New Axisymmetric Stationary Solutions of Five-dimensional Vacuum Einstein Equations with Asymptotic Flatness

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New axisymmetric stationary solutions of the vacuum Einstein equations in five-dimensional asymptotically flat spacetimes are obtained by using solitonic solution-generating techniques. The new solutions are shown to be equivalent to the four-dimensional multi-solitonic solutions derived from particular class of four-dimensional Weyl solutions and to include different black rings from those obtained by Emparan and Reall.

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Inspired by the new picture of our Universe including brane world models and the prediction concerning the production of higher-dimensional black holes in future colliders [1], the studies of the spacetime structures in higher-dimensional General Relativity revealing the rich structure have been performed recently with great intensity. For example, some qualitative features concerning the topology of event horizons were clarified [2]. Also several exact solutions involving black holes were obtained and the richness of the phase structure of black holes have been discussed. Particularly in the five-dimensional case, several researchers have tried to search new exact solutions since the remarkable discovery of a rotating black ring solution by Emparan and Reall [3]. For example, the supersymmetric black rings [4] and the black ring solutions under the influence of external fields [5] are found.

Despite these discoveries of black ring solutions, a systematic way of constructing new solutions in higher dimensions has not been fully developed as for the four-dimensional case, particularly for the non-supersymmetric spacetimes with asymptotic flatness. In the case of four dimensions solution-generating techniques were greatly developed and applied to construct new series of axisymmetric stationary solutions extensively [6]. The solutions corresponding to asymptotically flat spacetimes including the famous multi-Kerr solutions by Kramer and Neugebauer [7] were derived systematically, motivated by the discovery of Tomimatsu-Sato solutions [8].

In this letter, as a first step towards systematic construction of new solutions in higher dimensions and general understanding of the rich structure of higher-dimensional black objects, solution-generating techniques similar to those developed in the four-dimensional case are applied to five-dimensional General Relativity. (See Ref. [9] for the Kaluza-Klein compactification.)

We consider the spacetimes which satisfy the following conditions: (c1) five dimensions, (c2) asymptotically flat spacetimes, (c3) the solutions of vacuum Einstein equations, (c4) having three commuting Killing vectors including time translational invariance and (c5) having a single non-zero angular momentum component. Note that, in general, there can be two axes of rotation in the five-dimensional spacetime. Under these conditions, we show that five-dimensional solitonic solution-generating problems can be regarded as some four-dimensional problems. This means that we can use the knowledge obtained in the four-dimensional case. Then we can generate new solutions from seed solutions which correspond to known five-dimensional spacetimes. Here, for simplicity, we adopt the five-dimensional Minkowski spacetime as a seed solution. As a result, we obtain a new series of solutions which correspond to five-dimensional asymptotically flat spacetimes. Although the spacetimes found here have singular objects like closed timelike curves (CTC) and naked curvature singularities in general, we can see that a part of these solutions is a new class of black ring solutions whose rotational axes are different from those of Emparan and Reall’s [3].

Under the conditions (c1) – (c5), we can employ the following Weyl-Papapetrou metric form (for example, see the treatment in [10]),

\[
d s^2 = -e^{2U_0}(d\rho^0 - \omega d\phi)^2 + e^{2U_1}\rho^2( d\phi)^2 + e^{2U_2}( d\psi)^2 + e^{2(\gamma + U_1)}(d\rho^2 + dz^2),
\]

(1)

where \(U_0, U_1, U_2, \omega\) and \(\gamma\) are functions of \(\rho\) and \(z\). Then we introduce new functions \(S := 2U_0 + U_2\) and \(T := U_2\) so that the metric form (1) is rewritten into

\[
d s^2 = e^{-T} \left[ -e^S( d\rho^0 - \omega d\phi)^2 + e^{T+2U_1}\rho^2( d\phi)^2 + e^{2(\gamma + U_1)} + e^{2T}( d\psi)^2 \right] + e^{2T}( d\psi)^2.
\]

