THE SCHWARZSCHILD BLACK HOLE AS A POINT PARTICLE

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Abstract

The description of a point mass in general relativity (GR) is given in the framework of the field formulation of GR where all the dynamical fields, including the gravitational field, are considered in a fixed background spacetime. With the use of stationary (not static) coordinates non-singular at the horizon, the Schwarzschild solution is presented as a point-like field configuration in a whole background Minkowski space. The requirement of a stable $\eta$-causality stated recently in [J. B. Pitts and W. C. Schieve, Found. Phys. 34, 211 (2004)] is used essentially as a criterion for testing configurations.

Key words: general relativity, bimetric, black holes

1. INTRODUCTION AND MOTIVATION

During many decades up to the present [1], in numerous classical and quantum applications and developments, the Schwarzschild solution is the one of the most popular models in general relativity (GR). Usually the Schwarzschild solution is treated as a point mass solution in GR [2]. However, if one considers GR in the usual geometrical description, then this interpretation meets conceptual difficulties (for details see the paper by Narlikar [3] and a discussion in the paper [4]).

Such difficulties do not appear in Newtonian gravity, where a description of the distribution of masses and energy is very simple. The unique Poisson equation for the gravitational potential is considered within the matter and outside the matter. One can use integration over both the surface surrounding a source and the whole physical volume. The same formulae can be applied both to a continuous distribution and to a point mass. To describe a point particle one has to assume that a distribution has the form $m\delta(r)$ where $\delta$-function satisfies the ordinary Poisson equation, which in spherical coordinates is

\[
\nabla^2 \left( \frac{1}{r} \right) = \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \frac{1}{r} = -4\pi\delta(r).
\]

Then, the Newtonian potential will apply to the whole space including the point $r = 0$.

In [4], it was shown that, analogously to the Newtonian prescription, the point mass in GR can be described in a non-contradictory manner in the framework of a so-called field theoretical formulation (or simply “field formulation”) of GR, where all the dynamical fields, including the gravitational field, are considered in a background (fixed, auxiliary) spacetime (curved or flat). The field formulation was developed in [5] - [7] and is based on the famous paper by Deser [8], who has generalized the results of previous works of other authors deriving GR from the postulates of special relativity (see, for example, the paper by Kraichnan [9] as one of important papers). The field formulation is four-covariant and is very similar to a gauge invariant field theory in a fixed spacetime. At the same time, the field description can be constructed with the help of a simple decomposition of the variables of the geometrical formulation into a sum of background and dynamical variables of the field formulation [6]. Therefore, any solutions to GR can be treated in the framework of the field formulation, both the formulations of GR are equivalent locally, and they have to be equivalent in all the physical predictions. On the other hand, in the general case, a manifold which supports a physical metric has not to coincide with a manifold which supports a background auxiliary metric. As a result, non-physical “singularities”, “membranes”, “absolute voids”, etc., can appear in a field configuration propagating on the background. This can lead to cumbersome explanations, confused interpretations, etc. Considering the field formulation as a convenient tool for a resolution of several theoretical problems in GR it is reasonable to avoid such difficulties. Thus, here we exploit the model when a spacetime of the standard Schwarzschild solution and a background Minkowski space are in one-to-one correspondence.
The existence of the energy-momentum tensor (not pseudotensor) for the gravitational field and its matter sources is one of the advantages of the field formulation. This is the main reason why this formulation was used in [4] to consider the energy problem in GR. In particular, in [4] the Schwarzschild solution was presented as a gravitational field configuration in a background Minkowski space presented and described by the spherical Schwarzschild (static) coordinates. The concept of Minkowski space was extended from spatial infinity (frame of reference of a distant observer) up to the horizon \( r = r_g \) (in the Schwarzschild coordinate \( r \)), and even under the horizon including the worldline \( r = 0 \) of the true singularity. Then, the energy-momentum tensor was constructed, the energy distribution and the total energy with respect to the background were obtained. The configuration satisfies the Einstein equations at all the points of the Minkowski space, including \( r = 0 \). The energy distribution is presented by an expression proportional to \( \delta(r) \) and by free gravitational field outside \( r = 0 \). The picture is clearly interpreted as a point particle distribution in GR. Indeed, the configuration is essentially presented by \( \delta \)-function, one can use the volume integration over the whole Minkowski space and obtain the total energy \( mc^2 \) in the natural way. In spite of advantages, the interpretation of the point mass in [4] has open questions. At \( r = r_g \) both the gravitational potentials and the energy density have discontinuities. This highlights the fact that in the standard formulation of GR one has a coordinate singularity at \( r = r_g \) in the Schwarzschild coordinates. It is not a real singularity, and in the field formulation this break is interpreted as a “bad” fixing of gauge freedom. Nevertheless, a “visible” boundary between the regions outside and inside the horizon exists and does not allow to consider an evolution of events continuously.

