Gravitational radiation reaction and inspiral waveforms in the adiabatic limit

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We describe progress evolving an important limit of binary orbits in general relativity, that of a stellar mass compact object gradually spiraling into a much larger, massive black hole. These systems are of great interest for gravitational wave observations. We have developed tools to compute for the first time the radiated fluxes of energy and angular momentum, as well as instantaneous snapshot waveforms, for generic geodesic orbits. For special classes of orbits, we compute the orbital evolution and waveforms for the complete inspiral by imposing global conservation of energy and angular momentum. For fully generic orbits, inspirals and waveforms can be obtained by augmenting our approach with a prescription for the self force in the adiabatic limit derived by Mino. The resulting waveforms should be sufficiently accurate to be used in future gravitational-wave searches.

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The late dynamics of a merging compact binary remains one of the greatest challenges of general relativity (GR). GR doesn’t have a “two body” problem so much as it has a “one body problem” — essentially, an expansion in interaction potential $GM/rc^2$. PN theory works very well when the bodies’ separation $r$ is large. As the bodies come close, the expansion must be iterated to high order (though it may be possible to modify the expansion to improve its convergence [3]).

Strong field binaries can be modeled very accurately if one member is far more massive than the other. The spacecraft is then that of the larger body plus a perturbation due to the smaller body, with the mass ratio acting as an expansion parameter. This limit is not just of formal interest: binaries consisting of “small” compact bodies (white dwarfs, neutron stars, or black holes with mass $M \sim 1-100 M_\odot$) captured onto highly eccentric orbits of massive black holes ($M \sim 10^5-10^7 M_\odot$) are important targets for space-based gravitational-wave (GW) antennae, especially the LISA [4] mission. Such captures are estimated to occur with a rate density $\sim 10^{\pm1} \text{ Gpc}^{-3} \text{ yr}^{-1}$. Convolving with LISA’s planned sensitivity, one expects to measure dozens to thousands of events per year [5]. Precisely measuring GW phase as the smaller body spirals into the black hole (driven by GW back-reaction) and fitting to detailed models will determine system parameters with extraordinary accuracy. For example, the hole’s mass and spin should be determined to $\lesssim 0.1\%$ [6]. It should even be possible to “map” the black hole’s spacetime, testing whether it satisfies GR’s stringent requirements [7, 8].

Thanks to their extremal mass ratio, a formal prescription for modeling such systems now exists. At lowest order, the small body follows a geodesic orbit of the black hole [e.g., Ref. [3], Eqs. (33.32a)–(33.32d)]. This must be corrected by the small body’s interaction with its own spacetime distortion. The electromagnetic manifestation of this self force was given by Dirac [9], yielding the Abraham-Lorentz-Dirac equation of motion. Equations for a body’s curved spacetime self-interaction were worked out by Mino, Sasaki, and Tanaka [11] and by Quinn and Wald [12]; see Ref. [13] for an overview. Many researchers are now working to develop practical schemes to compute the self force in black hole spacetimes, which is quite a challenge.

Adiabatic radiation reaction (ARR). Astrophysical extreme-mass-ratio binaries allow a significant simplification for most of the inspiral, up to the last few orbits before the final plunge and merger. In this regime, the system evolves adiabatically: the radiation reaction time $T_{rad} \gg$ is much larger than the orbital time $T_{orb}$: $T_{orb}/T_{rad} \sim \mu/M \ll 1$. The inspiral can be approximated as a flow through a sequence of geodesic orbits. To compute the leading-order, adiabatic waveforms, it is necessary only to know the time-averaged rates of change of the three constants of motion: the energy $E$, axial angular momentum $L_z$, and Carter constant $Q$ [9, 10].

