 \]
\( \psi_{(m=2, n=1)} \)

\[ r \frac{\phi}{r} = \begin{cases} M^{(2)} & \text{for} \quad (m, n) = (2, 1) \\ M^{(1)} & \text{for} \quad (m, n) = (1, 1) \end{cases} \]

\( \gamma_{\text{lin}}^{(2)} = 5.5 \times 10^{-3} \)
\( \gamma_{\text{lin}}^{(1)} = 0.7 \times 10^{-3} \)

\( D_{12} = 0.31 \)

\( \gamma_{\text{lin}}^{(2)} = 2.5 \times 10^{-3} \)
\( \gamma_{\text{lin}}^{(1)} < 0 \)

\( D_{12} = 0.06 \)
\[
D_{12} = 0.06, \text{ Case (Ia)}
\]
\[
D_{12} = 0.21, \text{ Case (Ib)}
\]
\[
D_{12} = 0.31, \text{ Case (Ic)}
\]
(a) $D_{12} = 0.08$
- $D_{12} = 0.11$
- $D_{12} = 0.14$
- $D_{12} = 0.17$
- $D_{12} = 0.21$, Case (lb)

(b) $D_{12} = 0.08$
- $D_{12} = 0.11$
- $D_{12} = 0.14$
- $D_{12} = 0.17$
- $D_{12} = 0.21$, Case (lb)
\[ \gamma_{\text{lin}}(m, q_{\text{res}}) \]

- \( q_{\text{res}} = 1 \), Case (Ia)
- \( q_{\text{res}} = 2 \), Case (IIIb)
- \( q_{\text{res}} = 3 \), Case (V)

(a) (1,1) (2,1) (3,1) (3,2) (5,2)
\( S_{H_p} = 10^6 \)

\( \delta_\eta \approx 0.01 \)

\( \delta_\eta \approx 0.007 \)

\( \delta_\eta \approx 0.005 \)

\( \delta_\eta \approx 0.003 \)
$S_{Hp} = 10^6$

$S_{Hp} = 10^7$

$S_{Hp} = 10^8$

Case (Ia)
\[
\gamma_{\text{lin}}(S_{\text{Hp}}) = 0.96 \quad D_{12} = 0.21
\]

Case (Ib)

\[
\alpha = \frac{3}{5} \quad \alpha = \frac{1}{3}
\]
\[ M^{(2)} \]

\[ M^{(1)} \]

\[ D_{12} = 0.06 \]

\[ \gamma^{(2)}_{\text{lin}} = 2.5 \times 10^{-3} \]

\[ \gamma^{(1)}_{\text{lin}} = 4.4 \times 10^{-3} \]

\[ \gamma^{(2)}_{\text{lin}} = 9.3 \times 10^{-3} \]

\[ \gamma^{(1)}_{\text{lin}} < 0 \]
\( \gamma_{\text{lin}}(S_{H_p}) = 1.92 \)

\( D_{12} = 0.31 \)

\( q_{\text{min}} = 1.92 \)

\( D_{12} = 0.06 \)

\( q_{\text{min}} = 1.99 \)

\( \alpha = \frac{1}{3} \)

\( \alpha = \frac{3}{5} \)

\( \gamma^{(2)}(m=2) \)

\( \gamma^{(1)}(m=2) \)

\( \gamma^{(2)}(m=8) \)

Case (IIIa)

Case (IIIb)
\( S_{H^p}^{\gamma_{\text{lin}}(S_{H^p})} \) is plotted as a function of \( S_{H^p} \) for case (Ib). The data points are shown as circles, and the linear fit is represented by the line. The minimum decay length \( D_{12} \) is 0.21.

In case (Ic), the decay length \( D_{12} \) is 0.31.

The alpha values are \( \alpha = 1/3 \).
\[ S_{\text{Hp}} \gamma_{\text{lin}}(S_{\text{Hp}}) \leq 0.99 \]

\[ D_{12} = 0.06 \]

\[ q_{\text{min}} = 0.99 \]

Case (Ia)
Case (Ia)

\[ S_{Hp} \]

\[ q_{\text{min}} = 0.99 \]
\[ D_{12} = 0.06 \]

\[ \alpha = 3/5 \]
\[ \alpha = 1/3 \]