Non-Boltzmann Statistics as an Alternative to Holography

Marcelo Botta Cantcheff†,1 and Jose A.C. Nogales†,‡,2

†Centro Brasileiro de Pesquisas Físicas (CBPF),
Departamento de Teoria de Campos e Partículas (DCP),
Rua Dr. Xavier Sigaud 150, 22290-180, Rio de Janeiro, RJ, Brasil.
‡Instituto de Física, Facultad de Ciencias Puras y Naturales, Calle 27 Campus
Universitario, Universidad Mayor de San Andrés, Casilla 8635, La Paz, Bolivia

Abstract

An intriguing question related to black hole thermodynamics is that the entropy of a region shall scale as the area rather than the volume. In this essay we propose that the microscopical degrees of freedom contained in a given region of space, are statistically related in such a way that obey a non-standard statistics, in which case an holographic hypothesis would be not needed. This could provide us with some insight about the nature of degrees of freedom of the geometry and/or the way in which gravitation plays a role in the statistic correlation between the degrees of freedom of a system.
We start this note by making the following crucial observation: when gravitational interaction of a given system is considered, it is impossible to isolate two subsystems (unless they are infinitely separated)\(^3\), so, to define statistical independence of two subsystems is actually problematic in this context. On the other hand, it is known that for systems whose microscopical components cannot be considered statistically independent (and in particular, those being highly correlated), it shall be considered a non-standard statistics (non-additive or non-extensive)[1]. Consequently, we can expect a deviation of the standard statistics when gravity is considered an important part of a system.

In the last two decades, generalizations of the Boltzmann-Gibbs thermodynamics have been extensively explored [2].

In an ideal system in thermodynamic equilibrium, if the intensive variables are kept fixed and the side of the system is doubled, the extensive variables should then also double. This is not true for black holes [3, 4, 5, 6].

From the thermodynamic point of view, we mean macroscopic [7], the entropy of black holes is proportional to the area of the event horizon

\[ S \propto A. \]

We are going to analyze the statistical behavior of the degrees of freedom that compose this type of systems, in order to have consistency with the relation above.

Let us explain more carefully the difference between additivity and extensivity [8]. Consider a physical quantity \( W(i) \) related with the subsystem \( i \). For \( N \) of such

\(^3\)In contrast with ordinary thermodynamic systems.
subsystems, \( W \) is additive if we have:

\[
W(N) = \sum_{i=1}^{N} W(i),
\]

(1)

where \( W(N) \equiv W\left(\{i\}_{1}^{N}\right) \). Supposing that all subsystems are equal,

\[
W(N) = NW(1).
\]

(2)

In its turn, extensivity is defined by:

\[
\lim_{N \to \infty} \frac{|W(N)|}{N} < \infty.
\]

(3)

The physical quantities that do not fulfill this property are called non-extensive. We note that all quantities which are additive are also extensive, in fact

\[
\lim_{N \to \infty} \frac{|W(N)|}{N} = |\bar{W}|,
\]

(4)

where the mean value, \(|\bar{W}|\), is finite since \( W(i) < \infty \ \forall i \). In particular if the subsystems are equal, we get \( \lim_{N \to \infty} \frac{|W(N)|}{N} = W(1) \).

1 Blak Hole thermodynamics and the holographic interpretation

Roughly speaking, the argument which leads to the holographic principle is that if one assumes an uniform distribution of the gravitational degrees of freedom in a spatial region of volume \( V \), or in other words, if, as suggested by quantum gravity, the space-time is thought as filled of quantum geometric atoms like wormholes or more complex structures of Planck size \( \ell_{p} \) more or less uniformly distributed, then the number of
microstates is \( \propto M^{V} \), where \( M \) is the number of states reachable by each microscopical component. Then, following the Boltzmann (extensive) statistics, one obtains that entropy \( S \) must scale as the volume \( V \) which would lead to a contradiction with the Bekenstein-Hawking law for the black hole entropy, \( S = A/4 \) in Planck units [3, 5]. So, the holographic principle suggested by tHooft [9] essentially claims that all these degrees of freedom may be mapped to the boundary of that region, and consequently the corresponding entropy would scale as the area.

As we can notice, an important part of this argument is the extensive Boltzmann-Gibbs statistics, and our proposal is precisely based on alternatives to this, in order to match consistency with the area law and to render the holographic viewpoint not necessary \(^4\).

\section{2 Non-standard statistics and consistency with the area law.}

Now, we keep the assumption of uniformity in the distribution of the physical degrees of freedom in a spatial region of volume \( V \), and drop out the holographic point of view about the possibility of mapping them to the boundary.

According to this, if we suppose \( N \) identical degrees of freedom uniformly distributed in \( V \), we have \( N \propto V \). Following the area law, we must have \( S(N) \propto A \propto N^{2/3} \). Thus, our main result is:

\[ S(N) \sim N^{2/3} S(1) \], \hspace{1cm} (5)

\(^4\)In this approach, we are not worried with the aspects concerning the information paradox which, in principle, could be avoided by the holographic principle.
which differs from (2). Substituting this into (3) we get
\[
\lim_{N \to \infty} \frac{|S(N)|}{N} = 0 ,
\]
which shows the extensivity of the entropy.

Our conclusion is that the statistics that must be considered in a system where the gravitation plays an important role (as a Black Hole) is non-standard, in particular the entropy is a non-additive but extensive quantity. The law which rules the non additivity is given by (5).

3 Correlated Systems and Tsallis Statistics.

Finally, we are going to discuss here a possible realization of the framework described above. As discussed in [1], for certain systems where the hypothesis of probabilistic independence is not applicable, the number of states is
\[
\Gamma \sim N^\gamma \quad (\gamma > 0 , \; N \gg 1).
\]
On the other hand, the corresponding expression for the Tsallis entropy is
\[
S = k \frac{\Gamma^{1-q} - 1}{1-q} ,
\]
where \( q < 1 \) is a dimensionless parameter.

If the degrees of freedom, geometric or not, are correlated by gravitational interaction such that (7) holds, we obtain that
\[
S \sim N^{(1-q)\gamma} \; (N \gg 1) .
\]
Thus, by requiring consistency with (5), we obtain:
\[
(1-q)\gamma = 2/3.
\]
As a particular example, if we have a sort of Boolean system where each (distinguishable) degree of freedom may assume the states \( s_i = 0, 1 \), \( i = 1, \ldots, N \) [9], and the constraint

\[
\sum_{i=1}^{N} s_i = 1
\]

is also assumed, then, the possible states of the system are given by \( s_i = \delta_{ik} \), for some \( k \) (\( 1 \leq k \leq N \)). Then, we have \( \Gamma = N \). Substituting this into (10), as a result one has: \( \gamma \sim 1 \Rightarrow q = 1/3 \).

We conclude this work by stressing that also the problems associated with the black hole thermodynamical instability could be by-passed by considering certain non-standard statistics [10].

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References


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