Thus, the gauge fixing has to be improved. That is the break at \( r = r_g \) has to be countered with the use of an appropriate choice of a flat background, which is determined by related coordinates for the Schwarzschild solution. At least, the use of the coordinates without singularities at the horizon, like Novikov’s, Kruskal-Szekeres’s, etc., coordinates [2, 10], could resolve the problem locally at neighborhood of \( r = r_g \). Besides, we restrict ourself by the following. First, we represent a point particle at rest and in the whole Minkowski space; therefore it has to be natural to describe the true singularity by the world line \( r = 0 \) of the chosen polar coordinates. Second, the Schwarzschild solution in appropriate coordinates has to be asymptotically flat.

Third, we require a fulfilment of a so-called “\( \eta \)-causality” (property, when the physical light cone is inside the flat light cone) at all the points of the Minkowski space. It is necessary to avoid interpretation difficulties under the field theoretical presentation of GR. By this requirement all the causally connected events in the physical spacetime are described by the right causal structure of the Minkowski space. A related position of the light cones is not gauge invariant. Properties of the \( \eta \)-causality and gauge transformations conserving it were studied in detail recently by Pitts and Schieve [11]. We take the third requirement only to construct a more convenient in applications and interpretation field configuration for the Schwarzschild solution. To avoid ambiguities we stress again that, unlike Pitts and Schieve who gives a real sense to the background, we use it as an auxiliary construction. Thus, we agree with the assertion by Grishchuk [12] that changing the mutual disposition of the light cones one cannot change the physical properties of the solution. The requirement of the \( \eta \)-causality can be strengthened by the requirement of a “stable \( \eta \)-causality” [11]. The last means that the physical light cone has to be strictly inside the flat light cone, and this is important when quantization problems are under consideration. Indeed, in the case of tangency a field is on the verge of \( \eta \)-causality violation [11]. Returning to the presentation in the Schwarzschild coordinates in [4] we note that it does not satisfy the third requirement.

More appropriate coordinates are, first, the \( \textit{stationary} \) (not static) coordinates presented in [13, 14] (independently in [15, 16]), and recently improved in [11], second, contracting Eddington-Finkelstein coordinates in stationary form [10]. These coordinate systems belong to a parameterized family where all of systems satisfies all the above requirements. Qualitatively the aforementioned two systems present all the important properties of the family (see discussion at the end of the paper). Therefore, for the sake of simplicity and clarity we use just them to approach the goal of the present letter, that is to describe the Schwarzschild solution as a point particle in GR.

Except a pure theoretical interest the description given in the present paper could be interesting and useful for experimental gravity problems. Gravitational wave detectors such as LIGO and VIRGO will
2. ELEMENTS OF THE FIELD FORMULATION OF GR

At first, we briefly repeat the main notions of the field formulation of GR [5]. Here, it is enough to consider the equations for the gravitational field $h_{\mu\nu}$ on Ricci-flat backgrounds:

$$C^L_{\mu\nu} (h^{\alpha\beta}) = \kappa t^\text{tot}_{\mu\nu}. \quad (2)$$

The left hand side is linear in the symmetric tensor $h_{\mu\nu}$:

$$G^L_{\mu\nu} (h^{\alpha\beta}) \equiv \frac{1}{2} \left( h_{\mu\nu;\alpha} + \gamma_{\mu\nu} h^{\alpha\beta;\alpha\beta} - h^{\alpha\beta}_{\mu;\nu\alpha} - h^{\alpha\beta}_{\nu;\mu\alpha} \right) \quad (3)$$

