NON-MINIMAL COUPLING FOR THE GRAVITATIONAL
AND ELECTROMAGNETIC FIELDS: A GENERAL SYSTEM OF EQUATIONS

by

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Abstract We establish a new self-consistent system of equations for the gravitational and electromagnetic fields. The procedure is based on a non-minimal non-linear extension of the standard Einstein-Hilbert–Maxwell action. General properties of a three-parameter family of non-minimal linear models are discussed. In addition, we show explicitly, that a static spherically symmetric charged object can be described by a non-minimal model, second order in the derivatives of the metric, when the susceptibility tensor is proportional to the double-dual Riemann tensor.

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1 Introduction

The Einstein-Maxwell theory forms the basis for other gravitational-electromagnetic theories. The Einstein-Maxwell theory arises from the Einstein-Hilbert gravitational action plus the Maxwell action. It is minimally coupled because there is no coupling in the Lagrangian between the Maxwell part and the curvature part. It also gives equations which are second order in the derivatives of the metric (as opposed to higher order), because the Lagrangian does not contain generic products of curvature terms (the second derivatives of the metric that might appear in the Einstein-Hilbert gravitational Lagrangian form a divergence of some vector and do not contribute to give equations of higher order). In addition the Einstein-Maxwell theory is linear in the electrodynamics, which means that the Maxwell Lagrangian is quadratic in the Maxwell tensor. The property that most interest us here is the coupling between the electromagnetic and the gravitational parts. Thus we are led to classify gravitational-electromagnetic theories in a useful way into two classes, according to the theory is minimally coupled or non-minimally coupled.

The first class is minimally coupled gravitational-electromagnetism. It can have different subclasses. One can subdivide into two subclasses, whether the corresponding electrodynamics is linear or non-linear. One can then also subdivide into new subclasses according to the gravitational action, whether the gravitational part yields a second order theory (such as Einstein-Hilbert theory), or a higher order theory. For instance, a minimally coupled theory, linear in electrodynamics, and second order in the gravity part is the standard Einstein-Maxwell theory [1]. There are many exact results in the framework of this theory, such as the Reissner-Nordström solution for a charged black hole, gravity wave solutions in electrovacuum, cosmological models with a magnetic field, to name a few. Another instance is a minimally coupled theory, with non-linear electrodynamics, and second order Einstein gravity. The well-known models of Born and Infeld [2] and of Heisenberg and Euler [3] when coupled to gravity belong to this subclass. One of the most interesting problems in this theory is the search for the regular, non-singular, black holes. This search, started by Bardeen in 1968 [4], and developed by many authors (see, e.g., [5, 6, 7, 8]) led to the recent success of Ayón-Beato and García [9, 10, 11] in finding exact solutions to the Einstein equations coupled with specific four-parameter models of non-linear electrodynamics. And of course there are the other cases of higher order theories coupled to Maxwell, or to non-linear electrodynamic theories.

The second class is non-minimal coupled gravitational-electromagnetism. It can be subdivided according to whether the corresponding electrodynamics is linear or non-linear. Now, in non-minimal coupled models one can no longer divide, a priori, into second order and higher order theories, since by definition curvature terms appear in these models which in principle give rise in general to higher order terms in the equations. This second class includes non-minimal equations for the electrodynamics containing couplings with the Riemann and Ricci tensors and the Ricci scalar. This class is very wide and comprises several subclasses, such as: non-minimal linear electrodynamics plus Einstein-Hilbert term, non-minimal
non-linear electrodynamics plus Einstein-Hilbert term, non-minimal linear electrodynamics plus Einstein-Hilbert and other pure curvature terms, non-minimal non-linear electrodynamics plus Einstein-Hilbert and other pure curvature terms, and others. All these subclasses of models belong to this second class since they have one specific feature: the Lagrangian contains an interaction part with specific cross-terms, including scalar products of the Riemann tensor and its convolutions, with the Maxwell tensor.

Our goal is to study this last second class. This class is of great interest, since the appearance of cross-terms in the Lagrangian leads to modifications of the coefficients involving the second-order derivatives both in the Maxwell and Einstein equations. This means, in particular, that gravitational waves can propagate with a velocity different from the velocity of light in vacuum, in a similar fashion as electromagnetic waves propagate in a material medium. This new added feature has many interesting applications in various systems and models, such as cosmological scenarios, gravitational waves interacting with electromagnetic fields, and charged black holes. More specifically, in cosmology the evolution of the gravitational perturbations may have another rate and scale. In astrophysics, the interaction of gravitational with electromagnetic waves may lead to time delays in the arrival of those waves, and the gravitational waves themselves would change their own properties in a form noticeable in gravitational wave detection. It also leads to important modifications of the electromagnetic and gravitational structure of a charged black hole.

First we consider the simpler case of minimal coupling in the electromagnetic and gravitational parts, in Section 2. Then in Section 3 we study non-minimal coupled theories. In section 3.1 we obtain the structure of the master equations of the non-minimal gravitational-electromagnetic theory, for both non-linear and linear electrodynamics. In section 3.2 we consider in detail the linear version of the theory. In section 3.3 we briefly study an example.

2 Minimal coupling of gravity and electromagnetism

2.1 General formalism

In order to explain the novelty of our approach, let us first introduce the nomenclature in the well known case of gravitational-electromagnetic theories minimally coupled. We will consider, generically, high-order terms in the gravitational part, and non-linear terms in the electromagnetic part. The action functional is [1]

\[
S = \int d^4x \sqrt{-g} L_{\text{min}},
\]

where,

\[
L_{\text{min}} = \mathcal{L} \left[ \frac{R}{R}, R_{ik}R^{ik}, R_{ikmn}R^{ikmn}, \ldots \right] + \mathcal{L}(I(11), I(12)),
\]
is the determinant of the metric tensor $g_{ik}$, and $\mathcal{L}_{\text{min}}$ is the Lagrangian for the minimally coupled theory. It is composed of two distinct parts, which do not cross, the Lagrangian $\mathcal{L}$, related to the metric field, and the Lagrangian $\mathcal{L}$, related to the electromagnetic field. The Lagrangian $\mathcal{L} \left[ \frac{2}{\kappa}, R_{ik}R^{ik}, R_{ikmn}R^{ikmn}, \ldots \right]$ contains geometrical scalars only,

$$R, \quad R_{ik}R^{ik}, \quad R_{ikmn}R^{ikmn}, \ldots , \quad (3)$$

where $R$ is the Ricci scalar, $R_{ik}$ is the Ricci tensor, and $R_{ikmn}$ is the Riemann tensor. The constant $\kappa$ is equal to $\kappa = \frac{2G}{c^4}$.