This approximation can be understood by expanding the small body’s self acceleration $\mathbf{a}$. Define $\varepsilon \equiv \mu/M$. We can then write $\mathbf{a} = \varepsilon [\mathbf{a}_0^{\text{diss}} + \mathbf{a}_0^{\text{cons}} + \varepsilon (\mathbf{a}_1^{\text{diss}} + \mathbf{a}_1^{\text{cons}}) + \ldots]$. Terms labeled “diss” describe dissipative aspects of the self force; they drive the inspiral. Those labeled “cons” are conservative, representing non-dissipative components that contribute to the inspiraling body’s inertia. Dissipative terms accumulate secularly; conservative pieces do not. Their observable impact can be seen in the following expression obtained from a two-time expansion [15, 16] for the azimuthal orbital phase $\Phi(t)$ (and correspondingly, the GW phase of each harmonic component of the waveform):

$$\Phi(t) = \frac{1}{\varepsilon} \left[ \Phi_0(t, \varepsilon t) + \varepsilon \Phi_1(t, \varepsilon t) + O(\varepsilon^2) \right].$$ (1)

The leading-order, adiabatic waveforms contain only the term $\Phi_0$, and omit the subleading term $\Phi_1$, which contributes a phase correction of order unity over the entire inspiral. The leading-order term is determined by $a_0^{\text{diss}}$. Because it does not accu-
mulate secularly, $a_{\text{con}}^0$ does not contribute at leading order; it (along with $a_{\text{dis}}^0$) contributes to the subleading term $\Phi_1(t)$.

Estimates based on post-Newtonian theory suggest that the leading-order, adiabatic waveforms will be sufficiently accurate to detect waves in LISA data, and to provide an initial estimate of source parameters \[17\]. The phase mismatch between $\Phi_0(t)$ and the true signal is compensated for by adjustments in model parameters, which introduces systematic error in the inferred parameters. Eliminating this systematic error will require “measurement templates” that accurately model $\Phi_1(t)$ — a far more difficult task.

In this paper, we describe recent progress in computing adiabatic waveforms, using (i) an approach called “Poor man’s radiation reaction” (PMRR) for special classes of orbits, and (ii) augmenting PMRR using recent results of Mino \[19\] for fully generic orbits.

**Poor man’s radiation reaction.** PMRR only requires knowledge of the energy and angular momentum of the binary radiates; imposing global conservation, we evolve those quantities, approximating inspiral as a flow through a sequence of orbits. PMRR cannot rigorously evolve all the constants $E, L_z, Q$ describing black hole orbits, except in special cases. It is simple to evolve $E$ and $L_z$ — one computes the rate at which $E$ and $L_z$ are carried to infinity \[20\] and absorbed by the hole’s event horizon \[21\], and imposes global conservation \[22\]: $E_{\text{orb}} + E_{\text{rad}} = 0$, $L_{z\text{orb}} + L_{z\text{rad}} = 0$. Carter’s constant $Q$ cannot be so evolved, since there is no notion of Q-flux carried by radiation and no associated global conservation law. Orbits with zero eccentricity or inclination are sufficiently constrained that $Q$ is fixed by $E$ and $L_z$ \[23\].

The general case admits no such constraints. Nevertheless we can produce inspirals accurate enough for data-analysis algorithm development using simple, crude approximations based on limiting cases. For example, forcing a certain inclination angle to be constant determines the inspiral using only $E$ and $L_z$ fluxes \[24\]. Such inspirals are very unlikely to be accurate enough for detection templates.

The central engine of PMRR is a complex function $\psi_4$ representing the small body’s perturbation to the black hole’s curvature. This function is the Weyl (vacuum) curvature tensor $C_{\text{abcd}}$ projected onto a set of null vectors that are very convenient for describing outgoing radiation. Far away, $\psi_4$ is simply related to the two GW polarizations:

$$\psi_4(r \to \infty) = \frac{1}{2} \frac{\partial^2}{\partial t^2}(h_+ - ih_\times).$$  \hspace{1cm} (2)

From this one finds the outgoing fluxes of $E$ and $L_z$ \[20\]. This $\psi_4$ also completely describes the radiation’s interaction with the black hole, and thus encodes the rate at which the large black hole absorbs $E$ and $L_z$ \[21\] \[23\].

Teukolsky derived a “master equation” for black hole perturbations \[24\], and showed it separates by expanding in Fourier modes and spheroidal harmonics:

$$\psi_4 = \frac{1}{(r - ia\cos \theta)^4} \int d\omega \sum_{lm} R_{lm\omega}(r) S_{lm\omega}(\theta) e^{i(m\phi - \omega t)}.$$  \hspace{1cm} (3)

(The parameter $a = |\mathcal{S}|/M$ is the black hole spin per unit mass.) It is not necessary to expand in modes: one can leave the Teukolsky equation as a PDE coupled in $t, r, \theta$ (the $\phi$ dependence trivially decouples), and evolve initial data for $\psi_4$. This works so well modeling source-free radiation \[27\] that it has been argued time domain methods may replace frequency decomposition for most applications \[28\]. For point particle sources, frequency domain methods are currently considerably more accurate \[29\].