where $\gamma_{\mu\nu}$ is the background metric; $\gamma \equiv \text{det} \gamma_{\mu\nu}$; $(;\alpha)$ means the covariant derivative with respect to $\gamma_{\mu\nu}$. The total energy-momentum tensor

$$t^\text{tot}_{\mu\nu} \equiv t^g_{\mu\nu} + t^m_{\mu\nu} \quad (4)$$

is obtained after varying the action of GR in the field form with respect to $\gamma^{\mu\nu}$. The pure gravitational part of (4) has the form:

$$\kappa t^g_{\mu\nu} = - (KK)_{\mu\nu} + \frac{1}{2} \gamma_{\mu\nu} (KK)_{\alpha} + Q^\sigma_{\mu\nu;\sigma} \quad (5)$$

with the tensors

$$(KK)_{\mu\nu} \equiv K^\alpha_{\mu\nu} K^\beta_{\alpha\beta} - K^\gamma_{\mu\beta} K^\alpha_{\nu\alpha}, \quad (6)$$

$$Q^\sigma_{\mu\nu} \equiv - \frac{1}{2} \gamma_{\mu\nu} h^{\alpha\beta} (K^\sigma_{\alpha\beta} + \frac{1}{2} h_{\mu\nu} K^\alpha_{\gamma\sigma} - h^\sigma_{\mu (\gamma} K^\alpha_{\nu)\alpha}) + h^{\beta\gamma} K^\sigma_{(\nu\alpha)} + h^{\beta\sigma} (\mu K^\gamma_{\nu}) - h^\beta_{(\mu (\gamma} K^\sigma_{\nu)\alpha) K^\alpha_{\beta\rho} \gamma^{\rho\sigma}), \quad (7)$$

$$K^\alpha_{\beta\gamma} \equiv \Gamma^\alpha_{\beta\gamma} - C^\alpha_{\beta\gamma} \quad (8)$$

where $\Gamma^\alpha_{\beta\gamma}$ and $C^\alpha_{\beta\gamma}$ are the Christoffel symbols for the dynamic (physical) and background spacetimes respectively. Note that in fact the field configuration is defined by the components $h_{\mu\nu}$. However, sometimes variables of the 1-st order formalism are more convenient, thus in expressions (5) - (7) the components of the tensors $h_{\mu\nu}$ and $K^\alpha_{\beta\gamma}$ are used as independent variables (see for the details [6]). Note also that if Eq. (2) is satisfied, then the total energy-momentum tensor (4) can be obtained with the use of its left hand side, that is with the expression (3).

The equivalence between the field and the geometrical formulations of GR can be stated after the simple identifications

$$\sqrt{-\gamma} (\gamma^{\mu\nu} + h^{\mu\nu}) \equiv \sqrt{-g} g^{\mu\nu}$$

$$C^\alpha_{\beta\gamma} + K^\alpha_{\beta\gamma} \equiv \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} (g_{\rho\beta,\gamma} + g_{\rho\gamma,\beta} - g_{\beta\gamma,\rho}) \quad (9)$$

where $g \equiv \text{det} g_{\mu\nu}$. Then, the equations (2) change over to the usual form of the Einstein equations with the dynamic metric $g_{\mu\nu}$. The source energy-momentum tensor in (4) is connected with the usual matter energy-momentum tensor $T^\text{tot}_{\mu\nu}$ of GR as

$$t^m_{\mu\nu} = T^\text{tot}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\alpha\beta} g^{\alpha\beta} - \frac{1}{2} \gamma_{\mu\nu} \chi^{\alpha\beta} (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T_{\pi\rho} g^{\pi\rho}) \quad (10)$$
3. THE SCHWARZSCHILD SOLUTION IN A STABLY $\eta$-CAUSAL DESCRIPTION AND THE TRUE SINGULARITY

A Minkowski space related to the stationary coordinates $\{t, r^*, \theta, \phi\}$ constructed in [13, 14] for the Schwarzschild solution does not cover the region around the true singularity with the radius less $r_g/2$. After making a translation of the radial coordinate $r^* \rightarrow r = r^* + r_g/2$, as it was suggested in [11], a corresponding Minkowski space just covers the whole region of the standard Schwarzschild solution under the horizon including the singularity at $r = 0$. Thus, the stationary metric [13, 14] gets the modified form [11]:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{r_g^2}{r^2}cdt\,dr - \left(1 + \frac{r_g}{r}\right)\left(1 + \frac{r_g^2}{r^2}\right)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) .$$

