Elliptical orbits in the Bloch sphere

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As is well known, when an $SU(2)$ operation acts on a two-level system, its Bloch vector rotates without change of magnitude. Considering a system composed of two two-level systems, it is proven that for a class of nonlocal interactions of the two subsystems including $\sigma_i \otimes \sigma_j$ (with $i, j \in \{x, y, z\}$) and the Heisenberg interaction, the geometric description of the motion is particularly simple: each of the two Bloch vectors follows an elliptical orbit within the Bloch sphere. The utility of this result is demonstrated in two applications, the first of which bears on quantum control via quantum interfaces. By employing nonunitary control operations, we extend the idea of controllability to a set of points which are not necessarily connected by unitary transformations. The second application shows how the orbit of the coherence vector can be used to assess the entangling power of Heisenberg exchange interaction.

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I. INTRODUCTION

The Bloch vector, or vector of coherence [1], provides a geometric description of the density matrix of a spin-1/2 particle which is commonly used in nuclear magnetic resonance. Mathematically, the Bloch vector may be viewed as the adjoint representation of an $su(2)$ object in an $so(3)$ basis [2]. Extension of the notion of vector of coherence to two-spin systems [3, 4], and more generally to quantum spin systems of higher dimensions [5], has drawn attention in the contexts of quantum information theory and quantum computation. Specific motivations include the prospects of a useful quantification of entanglement for composite systems [3, 4, 5, 6, 7] and the quest for equations describing observables in quantum networks [4].

In the present work, the extension of the Bloch formalism to two spins is used to obtain a geometric representation of the orbits of the vector of coherence for each spin system in the case that a nonlocal interaction of the form $\sigma_i \otimes \sigma_j$ is introduced. We propose that this geometric picture will be useful in devising schemes for control of a quantum state via quantum interfaces [8], i.e., through the mediation of an ancillary system. In this vein, we investigate the limits of control of a quantum state $S$, mixed or pure, given a nonlocal interaction and an ancilla $Q$. The simple geometric picture developed below also applies to another special case of nonlocal interaction, namely the Heisenberg exchange Hamiltonian. As a second application of our formal results, we investigate the entangling power of the Heisenberg interaction.

II. PRODUCT OF OPERATOR BASIS FOR A DENSITY MATRIX

A. One qubit

The density matrix $\rho$ of a two-state system is a positive semi-definite Hermitian $2 \times 2$ matrix having unit trace. It can always be given expression in terms of the three trace-free Pauli matrices $\sigma_i$, $i = 1, 2, 3$, which are generators of $su(2)$, and $I/\sqrt{2}$ ($I$ being the unit matrix):

$$\rho = \frac{1}{2} I + v \sigma .$$

(1)

Here $v$ is the vector of coherence, whose magnitude is bounded by $0 \leq ||v|| \leq 1/2$ because $1/2 \leq \text{Tr}(\rho^2) \leq 1$. The two limiting values of the norm correspond to maximally mixed and pure states, respectively. The magnitude of the Bloch vector differs by a factor of 1/2 from that of the vector of coherence, as a matter of convention.

Unitary operations rotate the Bloch vector without changing its magnitude: $SU(2)$ operations on the qubit correspond to $SO(3)$ operations on the Bloch vector. The dynamical evolution of the Bloch vector under non-local operations is considered in the next section.
B. Two qubits and the correlation tensor

In analogy to the representation (1), we adopt the generators of $G = SU(4)$, i.e., the elements of the algebra $g = su(4)$ (together with the unit matrix), as an orthonormal basis for the $4 \times 4$ density matrix of the two-qubit system. We employ this basis as it appears in Ref. 6, noting that it differs from the basis used in Ref. 5 only in the coefficients.