The Lagrangian $\mathcal{L}(I_{(11)}, I_{(12)}^2)$ is an arbitrary function of the quantities $I_{(11)}$ and $I_{(12)}^2$. $I_{(11)}$ and $I_{(12)}$ form a first set (first subscript 1) of electromagnetic field invariants. This first set is composed of two invariants (denoted in the second subscript), the first $I_{(11)}$ and the second $I_{(12)}$ invariants. These invariants are quadratic in the anti-symmetric Maxwell tensor $F_{ik}$, and given by

$$I_{(11)} \equiv \frac{1}{2} F_{ik} F^{ik}, \quad I_{(12)} \equiv \frac{1}{2} F^{*}_{ik} F^{ik} . \quad (4)$$

The asterisk denotes the dualization procedure, defined as follows

$$F^{*ik} = \frac{1}{2} \epsilon^{ikls} F_{ls} . \quad (5)$$

Here $\epsilon^{ikls} = \frac{1}{\sqrt{-g}} \epsilon^{ikls}$ is the Levi-Civita tensor and $\epsilon^{ikls}$ is the completely anti-symmetric symbol with $\epsilon^{0123} = -\epsilon^{0123} = 1$. The Maxwell tensor satisfies the condition

$$\nabla_k F^{*ik} = 0 , \quad (6)$$

where $\nabla_k$ is the covariant derivative. Equation (4) can also be written as $\nabla_i F_{kl} + \nabla_l F_{ik} + \nabla_k F_{il} = 0$. Due to (6), the Maxwell tensor may be represented in terms of a four-vector potential $A_i$ as

$$F_{ik} = \nabla_i A_k - \nabla_k A_i = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} . \quad (7)$$

Now, the variation of the action functional (11) with respect to the four-vector potential $A_i$ gives the minimal vacuum Maxwell equations

$$\nabla_k \left[ \frac{\partial \mathcal{L}}{\partial I_{(11)}} F^{ik} + \frac{\partial \mathcal{L}}{\partial I_{(12)}} F^{*ik} \right] = 0 . \quad (8)$$

On the other hand, the variation of the action functional (11) with respect to $g^{ik}$ yields the gravitational equations

$$\frac{1}{\kappa} \text{Ein}_{ik} = T_{ik} , \quad (9)$$
where Ein_{ik} is the corresponding non-linear generalization of the Einstein tensor, \( G_{ik} = R_{ik} - \frac{1}{2} R g_{ik} \). The tensor \( T_{ik} \), defined by

\[
T_{ik} \equiv \frac{1}{2} \mathcal{L} g_{ik} - \frac{\partial \mathcal{L}}{\partial I_{(11)}} F_{in} F_{nk} - \frac{1}{2} \frac{\partial \mathcal{L}}{\partial I_{(12)}} (F_{il}^* F_{lk} + F_{il} F_{lk}^*),
\]

is the symmetric stress-energy tensor of the electromagnetic field in vacuum. The tensor \( T_{ik} \) is conserved in accordance with the Bianchi identities

\[
\nabla^k T_{ik} = 0.
\]

**2.2 Example: linear Einstein-Maxwell theory**

Before we leave this section, we give in this subsection, as a simple example, the usual linear Einstein-Maxwell theory. It can be obtained from equations (1)-(10) when the gravitational Lagrangian is given by the Einstein-Hilbert term \( \frac{R}{\kappa} \) only, and \( \mathcal{L}(I_{(11)}, I_{(12)}) \equiv I_{(11)} \). The relations (8)-(10) reduce, respectively, to

\[
\nabla_k F^{ik} = 0,
\]

and

\[
\frac{1}{\kappa} G_{ik} = T_{ik}^{(0)},
\]

where the superscript \( (0) \) denotes that the tensor \( T_{ik}^{(0)} \) is the simplest part of a more general electromagnetic stress-energy tensor. Such a formalism describes a minimal coupling of gravitation and electromagnetism, since the right-hand-side of the Einstein equations (9), as well as the Maxwell equations (8) and (9) contain metric couplings and covariant derivatives only, while the curvature tensor appears exclusively in the left-hand-side of (13).

In the following section, and in contrast to the minimal gravitational-electromagnetic equations discussed in this section, we consider in some detail, along the section, a non-minimal gravitational-electromagnetic theory both with non-linear and linear electrodynamics, generalizing the Einstein-Maxwell theory and other minimal theories. This approach deals with self-consistent modifications to both the Einstein and the Maxwell equations.
3 Non-minimal extensions of the Einstein-Maxwell Lagrangian

3.1 Full formalism and equations

3.1.1 Invariants containing the Maxwell and dual-Maxwell tensors coupled with the Ricci scalar and the Riemann and Ricci tensors

Let us introduce the invariant scalars, quadratic in the tensors $F^{ik}$ and $F^{*}_{ik}$ and containing the Riemann and Ricci tensors and the Ricci scalar. These scalars yield cross-terms, and are the appropriate quantities for the description of non-minimal interactions. They can be formally divided into five sets. The first set is the trivial one, containing $I_{(11)}$ and $I_{(12)}$ alone, as described before. The second set contains $I_{(11)}$ and $I_{(12)}$ multiplied by $R$,

$$I_{(21)} \equiv \frac{R}{2} g^{im} g^{kn} F_{ik} F_{mn} , \quad I_{(22)} \equiv \frac{R}{2} g^{im} g^{kn} F^{*}_{ik} F_{mn} .$$  \hfill (15)

The third set includes the Ricci tensor $R_{mn}$,

$$I_{(31)} \equiv \frac{1}{2} R^{im} g^{kn} F_{ik} F_{mn} , \quad I_{(32)} \equiv \frac{1}{2} R^{im} g^{kn} F^{*}_{ik} F_{mn} .$$  \hfill (16)