(11)

Coordinates $\{t, r, \theta, \phi\}$ of this metric are connected with the standard Schwarzschild coordinates $\{T, r, \theta, \phi\}$ by the transformation of the time coordinate only:

$$ct = cT + r_g \ln \left|1 - \frac{r_g}{r}\right| .$$

(12)

The important property of the solution (11) is that a falling test particle reaches the horizon $r = r_g$ in finite coordinate time $t$, under the horizon is always falling towards the singularity, gets arbitrarily close to it, but only hits it at $t = \infty$ (see [14]). However, in Minkowski space there are simply no events with $t \geq \infty$, as it was noted in [11].

As is seen, the metric (11) is left stationary due to the non-zero cross component $g_{01} = r_g^2/r^2$. Thus, analogously to the Kerr solution [2, 10] that presents the rotating dragging, or to the Lorentz transformed Schwarzschild solution [20] that presents the dragging in the direction of a velocity of distant observer, the solution (11) presents the dragging in the direction of the singularity. In this respect it is a place to note the Gullstrand-Painleve form of the Schwarzschild solution (see, e.g., a recent paper [21]). It is connected with the standard Schwarzschild metric by the transformation $ct_{GP} = cT + r_g (2\beta + \ln (1 + \beta)/(1 - \beta))$ with $\beta = (r_g/r)^{1/2}$ and is similar to (11). The Gullstrand-Painleve metric is also stationary and presents the dragging directed to the singularity. The last property is used to conceptualize a black hole as a river model [21]: the space itself flows like a river through a flat background, while objects move through the river according to the rules of special relativity. But this solution cannot be considered here because it does not satisfy to the third ($\eta$-causality) requirement.

Now let us present the solution (11) in the form of a field configuration in the Minkowski space with the metric in the polar coordinates:

$$ds^2 = c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

(13)

where we will numerate the coordinates as $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. The use of the physical metric (11) and the background metric (13) in the relations (9) and (8) give a possibility to construct the field configuration

$$h^{00} = \frac{r_g}{r} + \frac{r_g^2}{r^2} + \frac{r_g^3}{r^3}, \quad h^{01} = -\frac{r_g^2}{r^2}, \quad h^{11} = \frac{r_g}{r};$$

(14)

$$K^{00} = -K^{11} = \frac{1}{2} \frac{r_g}{r^4},$$

$$K^{01} = \frac{1}{2} \frac{r_g}{r^3} \left(1 + \frac{r_g}{r}\right) \left(1 + \frac{r_g^2}{r^2}\right),$$

$$K^{11} = \frac{1}{2} \frac{r_g^2}{r^3} \left(4 + \frac{r_g}{r} + 2 \frac{r_g^2}{r^2} + \frac{r_g^3}{r^3}\right),$$

$$K^{10} = \frac{1}{2} \frac{r_g}{r^2} \left(1 - \frac{r_g}{r}\right),$$

$$K^{03} = K^{02} \sin^2 \theta = -\frac{r_g}{r} \sin^2 \theta,$$

$$K^{13} = K^{12} \sin^2 \theta = r_g \sin^2 \theta .$$

(15)
Now we assume that the field configuration (14) and (15) satisfies the Einstein equations (2) at every point of Minkowski space, including \( r = 0 \). Then for calculating the components of the energy-momentum tensor (4) and its parts it is important to define the expression \( \nabla^2 (1/r^{k+1}) \) with integer \( k \geq 0 \). Recalling that for \( k = 0 \) it is already given in (1), we use the technique of the generalized functions [22]. Thus, considering the expression

\[
\Phi_i = \frac{\partial}{\partial x^i} \frac{1}{r^{k+1}} = -(k + 1) x_i \frac{1}{r^{k+2}}
\]