(2)

Using this metric form the Einstein equations are reduced
to the following set of equations,

(i) \[ \nabla^2 T = 0, \]

(ii) \[ \begin{align*}
\partial_\rho \gamma_T &= \frac{3}{4} \rho \left[ (\partial_\rho T)^2 - (\partial_2 T)^2 \right] \\
\partial_2 \gamma_T &= \frac{3}{2} \rho \left[ \partial_\rho T \partial_2 T \right],
\end{align*} \]

(iii) \[ \nabla^2 \mathcal{E}_S = \frac{2}{\mathcal{E}_S + \mathcal{E}_S} \nabla \mathcal{E}_S \cdot \nabla \mathcal{E}_S, \]

(iv) \[ \begin{align*}
\partial_\rho \gamma_S &= \frac{\rho}{2(\mathcal{E}_S + \mathcal{E}_S)} \left( \partial_\rho \mathcal{E}_S \partial_\rho \mathcal{E}_S - \partial_\rho \mathcal{E}_S \partial_2 \mathcal{E}_S \right) \\
\partial_2 \gamma_S &= \frac{\rho}{2(\mathcal{E}_S + \mathcal{E}_S)} \left( \partial_2 \mathcal{E}_S \partial_\rho \mathcal{E}_S + \partial_2 \mathcal{E}_S \partial_2 \mathcal{E}_S \right),
\end{align*} \]

(v) \[ (\partial_\rho \Phi, \partial_2 \Phi) = \rho^{-1} e^{2S} (-\partial_2 \omega, \partial_\rho \omega), \]

(vi) \[ \gamma = S + \gamma_T, \]

(vii) \[ U_1 = -\frac{S + T}{2}, \]

where \( \Phi \) is defined through the equation (v) and the function \( \mathcal{E}_S \) is defined by \( \mathcal{E}_S := e^{S} + i \Phi \). The equation (iii) is exactly the same as the Ernst equation in four dimensions [10], so that we can call \( \mathcal{E}_S \) the Ernst potential. The most non-trivial task to obtain new metrics is to solve the equation (iii) because of its non-linearity. To overcome this difficulty we can however use the methods already established in the four-dimensional case. Here we use the method similar to the Neugebauer’s Bäcklund transformation [12] or the HKX transformation [13], whose essential idea is that new solutions are generated by adding solitons to seed spacetimes. The applicability of this method to the five-dimensional problem is recognized by the following. The part in the bracket of Eq. (2) corresponds to a metric of a four-dimensional axisymmetric spacetime with a “dilaton field” \( T \), where the function \( T \) is a solution of the Laplace equation (1). Then the four-dimensional part is determined by a solution of the Ernst equation (iii).

For the actual analysis in the following, we follow the procedure given by Castejon-Amendo and Manko [14], in which they discussed a deformation of a Kerr black hole under the influence of some external gravitational fields. When a static seed solution \( e^{S(0)} \) for (iii) is obtained, a new Ernst potential can be written in the form

\[ \mathcal{E}_S = e^{S(0)} \frac{x(1 + ab) + iy(b - a) - (1 - ia)(1 - ib)}{x(1 + ab) + iy(b - a) + (1 - ia)(1 - ib)}, \]

where \( x \) and \( y \) are the prolate spheroidal coordinates: \( \rho = \sigma \sqrt{x^2 - 1} \sqrt{1 - y^2} \), \( z = \sigma xy \), and the functions \( a \) and \( b \) satisfy the following simple first-order differential equations

\[ (x - y)\partial_x a = a \left[ (xy - 1)\partial_x S(0) + (1 - y^2)\partial_y S(0) \right], \]

\[ (x - y)\partial_y a = a \left[ -(x^2 - 1)\partial_x S(0) + (xy - 1)\partial_y S(0) \right] \]

\[ (x + y)\partial_x b = -b \left[ (xy + 1)\partial_x S(0) + (1 - y^2)\partial_y S(0) \right], \]

\[ (x + y)\partial_y b = -b \left[ -(x^2 - 1)\partial_x S(0) + (xy + 1)\partial_y S(0) \right]. \]