The fourth set is based on the convolutions of the quadratic combinations of $F^{ik}$ and $F^{*}_{ik}$ with the Riemann tensor

$$I_{(41)} \equiv \frac{1}{2} R^{ikmn} F_{ik} F_{mn} , \quad I_{(42)} \equiv \frac{1}{2} R^{ikmn} F^{*}_{ik} F_{mn} .$$  \hfill (17)

The invariants $I_{(21)} - I_{(42)}$ are chosen to be linear in the curvature. Note also that the scalar $\frac{1}{2} R^{im} g^{kn} F^{*}_{ik} F^{*}_{mn}$ can be reduced to a linear combination of $I_{(21)}$ and $I_{(31)}$, and the scalar $\frac{1}{2} R^{ikmn} F^{*}_{ik} F^{*}_{mn}$ can be represented as a linear combination of $I_{(21)}$, $I_{(31)}$, $I_{(41)}$. Finally, the fifth set includes the various scalars nonlinear in the curvature. Below we introduce a few of them,

$$I_{(51)} \equiv \frac{1}{2} g^{im} g^{kn} F_{ik} F_{mn} R , \quad I_{(52)} \equiv \frac{1}{2} g^{im} g^{kn} F^{*}_{ik} F_{mn} F_{R} ,$$  \hfill (18)

$$I_{(53)} \equiv \frac{1}{2} R^{im} R^{kn} F_{ik} F_{mn} , \quad I_{(54)} \equiv \frac{1}{2} R^{im} R^{kn} F^{*}_{ik} F_{mn} ,$$  \hfill (19)

$$I_{(55)} \equiv \frac{1}{2} R^{ikab} R_{abmn} F_{ik} F_{mn} , \quad I_{(56)} \equiv \frac{1}{2} R^{ikab} R_{abmn} F^{*}_{ik} F_{mn} ,$$  \hfill (20)

$$I_{(57)} \equiv \frac{1}{2} R^{*ikab} R^{*}_{abmn} F^{*}_{ik} F_{mn} , \quad I_{(58)} \equiv \frac{1}{2} R^{*ikab} R^{*}_{abmn} F^{*}_{ik} F_{mn} ,$$  \hfill (21)

$$\quad I_{(59)} \equiv \frac{1}{2} R^{ikab} R_{abcd} R^{cdmn} F_{ik} F_{mn} , \ldots .$$  \hfill (22)
In equation (18), \( f_R \) and \( F_R \) denote arbitrary functions of the all possible independent non-linear scalar invariants of the gravitational field, such as \( R^2, R_{mn}R^{mn}, R_{ikmn}R^{ikmn}, ... \).

Thus, the non-minimal Lagrangian can be written in the form

\[
\mathcal{L}_{\text{non-min}} = \mathcal{L} \left( \frac{R}{\kappa}, R_{mn}R^{mn}, ... \right) + \mathcal{L}(I_{(11)}, I_{(12)}, I_{(21)}, I_{(22)}, I_{(31)}, I_{(32)}, I_{(41)}, I_{(42)}, ...) .
\]

This non-minimal Lagrangian is \( U(1) \) gauge invariant since it contains the Maxwell tensor \( F_{ik} \) only, and does not include the potential four-vector \( A^i \).

### 3.1.2 Non-minimal non-linear electrodynamics

The variation, with respect to the 4-vector \( A_k \), of the action functional with Lagrangian (23) yields the equation for the non-minimal non-linear electromagnetic field,

\[
\nabla_k H^{ik} = 0 ,
\]

where \( H^{ik} \) is the induction tensor given by

\[
H^{ik} = \gamma^{ikmn} F_{mn} + \mathcal{W}^{ikmn} F^{n}_{mn} ,
\]

with

\[
\gamma^{ikmn} = \frac{1}{2} (g^{im} g^{kn} - g^{km} g^{in}) \left[ \frac{\partial \mathcal{L}}{\partial I_{(11)}} + \frac{\partial \mathcal{L}}{\partial I_{(21)}} R \right] + \frac{1}{4} (R_{im} g^{kn} - R_{in} g^{km} + R_{kn} g^{im} - R_{km} g^{in}) \frac{\partial \mathcal{L}}{\partial I_{(31)}} + R_{ikmn} \frac{\partial \mathcal{L}}{\partial I_{(41)}} + ... ,
\]

and

\[
\mathcal{W}^{ikmn} = \frac{1}{2} (g^{im} g^{kn} - g^{km} g^{in}) \left[ \frac{\partial \mathcal{L}}{\partial I_{(12)}} + \frac{\partial \mathcal{L}}{\partial I_{(22)}} R \right] + \frac{1}{8} \left[ R(g^{im} g^{kn} - g^{km} g^{in}) - (R_{im} g^{kn} - R_{in} g^{km} + R_{kn} g^{im} - R_{km} g^{in}) \right] \frac{\partial \mathcal{L}}{\partial I_{(32)}} + \left[ R_{ikmn} + \frac{R}{4} (g^{im} g^{kn} - g^{km} g^{in}) - \frac{1}{2} (R_{im} g^{kn} - R_{in} g^{km} + R_{kn} g^{im} - R_{km} g^{in}) \right] \frac{\partial \mathcal{L}}{\partial I_{(42)}} + ... .
\]

### 3.1.3 The generalized higher order Einstein equations

The variation, with respect to the metric coefficients \( g^{ik} \), of the action functional with Lagrangian (23) yields the non-minimal extension of the Einstein equations,