(16)
as a generalized homogeneous function of the \(-k - 2\) degree in 3 dimensions one can apply to it the rules of a differentiation derived in [23] and based on the standard notions [22]. This gives

\[
\nabla^2 \frac{1}{r^{k+1}} = (k + 1) \left[ \frac{k}{r^{k+3}} - \frac{(-1)^k}{k!} \frac{\partial^k \delta (r)}{\partial r^k} n_{\alpha_1} \ldots n_{\alpha_k} \oint_\Gamma n^{\alpha_1} \ldots n^{\alpha_k} \, d\Omega \right].
\]

(17)

In this paragraph we use the related Cartesian coordinates \( \{x^i\}, i = 1, 2, 3 \), with that \( n^i = x^i / r \); \( \Gamma \) is a closed two-surface surrounding a singular point; \( d\Omega = r^{-2} n' ds_i \) where \( ds_i \) is the element of integration on \( \Gamma \). On the other hand, one can consider \( \Psi_i = r^k \Phi_i \) as a generalized homogeneous function of the \(-2\) degree and apply the rule of differentiation given in [22] to \( \Psi_i \). The final expression is

\[
\nabla^2 \frac{1}{r^{k+1}} = (k + 1) \left[ \frac{k}{r^{k+3}} - \frac{4\pi}{r^k} \delta (r) \right].
\]

(18)

Comparing (17) with (18) one finds an equivalence between the last terms in these formulae. Reducing this equivalence to the simplest case of 1 dimension, for example, one obtains the known relation [24]: \( \partial^k \delta (x) / \partial x^k = (-1)^k k! x^{-k} \delta (x) \). Here we prefer to use the formula (18) as a more convenient in calculations. Thus, e.g., it is easy to see that integration over a round ball of the r.h.s. of (18) gives two divergent integrals at \( r \to 0 \) that compensate one another. Then, a convergent part of this volume integral is equal to a value of a surface integral that follows after integration of the l.h.s. of (18), which is a divergence \( \nabla^2 = \partial_i \partial^i \). In [4] we use also the presentation (18).

For the calculations of \( t_{\mu \nu}^{\text{tot}} \) we use the expression (3), the non-zero components of which are

\[
t_{00}^{\text{tot}} = mc^2 \delta (r) + mc^2 \frac{r_g}{r} \left( 1 + \frac{3}{2} \frac{r_g}{r} \right) \delta (r) - \frac{mc^2}{4\pi} \frac{r_g}{r^4} \left( 1 + \frac{3}{2} \frac{r_g}{r} \right),
\]

\[
t_{11}^{\text{tot}} = -mc^2 \delta (r),
\]

\[
t_{AB}^{\text{tot}} = -\frac{1}{2} \gamma_{AB} mc^2 \delta (r); \quad A, B = 2, 3.
\]

(19)

For calculations of the free gravitational part in (4) we use the expressions (5) - (7):

\[
t_{00}^g = mc^2 \frac{r_g}{4r} \left( 6 + \frac{7}{2} \frac{r_g}{r} + \frac{r_g}{r^2} \right) \delta (r) - \frac{mc^2}{4\pi} \frac{r_g}{r^4} \left( 1 + \frac{3}{2} \frac{r_g}{r} \right),
\]

\[
t_{01}^g = mc^2 \frac{r_g}{2r^2} \delta (r),
\]

\[
t_{11}^g = mc^2 \frac{r_g}{2r} \left( 1 + \frac{3}{2} \frac{r_g}{2r^2} \right) \delta (r),
\]

\[
t_{AB}^g = \gamma_{AB} mc^2 \frac{r_g^2}{4r^2} \left( 1 + \frac{r_g}{r} \right) \delta (r).
\]

(20)

In calculations of the components (19) and (20) it was used the usual notations \( \kappa = 8\pi G / c^4 \) and \( r_g = 2mG / c^2 \).