The corresponding expressions for the metric functions can be obtained by using the formulas shown by [14]. Here we adopt the following metric form of the five-dimensional Minkowski spacetime as a seed solution,

\[ ds^2 = -\left( dx^0 \right)^2 + \left( \sqrt{\rho^2 + (z + \lambda \sigma)^2} - (z + \lambda \sigma) \right) d\phi^2 + \left( \sqrt{\rho^2 + (z + \lambda \sigma)^2} + (z + \lambda \sigma) \right) d\psi^2 + \frac{1}{2} \rho^2 (d\rho^2 + dz^2), \]

where \( \lambda \) and \( \sigma \) are arbitrary real constants. Introducing the new coordinates \( r \) and \( \chi \):

\[ \rho = r \chi, \quad z = \frac{1}{2} (\chi^2 - r^2) - \lambda \sigma, \]

the above metric (4) can be transformed into a simple form

\[ ds^2 = -(dx^0)^2 + (dr^2 + r^2 d\phi^2) + (d\chi^2 + \chi^2 d\psi^2). \]

From Eq. (4), we can derive the seed functions

\[ S(0) = T(0) = \frac{1}{2} \ln \left[ \sqrt{\rho^2 + (z + \lambda \sigma)^2} + (z + \lambda \sigma) \right]. \]

For the seed function (5) we obtain the solutions of the differential equations (3) as

\[ a = \alpha \frac{(x - y + 1 + \lambda) + \sqrt{x^2 + y^2 + 2\lambda xy + (\lambda^2 - 1)}}{2 \left( (xy + \lambda) + \sqrt{x^2 + y^2 + 2\lambda xy + (\lambda^2 - 1)} \right)^{1/2}}, \]

\[ b = \beta \frac{(x + y - 1 + \lambda) + \sqrt{x^2 + y^2 + 2\lambda xy + (\lambda^2 - 1)}}{2 \left( (xy + \lambda) + \sqrt{x^2 + y^2 + 2\lambda xy + (\lambda^2 - 1)} \right)^{1/2}}, \]

where \( \alpha \) and \( \beta \) are integration constants.

The explicit expression for the corresponding metric is

\[ ds^2 = -\frac{A}{B} \left[ dx^0 - \left( 2\sigma e^{-S(0)} C + C_1 \right) d\phi \right]^2 + \frac{B}{A} e^{-S(0) - T(0)} \rho^2 (d\phi^2) + e^{2T(0)} (d\psi)^2 + C_2 B \left( \frac{x - 1}{x + 1} \frac{Y_{\gamma, -\lambda \sigma}}{Y_{\gamma, -\lambda \sigma} \left[ (z^2 + \beta^2 + 2\lambda xy + (\lambda^2 - 1)) \right]^{1/2}} \right) \times \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right). \]
where \( Y_{\pm \sigma, -\lambda \sigma} \) are given by

\[
Y_{\pm \sigma, -\lambda \sigma} = \sigma^2 \left[ (x \mp y) \sqrt{x^2 + y^2 + 2\lambda xy + (\lambda^2 - 1)} + x^2 + y^2 + (\lambda \mp 1)xy \mp (\lambda \pm 1) \right],
\]

and \( A, B \) and \( C \) are defined with \( a \) and \( b \) as

\[
A := (x^2 - 1)(1 + ab)^2 - (1 - y^2)(b - a)^2, \\
B := [(x + 1) + (x - 1)ab]^2 + [(1 + y)a + (1 - y)b]^2, \\
C := (x^2 - 1)(1 + ab)[b - a - y(a + b)] + (1 - y^2)(b - a)[1 + ab + x(1 - ab)].
\]

In the following, the constants \( C_1 \) and \( C_2 \) are fixed as

\[
C_1 = \frac{2\sigma^{1/2}}{1 + \alpha \beta}, \quad C_2 = \frac{\sigma}{2(1 + \alpha \beta)^2},
\]

to assure that the spacetime should asymptotically approach the Minkowski spacetime globally. From the metric (6), we can easily see that the sequence of new solutions has four independent parameters: \( \lambda, \sigma, \alpha \) and \( \beta \).