\[
0 = \text{Ein}_{ik} - \frac{1}{2} \mathcal{L} g_{ik} + \left( \frac{\partial \mathcal{L}}{\partial I_{(11)}} + R \frac{\partial \mathcal{L}}{\partial I_{(21)}} \right) F_{in} F^{n}_k\]
+ \frac{1}{2} \left( \frac{\partial L}{\partial I_{(12)}} + R \frac{\partial L}{\partial I_{(22)}} \right) (F_{il}^* F_{il}^l + F_{kl}^* F_{kl}^l) + R_{ik} \left( I_{(11)} \frac{\partial L}{\partial I_{(21)}} + I_{(12)} \frac{\partial L}{\partial I_{(22)}} \right) \\
+ \left( g_{ik} \nabla^i \nabla_l - \nabla_i \nabla_k \right) \left( I_{(11)} \frac{\partial L}{\partial I_{(21)}} + I_{(12)} \frac{\partial L}{\partial I_{(22)}} \right) \\
+ \frac{1}{2} \frac{\partial L}{\partial I_{(31)}} F_{in}^{(R)} (R_{il} F_{kn} + R_{kl} F_{in}) + R_{im} F_{in} F_{kn} \right) + \frac{1}{4} g_{ik} \nabla_m \nabla_l \left( \frac{\partial L}{\partial I_{(31)}} F_{in}^{(F)} \right) \\
+ \frac{1}{4} \nabla^m \nabla_m \left( \frac{\partial L}{\partial I_{(31)}} F_{in}^{(F)} \right) - \frac{1}{4} \nabla_l \left[ \nabla_i \left( \frac{\partial L}{\partial I_{(31)}} F_{mn}^{(F)} \right) + \nabla_k \left( \frac{\partial L}{\partial I_{(31)}} F_{in}^{(F)} \right) \right] \\
+ \frac{1}{8} \nabla_m \left\{ \nabla_i \left[ \frac{\partial L}{\partial I_{(32)}} (F_{kn}^{*} F_{ln}^{*} + F_{kn}^{*} F_{sn}^{*} + F_{kn}^{*} F_{kn}^{*}) \right] + \nabla_k \left[ \frac{\partial L}{\partial I_{(32)}} (F_{kn}^{*} F_{ln}^{*} + F_{kn}^{*} F_{sn}^{*}) \right] \right\} + \\
+ \frac{1}{16} \frac{\partial L}{\partial I_{(32)}} R (F_{kn}^{*} F_{kn}^{*} + F_{kn}^{*} F_{kn}^{*} + F_{kn}^{*} F_{kn}^{*} + F_{kn}^{*} F_{kn}^{*}) + \\
+ \frac{3}{4} \frac{\partial L}{\partial I_{(41)}} F_{i}^{*} R_{kln}^{*} + F_{k}^{*} R_{imls}^{*} \right) + \frac{1}{2} \nabla_m \nabla_n \left[ \frac{\partial L}{\partial I_{(41)}} (F_{i}^{*} F_{m}^{*} + F_{k}^{*} F_{i}^{*}) \right] + \\
+ \frac{3}{8} \frac{\partial L}{\partial I_{(42)}} (F_{i}^{*} R_{kln}^{*} F_{i}^{*} + F_{i}^{*} R_{imls}^{*} F_{i}^{*} + F_{i}^{*} R_{kln}^{*} F_{i}^{*} + F_{i}^{*} R_{imls}^{*} F_{i}^{*}) + \\
+ \frac{1}{2} g_{ik} \frac{\partial L}{\partial I_{(42)}} I_{(42)} + \frac{1}{4} \nabla_m \nabla_n \left[ \frac{\partial L}{\partial I_{(42)}} (F_{i}^{*} F_{i}^{*} + F_{i}^{*} F_{i}^{*} + F_{i}^{*} F_{i}^{*} + F_{i}^{*} F_{i}^{*}) \right] + \ldots . \quad (28)

These equations can be rewritten in the well-known form $\text{Ein}_{ik} = \kappa T_{ik}^{\text{eff}}$, but it is not the canonic form when one is dealing with general non-minimal non-linear electrodynamics. The reason is the following: even if the Lagrangian for the pure gravitational field is of the Einstein-Hilbert form, the equation (28) contains higher order derivatives of the metric, coming from the curvature tensor in terms containing non-minimal scalars, $\frac{\partial L}{\partial I_{(ab)}}$. Thus, a generic non-minimal non-linear electrodynamics is associated with a higher order gravitation. One can then ask the question, whether or not non-minimal non-linear electrodynamics models exist for which gravity is of second order. We believe that this is possible for a special choice of the dependence $\mathcal{L}(I_{(ab)})$ and for specific symmetric space-times. Below we consider a simple example confirming such idea.
3.2 Non-minimal coupling models, with the coupling linear in the curvature, in Einstein-Hilbert gravity

3.2.1 The action

A special case worth of discussion is when one restricts the above theory to a Lagrangian that is Einstein-Hilbert in the gravity term, quadratic in the Maxwell tensor and the coupling between the electromagnetism and the metric is linear in the curvature. Thus the theory may contain the invariants \( I_{(11)}, I_{(21)}, I_{(31)}, I_{(41)} \), only. Such a Lagrangian takes the form

\[ \mathcal{L} = \frac{R}{\kappa} + \frac{1}{2} F_{mn} F^{mn} + \frac{1}{2} \chi^{ikmn} F_{ik} F_{mn} , \]

(29)

where the quantity \( \chi^{ikmn} \) is the susceptibility tensor. The origin of such a terminology is the following. One obtains from the Lagrangian (29) with the definition (25) that the induction tensor \( H^{ik} \) and the Maxwell tensor \( F_{mn} \) are linked by the linear constitutive law (see, e.g., [12, 13])

\[ H^{ik} \equiv F^{ik} + \chi^{ikmn} F_{mn} . \]

(30)

Another important tensor, appearing in the electrodynamics of continuum media, is the polarization-magnetization tensor \( M^{ik} \), defined by

\[ 4\pi M^{ik} \equiv H^{ik} - F^{ik} , \]

(31)

and equal to

\[ 4\pi M^{ik} = \chi^{ikmn} F_{mn} , \]

(32)

according to (30). In the standard terminology of continuum electrodynamics [14, 15] the proportionality coefficients \( \chi^{ikmn} \) form the so-called susceptibility tensor. Generally, it has the same symmetry of the indices transposition as the Riemann tensor, and has 21 independent components. In our case the susceptibility tensor is linear in the curvature.