The dynamical evolution of the system becomes more transparent if we choose basis elements of the algebra $g = su(4)$ in accordance with its Cartan Decomposition $g = p \oplus e$. The algebras $p$ and $e$ satisfy the commutations relations

$$[e, e] \subset e, \quad [p, e] \subset p, \quad [p, p] \subset e. \quad (2)$$

The basis elements, $W_j$, $j = 1, \ldots, 15$ of the orthogonal algebra pair $(e, p)$ are

$$e = \text{span} \frac{i}{2} \{\sigma_x \otimes 1, \sigma_y \otimes 1, \sigma_z \otimes 1, 1 \otimes \sigma_x, 1 \otimes \sigma_y, 1 \otimes \sigma_z\}, \quad (3)$$

$$p = \text{span} \frac{i}{2} \{\sigma_x \otimes \sigma_x, \sigma_x \otimes \sigma_y, \sigma_x \otimes \sigma_z, \sigma_y \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_y \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_z \otimes \sigma_y, \sigma_z \otimes \sigma_z\}. \quad (4)$$

The basis defined by Eqs. (3) and (4) is used to expand the density matrix as

$$\rho = \sum_{j=0}^{15} \text{Tr}(\rho X_j)X_j = \sum_{j=0}^{15} \rho_j X_j, \quad (5)$$

where $X_0 = I/\sqrt{4}$, $\rho_0 = 1/\sqrt{4}$, and $X_j = -iW_j$ ($j = 1, \ldots, 15$). In this representation, the density matrix is specified by three objects, namely the vectors of coherence $r_1$ and $r_2$ for the two subsystems along with the spin-spin correlation tensor $T_j^i$.

$$r_1 = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}, \quad r_2 = \begin{pmatrix} \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}, \quad T_j^i = \begin{pmatrix} \rho_7 & \rho_8 & \rho_9 \\ \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{13} & \rho_{14} & \rho_{15} \end{pmatrix}. \quad (6)$$

We note that the object $T_j^i$ has other names: Stokes tensor $\ |11\rangle$, entanglement tensor $\ |10\rangle$, and tensor of coherence (when combined with the coherence vectors in one object). Details of the properties of $T_j^i$ can be found in Ref. 6, where many prior studies are cited. This tensor contains information on the correlations between the two subsystems, of both classical and quantum nature. Necessary and sufficient conditions for separability of a pure state can be stated in terms of its properties, whereas in the case of a mixed state, only necessary conditions for separability can be given.

III. EVOLUTION UNDER LOCAL AND NON LOCAL OPERATIONS

As we have seen, the Lie algebra $g = su(4)$ possesses a Cartan decomposition $g = e \oplus p$, which informs us that there exists within the Lie group $G = SU(4)$ a subgroup of local operations $G_L = SU(2) \otimes SU(2)$ generated by $e$. All the other operations are nonlocal and members of the coset space $SU(4)/SU(2) \otimes SU(2)$, which does not form a subgroup of $SU(4)$. It is known (see Proposition 1 of Ref. 10) that any $U \in SU(4)$ can be written as

$$U = k_1 A k_2 \quad (7)$$

with

$$A = \exp \left[ \frac{i}{2} \left( c_1 \sigma_x^1 \sigma_x^2 + c_2 \sigma_y^1 \sigma_y^2 + c_3 \sigma_z^1 \sigma_z^2 \right) \right], \quad (8)$$

where $k_1, k_2 \in SU(2) \otimes SU(2)$ and $c_1, c_2, c_3 \in R$. In the following, we focus on the effect of nonlocal operations generated by a single operator among the possibilities for $\sigma_i \otimes \sigma_j$, where $i, j \in \{x, y, z\}$. (This consideration includes the special case in which two of the parameters $c_1, c_2$, and $c_3$ in the decomposition (7)-(8) are zero.) Such nonlocal operations will be called one-dimensional.
FIG. 1: Local operations on the two spin subsystems produce a rotation of the corresponding vectors of coherence around some direction \( \hat{n} \). The effect is the same for both pure states (a) and mixed states (b).