When \( \lambda = 1 \) and \( \beta = 0 \), the metric is reduced to the form found by Myers and Perry [15] which describes a one-rotational spherical black hole in five dimensions. In fact the metric has the following expression,

\[
ds^2 = \frac{p^2 x + q^2 y - 1}{p^2 x + q^2 y + 1} \left( dx^0 - 2\sigma^{1/2} \frac{q}{p} \frac{1 - y}{p^2 x + q^2 y - 1} d\phi \right)^2 + \sigma \frac{p^2 x + q^2 y + 1}{p^2 x + q^2 y - 1} (x - 1)(1 - y) d\phi^2 + \sigma (x + 1)(1 + y) dy^2 + \sigma \frac{p^2 x + q^2 y + 1}{2p^2} \left[ \frac{dx^2}{x^2 - 1} + \frac{dy^2}{y^2 - 1} \right],
\]

where \( p^2 = 1/(\alpha^2 + 1) \) and \( q^2 = \alpha^2/(\alpha^2 + 1) \). Introduce new parameters \( a_0 \) and \( m \), and new coordinates \( \tilde{r} \) and \( \theta \) through the relations,

\[
p^2 = \frac{4\sigma}{m^2}, \quad q^2 = \frac{a_0^2}{m^2}, \quad x = \frac{\tilde{r}^2}{2\sigma} - \lambda, \quad y = \cos 2\theta,
\]

so the metric (7) is transformed into

\[
ds^2 = -(1 - \Delta) \left[ dx^0 - \frac{a_0 \Delta \sin^2 \theta}{1 - \Delta} d\phi \right]^2 + \frac{1}{1 - \Delta} \left[ \tilde{r}^2 + (m^2 - a_0^2) \sin^2 \theta d\phi^2 + \tilde{r}^2 \cos^2 \theta d\psi^2 + (\tilde{r}^2 + a_0^2 \cos^2 \theta) \left( d\theta^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - (m^2 - a_0^2)} \right) \right],
\]

where \( \Delta := m^2/(\tilde{r}^2 + a_0^2 \cos^2 \theta) \). The line-element (8) is exactly the same form found by Myers and Perry.

We can show that the spacetimes are asymptotically flat. The asymptotic form of \( E_S \) near the infinity \( \tilde{r} = \infty \) becomes

\[
E_S = \tilde{r} \cos \theta \left[ 1 - \frac{\sigma}{\tilde{r}^2} \frac{P(\alpha, \beta, \lambda)}{(1 + \alpha \beta)^2} + \cdots \right] + 2i \sigma^{1/2} \left[ \frac{\alpha}{1 + \alpha \beta} - \frac{2\sigma \cos^2 \theta Q(\alpha, \beta, \lambda)}{\tilde{r}^2 (1 + \alpha \beta)^2} + \cdots \right], 
\]

where

\[
P(\alpha, \beta, \lambda) = 4(1 + \alpha^2 - \alpha^2 \beta^2) \\
Q(\alpha, \beta, \lambda) = \alpha(2\alpha^2 - \lambda + 3) - 2\alpha^2 \beta^3 \\
- \beta [2(2\alpha + 1)(\alpha^2 + 1) + (\lambda - 1)\alpha^2(\alpha + 2)].
\]

This means that, even if \( \lambda \neq 1 \) or \( \beta \neq 0 \), the asymptotic behavior of the spacetime is the same as the one-rotational spherical black hole. That is, the general spacetimes derived here keep the asymptotic flatness.