3.2.2 Susceptibility tensor

According to the specifications above, the susceptibility tensor has to be of the form

\[ \chi^{ikmn} \equiv \frac{q_1 R}{2} (g^{im} g^{kn} - g^{in} g^{km}) + \frac{q_2}{2} (R^{im} g^{kn} - R^{im} g^{km} + R^{kn} g^{im} - R^{km} g^{im}) + q_3 R^{ikmn} . \]

(33)

The parameters \( q_1, q_2, \) and \( q_3 \) are in general arbitrary. They have to be chosen by some ad hoc constraint, phenomenological or otherwise. For instance, the Lagrangian of the type given by equations (29) and (33), with \( q_1 = q_2 = 0, q_3 = -\lambda \), and \( \lambda \) a constant, has been proposed phenomenologically by Prasanna in the context of non-minimal modifications of the electrodynamics [16, 17]. Some general phenomenological properties of the Lagrangian (29) and (33) have been discussed.
by Goenner in [18]. The problem of a phenomenological introduction of non-minimal terms into the electrodynamic equations has been exhaustively studied by Hehl and Obukhov [19]. Drummond and Hathrell [20] have made a qualitatively new step, they obtained modified Maxwell equations from one-loop corrections of quantum electrodynamics in curved spacetime. Their model is not phenomenological and corresponds to the Lagrangian (29) and (33) with specific choices for $q_1, q_2,$ and $q_3,$ which involve the fine structure constant and the Compton wavelength of the electron. A quantum electrodynamics motivation for the use of generalized Maxwell equations can also be found, for instance, in the work of Kostelecky and Mewes [21]. Accioly, Azeredo, Aragão, and Mukai [22] used the Prasanna electrodynamic equations to construct a special example of a conserved non-minimal effective stress-energy tensor. Exact solutions of master equations of non-minimal electrodynamics in a non-linear gravitational wave background were obtained and discussed in [23]-[26].

The susceptibility tensor $\chi^{ikmn}$ has the same index symmetries as the Riemann tensor $R^{ikmn}$. Its convolutions yield

\[
g_{kn}\chi^{ikmn} = R^{im}(q_2 + q_3) + \frac{1}{2}Rg^{im}(3q_1 + q_2),
g_{kn}g_{im}\chi^{ikmn} = R(6q_1 + 3q_2 + q_3).
\] (34)

The coefficients $q_1$, $q_2$, and $q_3$ are considered to be independent phenomenological parameters. They introduce specific cross-terms, which describe non-minimal interactions of the electromagnetic and gravitational fields. Thus, one has a three-parametric family of non-minimal models. We now consider three specific variants in the choice of the set $q_1$, $q_2$ and $q_3$, and see how it influences the expression for the susceptibility tensor.

(a) The susceptibility tensor is proportional to the double-dual Riemann tensor

The gravitational analogue of the dual Maxwell tensor $F^*_ik$, is given by the double-dual Riemann tensor

\[
G_{ikmn} \equiv *R_{ikmn}^* \equiv \frac{1}{4} \epsilon_{ikab}R_{abcd}\epsilon_{cdmn}.
\] (35)

The analogy is due to the similarity of the identity $\nabla^n F^*_{in} = 0$ for the Maxwell tensor, with the identity $\nabla^n G_{ikmn} = 0$ for the double-dual Riemann tensor. The convolution of the double-dual Riemann tensor gives the Einstein tensor

\[
g^{kn}G_{ikmn} = R_{im} - \frac{1}{2}Rg_{im}.
\] (36)

Now, the double-dual Riemann tensor is given by

\[
G^{ikmn} \equiv -\frac{R}{2}(g^{im}g^{kn} - g^{in}g^{km}) + (R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in}) - R^{ikmn}.
\] (37)
Thus, if one imposes that the susceptibility tensor $\chi^{ikmn}$ is proportional to the double-dual Riemann tensor, i.e.,

$$\chi^{ikmn} = q \, g^{ikmn},$$  \hfill (38)

one obtains from equation (33) a one-parameter model with the following values for $q_1, q_2,$ and $q_3$: $q_1 = q_3 = -q,$ and $q_2 = 2q$. This can also be written as,

$$q_1 + q_2 + q_3 = 0, \quad 2q_1 + q_2 = 0. \hfill (39)$$

For this one-parameter model the non-minimal Lagrangian (29) can be rewritten in terms of the Ricci scalar, the Maxwell tensor, the dual Maxwell tensor and the standard Riemann tensor as follows,

$$L = R + \frac{1}{2} F_{mn} F^{mn} + \frac{q}{2} R^{ikmn} F^*_i F^*_m. \hfill (40)$$

(b) The susceptibility tensor is proportional to the Weyl conformal tensor

The Weyl tensor is given by

$$C^{ikmn} \equiv R^{ikmn} + \frac{R}{6} (g^{im} g^{kn} - g^{in} g^{km}) - \frac{1}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}). \hfill (41)$$

It has vanishing trace, i.e., $g^{kn} C^{ikmn} = 0$. If one imposes that the susceptibility tensor $\chi^{ikmn}$ is proportional to the Weyl tensor, i.e.,

$$\chi^{ikmn} = q \, C^{ikmn},$$  \hfill (42)

one obtains from equation (33) that

$$3q_1 + q_2 = 0, \quad q_2 + q_3 = 0. \hfill (43)$$

This is also a one-parameter model for which one can easily explicitly give the non-minimal Lagrangian (29).

(c) The susceptibility tensor is equal to the Drummond-Hathrell tensor

Drummond and Hathrell [20] have obtained modified Maxwell equations from one-loop corrections in quantum electrodynamics in curved spacetime. Their model corresponds to the Lagrangian (29), (33) with the following coefficients

$$2q_1 - q_3 = 0, \quad 13q_1 + q_2 = 0, \quad q_1 = -\frac{\alpha \lambda_e^2}{180 \pi}, \hfill (44)$$

where $\alpha$ is the fine structure constant and $\lambda_e$ is the Compton wavelength of the electron. This is also a one-parameter model for which one can easily explicitly give the non-minimal Lagrangian (29).
3.2.3 Non-minimal constitutive equations for the electromagnetic field

The relation (30) is of the type of a linear constitutive equation \[12, 13\]

\[
H^{ik} = C^{ikmn} F_{mn},
\]  
(45)

where the material tensor \(C^{ikmn}\) links the induction tensor with the Maxwell tensor. Comparing (30) with (45) one finds

\[
C^{ikmn} \equiv \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) + \chi^{ikmn}.
\]  
(46)

The material tensor \(C^{ikmn}\) describes the properties of the linear response of the material to an electromagnetic field, and contains the information about dielectric and magnetic permeabilities, as well as about the magneto-electric coefficients \(12, 15\).