A. Local operations

Local operations are operations \( g \in SU(2) \otimes SU(2) \) generated by the elements of \( e \). From the commutation relations \([e,e] \subset e\) and \([p,e] \subset p\) it is clear that the elements of the vectors \( r_i \) and tensor \( (T^l_j) \) do not mix and do not affect one another. Under local operations, the vectors behave just like ordinary Cartesian vectors. In particular, a vector of coherence is rotated about some vector \( \hat{n} \) as illustrated in Fig. 1, i.e.,

\[
(r'_i)_1 = R_{ij} r_j^1, \quad (r'_i)_2 = R_{ij} r_j^2. \tag{9}
\]

On the other hand, the correlation tensor transforms like a mixed Cartesian tensor,

\[
(T'_j)_l = R_{mn}(R^l_j)^m T^m_n. \tag{10}
\]

The magnitude of each object remains invariant under local operations. In addition, there exist fifteen more invariants which can be constructed from the vectors and the tensor [12].

B. One-dimensional nonlocal operations

The nonlocal operations in the coset space \( SU(4)/SU(2) \otimes SU(2) \) require, in their construction, exponentiation of at least one of the elements of \( p \). Hence, under these operations the elements of the tensor and vectors of coherence are mixed, due to the commutation relations \([p,e] \subset p\) and \([p,p] \subset e\). We shall establish that the one-dimensional nonlocal operations generated by the chosen interaction \( \sigma_i \otimes \sigma_j \) give rise to elliptical orbits for the vectors of coherence of the subsystems. The characteristics of these elliptic paths depend on the indices \( i \) and \( j \), on the initial states of the subsystems, and on the degree of correlations between them. These orbits can be described by non-unitary transformations on each of the individual subsystems when one traces over the other’s degrees of freedom.

Accordingly, we take the interaction Hamiltonian between the two spins to be \( H_I = \sigma_i \otimes \sigma_j / 2 \), and, for reasons of simplicity, we suppose that the internal Hamiltonians for the two spins may be ignored. Assuming that the duration of the interaction is \( \phi \), and appealing to (i) the commutation relations as summarized in Ref. [10] and (ii) the identity

\[
\exp [-i(\phi/2)\sigma_i \otimes \sigma_j] = \cos(\phi/2)I - i \sin(\phi/2)\sigma_i \otimes \sigma_j, \tag{11}
\]

we can make the following observations:

(i) The components \( r^1_i \) and \( r^2_j \) of the vectors of coherence remain unaffected; hence the vectors are confined to planes perpendicular to the \( i \)-axis and \( j \)-axis respectively.

(ii) Of the nine elements of the correlation tensor \( T^k_l \), only four experience changes. The five that are unchanged under the action of \( \sigma_i \otimes \sigma_j \) are \( T^j_j \) and \( T^k_k \) with \( k \neq l \).

\[
\exp [-i(\phi/2)\sigma_i \otimes \sigma_j] = \cos(\phi/2)I - i \sin(\phi/2)\sigma_i \otimes \sigma_j, \tag{11}
\]
(iii) The vectors \( r^m + T^m \) and \( r^m + T^m \), are rotated, without change of magnitude, through an angle \( \phi \) about the \( i \) and \( j \) axes, respectively. (Here \( m \) ranges freely over \( \{x, y, z\} \)).