Now, for the calculation of \( t_{\mu \nu}^m \), of the matter part we use the difference between (19) and (20):

\[
t_{00}^m = mc^2 \delta (r) - mc^2 \frac{r_g}{2r} \left( 1 + \frac{r_g}{2} + \frac{r_g}{2r^2} \right) \delta (r),
\]

\[
t_{01}^m = -mc^2 \frac{r_g^2}{2r^2} \delta (r),
\]

\[
t_{11}^m = mc^2 \frac{r_g}{2r} \left( 1 + \frac{3}{2} \frac{r_g}{2r^2} \right) \delta (r),
\]

\[
t_{AB}^m = \gamma_{AB} mc^2 \frac{r_g^2}{4r^2} \left( 1 + \frac{r_g}{r} \right) \delta (r).
\]
\[ t_{11}^{\text{m}} = -mc^2 \delta(r) - mc^2 \frac{T_r}{2r} \left( 1 + \frac{r_g}{2r} + \frac{r^2}{2r^2} \right) \delta(r), \]
\[ t_{AB}^{\text{m}} = -\frac{1}{2} \gamma_{AB} mc^2 \left( 1 + \frac{r_g}{2r^2} + \frac{r^2}{2r^2} \right) \delta(r). \] (21)

The components (21) can be obtained directly. With the physical metric (11) one has to calculate the Einstein tensor \( G_{\mu\nu} \) everywhere including \( r = 0 \) and, thus, define the components of the matter tensor \( T_{\mu\nu} \), which could be a source for \( G_{\mu\nu} \). Then with using the relation (10) the components (21) are obtained again. However one has to note, in this case components \( T_{\mu\nu} \), obtained in the framework of the ordinary geometrical formulation of GR do not have a good interpretation [3].

Let us discuss properties of the field presentation of the solution (11). First, it is in the spirit of GR that \( t_{\mu\nu}^{\text{m}} \) can not be considered separately from \( t_{\mu\nu}^{\text{g}} \). Thus, it is more right to consider the total components (19).

The energy distribution is described by the \( 00 \)-component of the energy-momentum tensor. Then the total energy of the system is obtained with the use of the volume integration:

\[ E_{\text{tot}} = \lim_{r \to \infty} \int_V t_{00}^{\text{tot}} r^2 \sin \theta d\theta d\phi = mc^2. \] (22)

It is defined only by the first term \( mc^2 \delta(r) \) in \( t_{00}^{\text{tot}} \) that follows from the matter component \( t_{00}^{\text{m}} \) only. The other contributions into (22) from the \( \delta \)-functions in \( t_{00}^{\text{tot}} \) are infinite, but they are compensated by the energy distribution without \( \delta \)-functions that is a part of the gravitational component \( t_{00}^{\text{g}} \). Due to (2) the volume integration can be exchanged by the surface integration over the 2-sphere with the constant \( r = r_0 \):

\[ E_{\text{tot}} = \lim_{r_0 \to \infty} \frac{1}{2 \kappa} \oint_{\partial V} \left( h_{00}^{1/2} + \gamma_{00} h^{1/2} ; \alpha - 2 h_{00}^{1/2} \right) r^2 \sin \theta d\theta d\phi = mc^2. \] (23)

The other components \( t_{11}^{\text{tot}} \) and \( t_{AB}^{\text{tot}} \) in (19) formally could be interpreted as related to the “inner” properties of the point. Indeed, they are proportional only to \( \delta(r) \) and, thus, describe the point “inner radial” and “inner tangent” pressure.

Second, after transformation from the spherical coordinates in (11) to the corresponding Cartesian coordinates one can see that the metric (11) and the configuration (14) are asymptotically flat with the \( 1/r \)-like falloff at spatial infinity. As it was stated in [25], where the gauge invariance of integrals of motion of an isolated system was studied, the \( 1/r \)-like asymptotic behaviour just guarantees the satisfactory results (22) and (23) for the total energy. Third, the metrics (11) and (13) satisfy the requirement of the stable \( \eta \)-causality at all the points of the Minkowski space down to the true singularity at \( r = 0 \). Thus, all the requirements are satisfied.

The presented picture is more complicated than in the case of the point mass in the Newtonian gravity. Neverthless, the problem of the point mass is resolved enough simply. Indeed, the energy-momentum tensors contain \( \delta \)-functions at \( r = 0 \), and, like in the Newtonian case, the volume integration over the whole space gives a satisfactory total energy. On the other hand, the presented here description is significantly simpler and more appropriate than in [4]. The field configuration (14), unlike [4], is continuous at all the point of the Minkowski space except the true singularity \( r = 0 \), that is natural. A falling test particle approaches and intersects the horizon \( r = r_g \) in finite Minkowski time \( t \). The components \( t_{00}^{\text{tot}} \) and \( t_{00}^{\text{g}} \) have no breaks outside \( r = 0 \), and all the other energy-momentum components in (19) - (21) are defined only by a \( \delta \)-function.