From the asymptotic behavior, we can compute the mass parameter \( m^2 \) and rotational parameter \( m^2 a_0 \):

\[
m^2 = \sigma \frac{P(\alpha, \beta, \lambda)}{(1 + \alpha \beta)^2}, \quad m^2 a_0 = 4\sigma^{3/2} Q(\alpha, \beta, \lambda) (1 + \alpha \beta)^3.
\]

The spacetimes generally have some local gravitational objects which one may regard as black holes. Figure 1 shows the schematic diagram of the local gravitational object in the case: \( \lambda > 1 \). At a glance it seems to be a black ring. As naturally expected from the presence of the rotation, the new rings have ergo-regions as depicted in FIG.2.

The following special combination of parameters,

\[
\beta = -\frac{2 + \alpha^2(\lambda + 1) - \sqrt{\alpha^4(\lambda + 1)^2 - 4\alpha^2(\lambda - 3) + 4}}{4\alpha},
\]

makes the singular structure of the spacetimes fairly mild. For example, looking at the behavior of the \( \phi - \phi \) component of the metric, the CTC-region which generally appears near the horizon seems to disappear completely as seen in FIG.3.

Even for this case, there exists a kind of strut structure in this spacetime. The reason for this is that the effect
of rotation cannot compensate for the gravitational attractive force. The periods of the coordinates \( \psi \) and \( \phi \) should be defined as

\[
\Delta \psi = 2\pi \lim_{\chi \to 0} \sqrt{\frac{\chi^2 g_{\chi\chi}}{g_{\psi\psi}}}
\]

and

\[
\Delta \phi = 2\pi \lim_{r \to 0} \sqrt{\frac{r^2 g_{rr}}{g_{\phi\phi}}}
\]

to avoid a conical singularity. Both \( \Delta \psi \) and \( \Delta \phi \) for \( y = 1 \), i.e. outside the ring, are \( 2\pi \). While the period of \( \phi \) inside the ring becomes

\[
\Delta \phi = 2\pi \frac{\lambda - 1 + (\lambda + 1)\alpha\beta}{\sqrt{\lambda^2 - 1(1 + \alpha\beta)}}
\]

which is less than \( 2\pi \) for \( 1 < \lambda < \infty \) with real \( \alpha \). Hence, two-dimensional disk-like struts, which appear in the case of static rings [16], are needed to prevent the collapse of the rings.

In some limiting cases with the relation (11), the corresponding solutions are reduced to the well-known solutions like the static black rings or the rotational black strings corresponding to (four-dimensional Kerr spacetime) \( \times R \). The former case is realized when we take the limit \( \alpha \to 0 \) and the latter is realized when the parameter \( \lambda \) goes to infinity under the condition: \( \alpha = \tilde{\alpha} \times \sqrt{2/\lambda} \) with \(-1 < \tilde{\alpha} < 1\).

Finally we comment on the four independent parameters. The parameters \( \lambda \) and \( \sigma \) characterize the size and mass of the local object which resides in the spacetime. Appropriate combinations of \( \alpha \) and \( \beta \) can be considered as the Kerr-parameter and the NUT-parameter in four-dimensional case.

In this letter, we generated the new axisymmetric stationary solutions of five-dimensional vacuum Einstein equations from the five-dimensional Minkowski spacetime as the simplest seed spacetime. In particular we found a candidate of another branch of one rotational “black rings”. More detailed and systematic analysis of the new solutions will be presented [17].

In the method presented here we can also adopt other seed spacetimes, so that we can find some new spacetimes. However it should be noticed that the method introduced here can not be used for the solution-generation of two rotational black rings because of the metric form (1). For this purpose other methods may be used. One of the most powerful methods would be the inverse scattering method [18], which was applied to a five-dimensional string theory system [19] and static five-dimensional cases [20].

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