(iv) More explicitly, the components of the vectors transform according to

\[
\begin{align*}
    r^i_1 &\rightarrow (r')^i_1 = r^i_1, & r^j_2 &\rightarrow (r')^j_2 = r^j_2, \\
    r^k_3 &\rightarrow (r')^k_3 = r^k_3 \cos \phi - T^k_1 \sin \phi, & r^m_3 &\rightarrow (r')^m_3 = r^m_3 \cos \phi - T^m_1 \sin \phi, \\
    r^i_1 &\rightarrow (r')^i_1 = T^i_j \sin \phi + r^i_1 \cos \phi, & r^j_2 &\rightarrow (r')^j_2 = T^j_j \sin \phi + r^j_2 \cos \phi, \quad (12)
\end{align*}
\]

and the components of the tensor of coherence, according to

\[
\begin{align*}
    T^i_j &\rightarrow (T')^i_j = T^i_j, & T^j_j &\rightarrow (T')^j_j = T^j_j, \\
    T^k_3 &\rightarrow (T')^k_3 = T^k_3 \cos \phi - r^k_1 \sin \phi, & T^m_3 &\rightarrow (T')^m_3 = T^m_3 \cos \phi - r^m_1 \sin \phi, \\
    T^i_1 &\rightarrow (T')^i_1 = r^i_1 \sin \phi + T^i_j \cos \phi, & T^n_3 &\rightarrow (T')^n_3 = r^n_2 \sin \phi + T^n_3 \cos \phi, \quad (13)
\end{align*}
\]

so that the two sets of coordinates, \( \{r_1, \phi_1\} \) and \( \{r_2, \phi_2\} \), are related by

\[
\begin{align*}
    T^i_j &\rightarrow (T')^i_j = T^i_j, & T^j_j &\rightarrow (T')^j_j = T^j_j, \\
    T^k_3 &\rightarrow (T')^k_3 = T^k_3 \cos \phi - r^k_1 \sin \phi, & T^m_3 &\rightarrow (T')^m_3 = T^m_3 \cos \phi - r^m_1 \sin \phi, \\
    T^i_1 &\rightarrow (T')^i_1 = r^i_1 \sin \phi + T^i_j \cos \phi, & T^n_3 &\rightarrow (T')^n_3 = r^n_2 \sin \phi + T^n_3 \cos \phi,
\end{align*}
\]

with no change in the tensor’s other elements. The ordered sets of indices \( \{i, l, k\} \) and \( \{j, m, n\} \) belong to \( \{(x, y, z), (y, z, x), (z, x, y)\} \).

Given this behavior, it is not difficult to show that \( r_1(\phi) \) and \( r_2(\phi) \) follow elliptical orbits. Since the 1,2 labeling is arbitrary, it suffices to demonstrate this property for the vector \( r_1(\phi) \).

**Proof.** Referring to Fig. 2(a), the coordinates for a vector \( s \) tracing an ellipse in the \( x - y \) plane, with principal axes \( a \) and \( b \), rotated by an angle \( \psi \), are

\[
\begin{align*}
    s_x(\phi) &= a \cos \phi \cos \psi + b \sin \phi \sin \psi, \\
    s_y(\phi) &= -a \cos \phi \sin \psi + b \sin \phi \cos \psi. \quad (14)
\end{align*}
\]

The angle \( \phi \) is zero when the vector \( s \) is aligned with the principal axis \( a \).

The coordinates of the vector of coherence \( r_1 \) moving in the \( k - l \) plane are given by

\[
\begin{align*}
    r^i_1(\phi') &= r^i_1(0) \cos \phi' - T^i_j(0) \sin \phi', \\
    r^j_2(\phi') &= T^j_j(0) \sin \phi' + r^j_2(0) \cos \phi'. \quad (15)
\end{align*}
\]

Of course, for the vector of coherence, \( \phi' = 0 \) does not in general correspond to the principal axis \( a \) (see Fig. 2(a)). In fact, \( \phi' = \phi + \chi \), and the coordinates of \( r_1 \) can be rewritten as follows in terms of the phase difference \( \chi \):

\[
\begin{align*}
    r^i_1(\phi) &= (r^i_1(0) \cos \chi - T^i_j(0) \sin \chi) \cos \phi + (-T^i_j(0) \cos \chi - r^i_1(0) \sin \chi) \sin \phi, \\
    r^j_2(\phi) &= (T^j_j(0) \sin \chi + r^j_2(0) \cos \chi) \cos \phi + (r^j_2(0) \sin \chi + T^j_j(0) \cos \chi) \sin \phi. \quad (16)
\end{align*}
\]