Using the medium four-velocity \(U^i\), normalized such that \(U^i U_i = 1\), one can decompose \(C^{ikmn}\) uniquely as

\[
C^{ikmn} = \frac{1}{2} \left( \varepsilon^{im} U^k U^n - \varepsilon^{in} U^k U^m + \varepsilon^{kn} U^i U^m - \varepsilon^{km} U^i U^n \right) + \frac{1}{2} \left[ -\eta^{ikl} (\mu^{-1})_{ls} \eta^{mns} + \eta^{ikl} (U^m \nu^l_n - U^n \nu^l_m) + \eta^{lmn} (U^i \nu^k_l - U^k \nu^i_l) \right].
\]  
(47)

Here \(\varepsilon^{im}\) is the dielectric tensor, \((\mu^{-1})_{pq}\) is the magnetic permeability tensor, and \(\nu^p_m\) is the magneto-electric coefficients tensor. These quantities are defined through

\[
\varepsilon^{im} = 2 C^{ikmn} U_k U_n,
\]

\[
(\mu^{-1})_{pq} = -\frac{1}{2} \eta_{pik} C^{ikmn} \eta_{mnq},
\]

\[
\nu^p_m = \eta_{pik} C^{ikmn} U_n = U_k C^{mkl} \eta_{lpn}.
\]  
(48)

The tensors \(\eta_{mn}l\) and \(\eta^{ikl}\) are anti-symmetric tensors orthogonal to \(U^i\) and defined as

\[
\eta_{mn}l \equiv \varepsilon_{mnls} U^s, \quad \eta^{ikl} \equiv \varepsilon^{ikls} U^s.
\]  
(49)

They are connected by the useful identity

\[
-\eta^{ikp} \eta_{mnp} = \delta^{ikl} U_l U^s = \Delta^i_m \Delta^k_n - \Delta^i_n \Delta^k_m,
\]  
(50)

where the projection tensor \(\Delta^{ik}\) is defined as

\[
\Delta^{ik} = g^{ik} - U^i U^k.
\]  
(51)

The generalized 6-indices \(\delta\)-Kronecker tensor \(\delta_{mn}^{ikl}\) (see, e.g., \[1\]) may be defined by a recurrent formula through the \(\delta\)-Kronecker tensor with four indices, \(\delta_{mn}^{ik}\), as

\[
\delta_{mn}^{ikl} \equiv \delta_{m}^{i} \delta_{ns}^{kl} + \delta_{n}^{i} \delta_{sm}^{kl} + \delta_{s}^{i} \delta_{mn}^{kl}; \quad \delta_{mn}^{ik} \equiv \delta_{m}^{i} \delta_{n}^{k} - \delta_{n}^{i} \delta_{m}^{k}.
\]  
(52)
Upon contraction, equation (50) yields another useful identity
\[
\frac{1}{2} \delta_{ikl} = -\delta_{ilm} U_l U^s = -\Delta_i^m. \tag{53}
\]
The tensors \( \varepsilon_{ik} \) and \((\mu^{-1})_{ik}\) are symmetric, but \(\nu^k_l\) is in general non-symmetric. The dot denotes the position of the second index when lowered. These three tensors are orthogonal to \(U^i\),
\[
\varepsilon_{ik} U_k = 0, \quad (\mu^{-1})_{ik} U_k = 0, \quad \nu^k_l U_l = 0 = \nu^k_l U_k. \tag{54}
\]
Using the equation (46), one can show through straightforward calculations that
\[
\varepsilon^{im} = \Delta^{im} + 2 \chi^{ikmn} U_k U_n, \quad (\mu^{-1})_{pq} = \Delta_{pq} - \frac{1}{2} \delta_{ik} \chi^{ikmn} \eta_{mnq} = \Delta_{pq} - 2 \chi^{s}_{plq} U^l U^s, \quad \nu^m_{p} = \eta_{ik} \chi^{ikmn} U_n = -\chi^{*}_{pmn} U^l U^m, \tag{55}
\]
which in turn satisfy the relations (54). From the relations given in (55), one sees that the non-minimal interaction of the gravitational and electromagnetic fields effectively changes the dielectric and magnetic properties of the vacuum, and produces a specific magnetoelectric interaction. In this sense, under the influence of non-minimal interactions the vacuum behaves as a material medium, called a quasi-medium. Note from (55) that the tensor \(\chi^{ikmn}\) predetermines the changes in the dielectric properties of this quasi-medium, the double-dual tensor \(\chi^{*}_{plq}\) influences its magnetic properties, while the dual tensor \(\chi^{*}_{pmn} U^l U^m\) produces magneto-electric effects.

In order to complete this analogy, one can write the relationships between the four-vectors electric induction \(D^i\) and magnetic field \(H^i\), on one hand, and the four-vectors electric field \(E^i\) and the magnetic induction \(B^i\) on the other hand. These relations are
\[
D^i = \varepsilon^{im} E_m - B^i \nu^i_l U_l, \quad H^i = \nu^i_{m} E_m + (\mu^{-1})_{im} B^m. \tag{56}
\]
The vectors \(D^i, H^i, E^i\) and \(B^i\) are defined by the following formulae:
\[
D^i = H^ik U_k, \quad H^i = H^{*ik} U_k, \quad E^i = F^{ik} U_k, \quad B^i = F^{*ik} U_k. \tag{57}
\]
These vectors are orthogonal to the velocity four-vector \(U^i\),
\[
D^i U_i = 0 = E^i U_i, \quad H^i U_i = 0 = B^i U_i, \tag{58}
\]
and form the basis for the \(F_{mn}\) and \(H_{mn}\) tensors decomposition
\[
F_{mn} = E_m U_n - E_n U_m - \eta_{mn} B^l, \quad H_{mn} = D_m U_n - D_n U_m - \eta_{mn} H^l. \tag{59}
\]
3.2.4 Master equations for the gravitational field

We are working with a non-minimal electro-gravitational system, with the coupling terms linear in curvature, with the additional restrictions that the system is also linear in the Maxwell tensor, and the gravity part is Einstein-Hilbert. In this non-minimal theory, linear in the curvature terms, the equations for the gravity field (28) can be written in such a way as to look like the standard form of Einstein equation, i.e., as

\[ R_{ik} - \frac{1}{2} R g_{ik} = \kappa T_{ik}^{(\text{eff})}. \] (60)