4. A FIELD THEORETICAL REFORMULATION OF THE CONTRACTING EDDINGTON-FINKELSTEIN METRIC

Now let us examine the contracting Eddington-Finkelstein metric for the Schwarzschild geometry [10]:

\[ ds^2 = \left( 1 - \frac{r_g}{r} \right) c^2 dt^2 - 2 \frac{T_r}{c} \, c \, dt \, dr - \left( 1 + \frac{r_g}{r} \right) \, dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \] (24)
Notice that a transformation was made from the standard null coordinate $\tilde{V}$ to time coordinate $\tilde{t}$: $\tilde{t} = \tilde{V} - r$. If the flat background, analogously to (13), is described by the coordinates $c\tilde{t}$, $r$, $\theta$ and $\phi$, then the gravitational field configuration corresponding to (24) is

$$\begin{align*}
h_{00} &= \frac{rg}{r}, \\
h_{01} &= -\frac{rg}{r}, \\
h_{11} &= \frac{rg}{r}.
\end{align*}$$

The properties of the solutions (11) and (24) are very close. Both metrics are stationary and asymptotically flat. In the whole Minkowski space they induce asymptotically flat and continuous (except $r = 0$) configurations (14) and (25). Falling test particles intersect the horizon $r = r_g$ in finite times $t$ and $\tilde{t}$, but in the case (24) test particles even reach the true singularity in finite time $\tilde{t}$. This is the result of the time transformation for (24) [10]: $c\tilde{t} = cT + r_g \ln |1 - r/r_g|$ instead of (12).

The components of the total energy-momentum tensor for the configuration (25) are

$$\begin{align*}
t_{00}^{\text{tot}} &= mc^2\delta(r), \\
t_{11}^{\text{tot}} &= -mc^2\delta(r), \\
t_{AB}^{\text{tot}} &= -\frac{1}{2}\gamma_{AB} mc^2\delta(r).
\end{align*}$$

This energy-momentum, unlike (19), is concentrated only at $r = 0$. Of course, the volume integration of $t_{00}^{\text{tot}}$ from (26) again, like (22), gives $E^{\text{tot}} = mc^2$, and the surface integration (23) with the configuration (25) gives it also. However, unlike (23), now $mc^2$ follows with arbitrary radius of the 2-sphere $r_0$ (it is not necessary $r_0 \to \infty$), like for the electric charge in electrodynamics and for the point mass in Newtonian gravity. This situation is very close to the Penrose charge integral prescription [26] for the “quasi-local mass” $m_P = m(\partial V)$ surrounded by a 2-sphere $\partial V$. Tod [27] has adopted the Penrose construction for 2-surfaces of spherical symmetry in spherically symmetric spacetimes. Thus, the Schwarzschild mass parameter $m = m_P$ is obtained independently of radius of $\partial V$. As is seen, with the solution (24) the description of the point mass in GR looks also quite appropriate.

The transformation $c\tilde{t}' = cT + r_g \ln |(r/r_g - 1)(r_g/r)|^\alpha$ gives a parameterized by $\alpha \in [0, 2]$ family of metrics, all of which satisfies all our requirements, the cases $\alpha = 0$ and $\alpha = 1$ correspond to (24) and (11). At this the requirement of the stable $\eta$-causality is not satisfied with $\alpha = 0$ at $0 \leq r \leq \infty$. Thus, all the configurations $\alpha \in (0, 2]$ are appropriate for the study both classical and quantum problems, whereas the case $\alpha = 0$ could not be useful for the study quantized fields. Properties of field configurations corresponding to $\alpha \in (0, 2]$ qualitatively are the same as for $\alpha = 1$. In the terms of the field approach [5], all the field configurations for $\alpha \in [0, 2]$ are connected by gauge transformations and are physically equivalent. Thus, inside this family, $\eta$-causal description with (25) can be converted into a stably $\eta$-causal description explicitly. Note also that a technique of infinitesimal gauge transformation developed in [11] permits do this conversion approximately without relation to this family.

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References