Comparison of the two sets of coordinates \( \{s_x, s_y\} \) and \( \{r^i_1(\phi), r^j_2(\phi)\} \) shows that a match can always be made, such that the parameters \( a, b, \psi, \) and \( \chi \) can be determined by solving the system of equations

\[
\begin{align*}
    a \cos \psi &= r^i_1(0) \cos \chi - T^i_j(0) \sin \chi, \\
    b \sin \psi &= -T^j_j(0) \cos \chi - r^j_2(0) \sin \chi, \\
    a \sin \psi &= -T^i_j(0) \sin \chi - r^j_2(0) \cos \chi, \\
    b \cos \psi &= -r^i_1(0) \sin \chi + T^j_j(0) \cos \chi. \quad (17)
\end{align*}
\]

This completes the proof. It is important to note that the shape of the ellipse depends explicitly on the spin-spin correlation tensor.

Solving Eqs. (17) for the angle \( \chi \), we find

\[
\tan(2\chi) = \frac{2 \left[ r^i_1(0)T^j_j(0) - r^j_2(0)T^i_j(0) \right]}{-\left( r^i_1(0) \right)^2 + \left( T^j_j(0) \right)^2 - \left( r^j_2(0) \right)^2 + \left( T^i_j(0) \right)^2}, \quad (18)
\]

which specifies the initial orientation of the coherence vector \( r_1 \) with respect to the principal axis \( a \). Suppose now the two-spin system is initially in a product state. For this case it is easy to prove these corollaries to our principal result:

(1) The phase difference \( \chi \) is zero. This means that the initial positions of both coherence vectors lie on the \( a \) principal axis (as in Fig. 3(a)). It follows that the linear entropy of the state of each of the subsystems (defined by \( 1 - \text{Tr} \rho^2 \)) can only decrease, showing it is possible to increase the entanglement of the system with this interaction. (This will depend on initial conditions. See Section IV B.)
(2) The length of the semi-minor axis of the ellipse followed by subsystem 1 is given by
\[ b_1 = |r_2^j(0)|[|r_1^k(0)|^2 + (r_1^l(0))^2]^{1/2} \]
and likewise for subsystem 2 with \( 1 \to 2 \) and \( \{j, k, l\} \to \{i, n, m\} \). It follows that for an initially pure state and the assumed single interaction \( \sigma_i \otimes \sigma_j \), the maximum attainable entanglement is achieved at \( \phi \) values of \( \pi/2 \) and \( 3\pi/2 \).

For the case of a initial state that is not pure but still separable, the phase difference \( \chi \) does not vanish, in general (see figure 3(b)). Accordingly, the implied dynamical behavior of a classically correlated system distinguishes it from an uncorrelated system, but not from a system experiencing quantum entanglement. Moreover, the linear entropy of each subsystem can either increase or decrease, showing it is possible to increase or decrease the amount of entanglement in the system.

C. General nonlocal operations

From Proposition 1 of Ref. 11, any nonlocal operation can be decomposed as a product of two local operations and an operation of the form

\[ A = \exp \left[ \frac{i}{2} (c_1 \sigma_x^1 \sigma_x^2 + c_2 \sigma_y^1 \sigma_y^2 + c_3 \sigma_z^1 \sigma_z^2) \right]. \]