The effective stress-energy tensor \( T_{ik}^{(\text{eff})} \) in the right-hand-side of (60) is quadratic in the Maxwell tensor and takes the following form

\[ T_{ik}^{(\text{eff})} = T_{ik}^{(0)} + q_1 T_{ik}^{(1)} + q_2 T_{ik}^{(2)} + q_3 T_{ik}^{(3)}. \] (61)

The linear part of the electromagnetic stress-energy tensor \( T_{ik}^{(0)} \) is given in equation (14). The definitions for the other three parts of the stress-energy tensor, \( T_{ik}^{(1)} \), \( T_{ik}^{(2)} \) and \( T_{ik}^{(3)} \), are

\[ T_{ik}^{(1)} = R T_{ik}^{(0)} - \frac{1}{2} R_{ik} F_{mn} F^{mn} - \frac{1}{2} g_{ik} \nabla^l \nabla_l (F_{mn} F^{mn}) + \frac{1}{2} \nabla_i \nabla_k (F_{mn} F^{mn}), \] (62)

\[ T_{ik}^{(2)} = -\frac{1}{2} g_{ik} \left[ \nabla_m \nabla_l (F_{mn} F^l) - R_{lm} F^m F^l \right] - F^l \left( R_{il} F_k + R_{kl} F_{in} \right) - R^{mn} F_{im} F_{kn} - \frac{1}{2} \nabla^l \nabla_l (F_{mn} F^{mn}) + \nabla_i (F_{mn} F^l) + \nabla_k (F_{mn} F^l), \] (63)

\[ T_{ik}^{(3)} = \frac{1}{4} g_{ik} F^{mnls} F_{mn} F_{ls} - \frac{3}{4} F^{ls} (F_{m} F_{knls} + F_{k} F_{m} R_{inls}) - \frac{1}{2} \nabla_m \nabla_n (F_{m} F_{m} + F_{k} F_{m}). \] (64)

Note that \( T_{ik}^{(3)} \) in equation (64) takes the same form as the stress-energy tensor constructed in [22].

While the stress-energy tensor of the electromagnetic field, \( T_{ik}^{(0)} \), has zero trace, the effective stress-energy tensor \( T_{ik}^{(\text{eff})} \) has a nonvanishing trace. Indeed, \( T^{(\text{eff})} \equiv g^{ik} T_{ik}^{(\text{eff})} \), is given by

\[ T^{(\text{eff})} = -q_1 \left[ \frac{1}{2} R F_{mn} F^{mn} + \frac{3}{2} \nabla^k \nabla_k (F_{mn} F^{mn}) \right] \]

\[ -q_2 \left[ R^{mn} F_{m} F_{kn} + \frac{1}{2} \nabla^k \nabla_k (F_{mn} F^{mn}) + \nabla^m \nabla_n (F_{kn} F_{km}) \right] \]

\[ -q_3 \left[ \frac{1}{2} R^{mnls} F_{mn} F_{ls} + \nabla^m \nabla_n (F_{kn} F_{km}) \right] \]

\[ = \frac{1}{2} \chi^{mnls} F_{mn} F_{ls} - (q_2 + q_3) \nabla^m \nabla_n (F_{kn} F_{km}) \]

\[ -\frac{1}{2} (3q_1 + q_2) \nabla^k \nabla_k (F_{mn} F^{mn}). \] (65)
Note that the sign of the trace is not defined a priori, depends on the specific model one uses. This feature also happens in non-linear electrodynamic models (see, e.g., [27]).

Equations (60)-(64) contain covariant derivatives of the Maxwell tensor only, and do not involve derivatives of the Riemann tensor, Ricci tensor and Ricci scalar. Thus for a given electromagnetic field they form a system of differential equations containing second order partial derivatives in the metric. Nevertheless, these equations have to be completed by the self-consistent equations of non-minimal electrodynamic (6), (24), and (30), which contain the covariant derivatives of the Riemann tensor, Ricci tensor and Ricci scalar. In general, the Maxwell tensor, envisaged as a solution to equations (6), (24), and (30), depends on the second order partial derivatives of the metric. Thus, in general, the equations for the gravitational field become of fourth order. However, the parameters $q_1$, $q_2$ and $q_3$ are arbitrary and may be fixed in an appropriate way. So, the question of whether or not there are models which are effectively of second order in the derivatives of the metric is pertinent. Below in section 3.3. we show explicitly one such a model.

3.2.5 Bianchi identities

Since the Einstein tensor in the left-hand-side of equation (60) is divergence-free, the effective stress-energy tensor (61)-(64) has to be conserved, i.e.,

$$\nabla^k T_{ik}^{(\text{eff})} = 0.$$  \hspace{1cm} (66)

In order to check directly that this is true, one has to use, first, the Maxwell equations (6) and (24) with (30), and second, the Bianchi identities and the properties of the Riemann tensor, $\nabla_i R_{klmn} + \nabla_l R_{ikmn} + \nabla_k R_{limn} = 0$ and $R_{klmn} + R_{mkln} + R_{lmkn} = 0$, as well as the rules for the commutation of covariant derivatives, which for vectors yields $(\nabla_i \nabla_k - \nabla_k \nabla_i) W^l = W^m R^l_{mkl}$.

3.3 An example: static spherically symmetric gravitational and electromagnetic fields non-minimally coupled

The line element for the static spherically symmetric model has the form

$$ds^2 = B(r) c^2 dt^2 - A(r) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$  \hspace{1cm} (67)

Assume also that the electromagnetic field inherits the static and spherical symmetries. Then the electric field potential $A_i$ has the form $A_i = \varphi'(r) \delta_i^0$. The Maxwell tensor happens to be equal to $F_{ik} = \varphi' (\delta_i^0 \delta_k^0 - \delta_i^0 \delta_k^0)$, where a prime denotes the derivative with respect to $r$. To characterize the electric field, it is convenient to introduce a new scalar quantity $E(r)$ as follows,

$$E^2(r) \equiv -E^i E_i = -\frac{1}{2} F_{ik} F^{ik} = \frac{1}{AB} F^2_{r0} = \frac{1}{AB} \varphi'^2,$$  \hspace{1cm} (68)
where the four-vector $E^i$ is defined in equation (57), and the velocity four-vector is chosen to be equal to $U^i = \delta^i_0 B^{-\frac{1}{2}}$.