The operators \( \{Y_i\} = \{i\sigma_x^i \sigma_x^2/2, i\sigma_y^i \sigma_y^2/2, i\sigma_z^i \sigma_z^2/2\} \) span a maximal Abelian subalgebra of \( P \), and the relations

\[ [Y_i, Y_j] = 0, \quad [Y_i, Y_j]^+ = -i|\epsilon_{ijk}|Y_k - \frac{1}{2} \delta_{ij} \]

hold, where \([·, ·]^+\) denotes the anticommutator. Consequently, \( A \) of Eq. (19) can be written in product form,

\[ A = \exp \left[ \frac{i}{2} (c_1 \sigma_x^1 \sigma_x^2) \right] \exp \left[ \frac{i}{2} (c_2 \sigma_y^1 \sigma_y^2) \right] \exp \left[ \frac{i}{2} (c_3 \sigma_z^1 \sigma_z^2) \right]. \]
FIG. 3: The initial position of the vector of coherence of subsystem 1 or 2 is shown, together with its time evolution under a one-dimensional nonlocal interaction (dashed line). If the initial state of the two-spin system is a product state, then the initial position is on the principal axis of the elliptical path, as in (a). In general this agreement no longer occurs if the subsystems are initially correlated, either classically or quantum mechanically, as in (b).

The property (21) tells us that any nonlocal operation can be decomposed into a sequence of operations effecting a succession of circular and elliptic paths in Bloch space. This result facilitates the calculation of the final states of the subsystems, but gives only limited insight into the geometric characteristics of the coherence vectors’ time orbits. For all $c_i$ distinct, two general observations can be made:

1. The motion of the vectors of coherence is no longer restricted to a plane, since there is no linear combination of $\{\sigma_x \otimes 1, \sigma_y \otimes 1, \sigma_z \otimes 1\}$ or $\{1 \otimes \sigma_x, 1 \otimes \sigma_y, 1 \otimes \sigma_z\}$ that is invariant under the action of $A$.

2. Characteristics of the trajectories such as periodicity depend in detail on the parameters $c_1, c_2$, and $c_3$. A trajectory is periodic only if $c_2/c_1$ and $c_3/c_1$ are both rational numbers. We note also that the set of parameters $\{c_1, c_2, c_3\}$ has been used to determine the equivalence classes of nonlocal interactions [10] as well as the invariants of the nonlocal interactions [12].

D. Special case of the Heisenberg Hamiltonian

The Heisenberg exchange Hamiltonian, corresponding to $c_1 = c_2 = c_3 = -c/2$, is not included in our general observations on nonlocal interactions (made for all $c_i$ distinct), but like the one-dimensional Hamiltonians, it admits a simple geometric picture. This interaction is the primary two-qubit interaction in several experimental proposals for quantum-dot qubits [13, 14, 15]. It can also be used for universal quantum computing on encoded qubits of several types [16, 17, 18, 19, 20]. For these reasons, it warrants special attention.

Introducing the time parameter $\phi$, the operator $A$ of Eq. (21) now takes the form

$$A(\phi) = \exp[-i(c\phi/2)(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)]$$

$$= \left[\cos^3(c\phi/2) - i\sin^3(c\phi/2)\right] I \otimes I$$

$$- (i/2)e^{ic\phi/2}\sin(c\phi)(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z).$$

The time development of the density matrix under the operator $A$ is given $\rho(\phi) = A(\phi)\rho(0)A^\dagger(\phi)$ and the corresponding coherence vectors change according to

$$r^i_1(\phi) = \frac{1}{2}[r^i_1(0) + r^j_2(0) + (r^i_1(0) - r^j_2(0))\cos(2c\phi) + (T^j_k(0) - T^k_j(0))\sin(2c\phi)],$$

$$r^i_2(\phi) = \frac{1}{2}[r^i_1(0) + r^j_2(0) + (r^i_1(0) - r^j_2(0))\sin(2c\phi) + (r^j_k(0) - r^k_j(0))\cos(2c\phi)].$$

where $i, j, k = 1, 2, 3$ and cyclic permutations are implied. Similarly, for the coherence tensor we have

$$T^i_j(\phi) = \frac{1}{2}[T^i_j(0) + T^j_i(0) + (T^i_j(0) - T^j_i(0))\cos(2c\phi) + (r^j_k(0) - r^k_j(0))\sin(2c\phi)].$$
The elements of \( \mathbf{r}_2(\phi) \) are found by symmetry 1 \( \leftrightarrow \) 2. The quantities \( r_1^i + r_2^i, T_1^j + T_2^j, \) and \( T_1^i \) are unchanged by the operation, and the form of the one-parameter set that describes the time-evolving coherence vector is

\[
\mathbf{r}_1(\phi) = \mathbf{R} + \mathbf{S}\cos(2c\phi) + \mathbf{V}\sin(2c\phi),
\]

where \( \mathbf{R} = r_1(0) + r_2(0), \mathbf{S} = r_1(0) - r_2(0), \) and

\[
\mathbf{V} = \begin{pmatrix}
T_2(0) - T_2^*(0) \\
T_1(0) - T_1^*(0) \\
T_2^*(0) - T_2(0)
\end{pmatrix}.
\]

Clearly the vector traces out an ellipse lying in the plane spanned by \( \mathbf{S} \) and \( \mathbf{V} \), defined by \( \mathbf{S} \times \mathbf{V} \), and passing through the point \( \mathbf{R} \).

IV. APPLICATIONS

We shall now illustrate some of the results of Section III with two examples. The first provides a controllability result for nonlocal unitary interactions and the second demonstrates how the orbit of the coherence vector can be used to describe the entangling power of the Heisenberg exchange interaction.

A. Quantum control via quantum controllers and one-dimensional nonlocal interactions

Let us now consider the implications of the findings of the preceding sections for the problem of quantum control. To this end, we adopt the nomenclature of Lloyd [8] and identify spin 1 with the system \( Q \) wish to control, and spin 2 with the quantum controller or interface \( I \). It has been established in Sec. III that when the interaction Hamiltonian \( H_I \) is in play between \( S \) and \( Q \) and \( (i) \) system \( Q \) is completely controllable via control Hamiltonians \( \{H_Q^m\} = \{1 \otimes \sigma_x, 1 \otimes \sigma_y, 1 \otimes \sigma_z\} \) that span the \( su(2) \) algebra. The initial state of the bipartite system is taken to be a product state in the ensuing analysis.

Suppose that the interaction Hamiltonian is nonlocal, but takes the one-dimensional form \( H_I = \sigma_i \otimes \sigma_j \). Then the set \( \{H_Q^m, H_I\} = \{1 \otimes \sigma_x, 1 \otimes \sigma_y, 1 \otimes \sigma_z, \sigma_i \otimes \sigma_x, \sigma_i \otimes \sigma_y, \sigma_i \otimes \sigma_z\} \) comprises a closed six-element subalgebra \( G_6 \) of \( G \). Given this set of operations, the vector of coherence \( \mathbf{r}_S \) of system \( S \) remains in the plane perpendicular to the \( i \)-axis.

It has been established in Sec. III that when \( H_I = \sigma_i \otimes \sigma_j \) is the only element of the algebra \( su(4) \) affecting the two-spin system, the vectors of coherence \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) are constrained to move in elliptical orbits. Now, with the six-element subalgebra \( G_6 \) available to the two-spin system \( S + Q \), the reachable set of the system \( S \) is enlarged to an elliptical disk (see Fig. 4). The principle axis of the disk coincides with the initial coherence vector \( \mathbf{r}_S(0) \) of the \( S \) system, while the length of its semiminor axis is given by \( b = [(r_2^S(0))^2 + (r_2^I(0))^2]^{1/2} |r_Q(0)| \), where \( r_Q(0) \) is the initial coherence vector of system \( Q \).

Proof. First, if one implements the two-step sequence of a local operation \( \in 1 \otimes su(2) \) on system \( Q \) followed by the nonlocal operation \( H_I = \sigma_i \otimes \sigma_j \) on \( S + Q \), the orbit of \( \mathbf{r}_S \) is necessarily an ellipse whose principle axis lies along the initial coherence vector \( \mathbf{r}_S(0) \) and whose semimajor axis \( b \) is restricted by \( 0 \leq b \leq [(r_2^S(0))^2 + (r_2^I(0))^2]^{1/2} |r_Q(0)| \).