To fix the sign we choose $F_{r0} = -(AB)^{\frac{1}{2}} E(r)$ and $F^{r0} = (AB)^{-\frac{1}{2}} E(r)$. For this electromagnetic field the Maxwell equations (3) are satisfied identically, while equations (24) and (30) give only one non-trivial equation when $i = 0$,

$$
[r^2 E(r) \left( 1 + 2 \chi^{0r}_{0r}(r) \right)]' = 0.
$$

The function $E(r)$ can then be found to be

$$
E(r) = \frac{Q}{r^2 \varepsilon_r(r)}, \quad \text{where} \quad \varepsilon_r(r) \equiv 1 + 2 \chi^{0r}_{0r}(r),
$$

and $Q$ is a constant. Assume now that the space-time with metric (67) is asymptotically flat and $\chi^{0r}_{0r}(\infty) = 0$. Then, the constant $Q$ in (70) coincides with the total charge of the object if $\varphi(r) \to \frac{Q}{r}$ at $r \to \infty$. Using (33) one can compute the term $\chi^{0r}_{0r}(r)$,

$$
\chi^{0r}_{0r}(r) = (q_1 + q_2 + q_3) \left[ \frac{B''}{2AB} - \frac{(B')^2}{4AB^2} - \frac{A'B'}{4A^2B} \right] + \\
+ (2q_1 + q_2) \frac{1}{2rA} (\frac{B'}{B} - \frac{A'}{A}) - q_1 \frac{1}{r^2} \left( 1 - \frac{1}{A} \right).
$$

So, from equation (70), one sees that generally, $E(r)$ contains derivatives of the metric up to the second order. With such an electric field, equations (60)-(64) for the gravitational field become of the fourth order. To illustrate this statement take the trace of equation (60), $R = -\kappa T^{(\text{eff})}$, where the trace $T^{(\text{eff})}$ is given in (65). For the metric (67) and the electric field (70)-(71) the trace equation takes the form

$$
\frac{1}{\kappa} \left[ \frac{B''}{B} - \frac{(B')^2}{2B^2} - \frac{A'B'}{2AB} + \frac{2}{r} \left( \frac{B'}{B} - \frac{A'}{A} \right) - \frac{2}{r^2} (A-1) \right] =
$$

$$
= (E^2)''(3q_1 + 2q_2 + q_3) + (E^2)' \left[ (3q_1 + 2q_2 + q_3) \left( \frac{B'}{2B} + \frac{2}{2A} \right) + \frac{2}{r} (q_2 + q_3) \right] +
$$

$$
+ E^2 \left[ (q_1 + q_2 + q_3) \left( \frac{B''}{B} + \frac{(B')^2}{2B^2} + \frac{A'B'}{2AB} + \frac{2}{r^2} \right) - \right.
$$

$$
- \frac{(2q_1 - q_3)}{r} \left( \frac{B'}{B} - \frac{A'}{A} \right) + \frac{2q_1}{r^2} (A-2) \right].
$$

Generally, equation (72) includes the first and the second derivatives of the square of the electric field $E(r)$, which contains, in its turn, the first and the second derivatives of the metric coefficients. Thus, for generic $q_1$, $q_2$ and $q_3$ we obtain a fourth order scalar equation for the gravity field. Direct calculations show that the equations derived from (60) for the sets of indices $tt$, $rr$, $\theta\theta$, $\varphi\varphi$ display the same features.
Now, when the susceptibility tensor is proportional to the double-dual Riemann tensor, i.e., \( q_1 + q_2 + q_3 = 0 \) and \( 2q_1 + q_2 = 0 \) or \( q_1 = q_3 = -q \) and \( q_2 = 2q \), all the derivatives disappear from the expression for \( E(r) \), providing the formula

\[
E(r) = \frac{Q}{r^2 + 2q\left(1 - \frac{1}{A}\right)}.
\]  

(73)

Thus, we recover the result obtained by Müller-Hoissen and Sippel in [28] for the special model with \( q_1 = q_2 = \gamma, q_2 = -2\gamma \). Moreover, equations (60)-(64) simplify significantly, in particular, equation (72) yields

\[
R = \frac{2\kappa q}{r^2A} \left[ r(E^2)' + E^2(2 - A) + \frac{r}{2}E^2 \left( \frac{B'}{B} - \frac{A'}{A} \right) \right].
\]  

(74)

This equation is, evidently, of second order with respect to the derivative \( d/dr \). For \( A(\infty) = 1 \) this electric field is asymptotically Coulombian. Formally, (73) has a form of the type discussed in [4, 9, 10, 11]. We intend to consider such a model in a future work.

4 Conclusion

We have established a new self-consistent system of equations for the gravitational and electromagnetic fields. The procedure was based on a non-minimal and non-linear extension of the standard Einstein-Hilbert–Maxwell Lagrangian. The class of systems we have studied includes non-minimal electrodynamic equations, containing the Riemann and Ricci tensors and the Ricci scalar both in the non-linear and linear versions.

This class of models of non-minimal and non-linear coupling of the gravitational and electromagnetic fields is of great interest, since the appearance of cross-terms in the Lagrangian leads to modifications of the coefficients involving the higher-order derivatives both in the Maxwell and Einstein equations. This means, in particular, that the velocity of the coupled gravito-electromagnetic waves should differ from the speed of light in vacuum.

The general field equations obtained in the paper can in principle be classified using the explicit dependence of \( \mathcal{L}(I_{(ab)}) \) on \( I_{(ab)} \) in the non-linear theory, whereas in the linear theory one uses the phenomenological parameters \( q_1, q_2 \) and \( q_3 \). This is important for two reasons. First, one should search for non-minimal models in which the gravitational field is described by equations of the second order in the derivatives of the metric. We have shown explicitly, that static spherically symmetric configurations satisfy such a requirement if the susceptibility tensor is proportional to the double-dual Riemann tensor. This model requires a detailed analysis and we intend to consider it in a separate paper. Second, for the non-minimal non-linear coupling between electrodynamics and gravitation one should search for master equations (no matter whether they are of second or of higher order) admitting non-singular, regular, solutions for the gravitational and electromagnetic fields.
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