Hence the set reachable by this two-step procedure is the elliptic disk in question. Second, using the Baker-Campbell-Hausdorff formula one can show that all the elements of the six-element subalgebra \( G_6 \) can be constructed by this two step sequence, so their reachable sets are the same.

From this result we infer that the entropy of system \( S \) cannot be decreased by intervention of the quantum interface \( Q \) if the interaction Hamiltonian is limited to the form \( H_I = \sigma_i \otimes \sigma_j \). Noting that \( |r_Q| \leq 1/2 \), it follows that \( a \geq b \), where \( a \) and \( b \) are respectively the magnitudes of the semimajor and semiminor axes of the elliptical reachable set. Furthermore, it is seen that the systems \( S \) and \( Q \) become maximally entangled if the initial state of the system \( S \) is situated on the equatorial plane perpendicular to \( i \)-axis.

B. Entanglement power of Heisenberg interaction

Upon examining Eq. (22), we see that the maximum entanglement, realized in a maximally entangled pure state, can be achieved if \( r_1(0) = -r_2(0), |r_1| = 1/2, \) and \( c\phi = \pm \pi/4 \). Otherwise, the state is not perfectly entangled since the linear entropy \( 1-\text{Tr}(\rho^2) \) is not minimized. This conclusion agrees with the result of Zhang et al. [10] that the only perfect entanglers that can be achieved with the Heisenberg Hamiltonian are the square-root of swap and its inverse.
FIG. 4: The gray area is the set of reachable states for the system $S$ if one has full control of the controller $Q$ and the interaction $\sigma_i \otimes \sigma_j$ is available. This elliptical disk is characterized by a semimajor axis coincident with the initial vector of coherence for $S$ and a semiminor axis with $b = [(r_{S}^k(0))^2 + (r_{S}^j(0))^2]^{1/2}|r_Q(0)|$.

However, suppose that the initial state of the two-spin system is represented by

$$\rho(0) = \frac{1}{4}(I + \sigma_z) \otimes (I + \sigma_z),$$

which is a pure-state density matrix for which $r_1^z = 1/2 = r_2^z$ and $T_z^z = 1/2$, all other elements of the coherence vectors and coherence tensor being zero. Then

$$r_1^z(\phi) = r_1^z(0) \cos^2(c\phi) + r_2^z(0) \sin^2(c\phi),$$

while all other components of $r_1$ and $r_2$ vanish at time $\phi$, and all other $r_\alpha^z(0) = 0$. In this case the ellipse collapses to a line and the coherence vector simply oscillates between two values along that line. The only element of the correlation tensor that changes is

$$T_2^1 = \frac{1}{2} \sin(2c\phi)(r_1^z(0) - r_2^z(0)),$$

which vanishes for an initial tensor product of pure states for which the subsystems are polarized in the $+z$ direction. Therefore one cannot create maximally entangled states with these initial conditions.

V. CONCLUSIONS

In this paper we have developed a geometric representation for the orbits of the coherence vectors of a two-qubit system. In various circumstances we have shown that their evolution is described by elliptical orbits lying within the surface of the Bloch sphere. Importantly, every two-qubit unitary operation can be expressed as a combination of one of the evolutions we have considered, together with “pre” and “post” local single-qubit rotations. We anticipate that this geometric picture will be helpful in devising schemes for control of a quantum state via quantum interfaces, and we have obtained a controllability result appropriate for such applications. Given the utility of the coherence-vector picture for modeling quantum systems and describing their entanglement, further studies along similar lines may be fruitful. Such work could include analysis of the orbits of higher-dimensional quantum states, as well as consideration of the effects of measurement operations on controllability.
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