The radial functions $R_{lm\omega}(r)$ are obtained by solving

$$\Delta^2 \left( \Delta^{-1} R_{lm\omega}^\prime \right) - V_{lm\omega}(r) R_{lm\omega} = -T_{lm\omega}(r),$$  \hspace{1cm} (4)

where prime denotes $d/dr$, $\Delta = r^2 - 2Mr + a^2$, $V_{lm\omega}$ is a potential \[e.g., \[30\], Eq. (4.3)] and $T_{lm\omega}$ is a source constructed from the small body’s stress energy tensor. We build a Green’s function $G_{lm\omega}(r, r')$ from solutions to the homogeneous wave equation (setting $T_{lm\omega} = 0$), using certain analytic transformations \[31\] that allow high numerical accuracy. We then find $R_{lm\omega}(r)$ by integrating $G_{lm\omega}(r, r')$ over the source,

$$R_{lm\omega}(r) = -\int dr' G_{lm\omega}(r, r') T_{lm\omega}(r').$$  \hspace{1cm} (5)

Our particular interest is in the limits $r \to \infty$ and $r \to$ the event horizon: from the function in these limits, we extract the $E$ and $L_z$ fluxes mode-by-mode. Using many modes, we assemble the adiabatic rates of change $\langle \dot{E} \rangle$ and $\langle \dot{L}_z \rangle$.

For bound orbits, the source is nicely described by a harmonic expansion. The continuous frequency $\omega$ goes over to a discrete set $\omega_{knm} = n\Omega_\phi + k\Omega_\theta + m\Omega_r$, where $\Omega_{\phi, \theta, r}$ describe motion in $\phi, \theta,$ and $r$ \[32\] \[33\]. Our modes become 4 index objects, $(l m kn)$, there is no mode-mode coupling, so this problem is ideally suited to parallel computation: different modes can be sent to different processors with little communication cost. Figure 11 illustrates parallelization. The top panel shows the average CPU time per mode $(lmkn)$ as a function of number of processors $N$. We find nearly linear scaling in $1/N$, demonstrating that we have little parallelization overhead. Typical CPU time per mode is quite short, so that $\sim 10^4 - 10^5$ modes can be computed in a reasonable time.

The bottom panel shows the convergence of energy flux from a strong field orbit. We write the orbit’s radial motion as $r = pM/(1 + e\cos \psi)$; for this plot, we put eccentricity $e = 0.5$ and semi-latus rectum $p = 4$. The inclination $i = 45^\circ$; the black hole’s spin is $a = 0.9M$. (Precise definitions of $p, e,$ and $i$ can be found in \[33\]; their key property for this paper is that they are simply related to the constants $E, L_z,$ and $Q$.) We plot

$$\dot{E}_l = \sum_{m=-l}^l \sum_{k=-K}^K \sum_{n=-N}^N \dot{E}_{lmkn}$$  \hspace{1cm} (6)
where $\dot{E}_{lmkn}$ is the modal energy flux. The cutoff values $N$ and $K$ (which formally are $\infty$), are set large enough that neglected terms contribute a fractional amount less than $10^{-4}$ to the sum; details of this truncation will be presented elsewhere [34]. We generically find that $\dot{E}_l$ falls exponentially with $l$.

The orbital parameter’s influence on the rate of this falloff, and on other convergence criteria, will be presented in [34]. Most importantly, the convergence of these fluxes is fairly quick for $e \lesssim 0.7$, a very astrophysically important range [5].

**FIG. 1:** Top: Average CPU time per mode versus number of processors $N$ for two strong field orbits [circular ($e = 0$) and eccentric ($e = 0.5$)]. Our scaling shows $T_{\text{cpu}} \propto 1/N$, showing how well PMRR parallelizes. The typical CPU time per mode is fairly short, though eccentric orbits are about 5–10 times slower than circular. Bottom: Flux $dE_l/\text{d}t$ (defined in the text) versus $l$ for an orbit with $p = 4$, $e = 0.5$, and $i = 45^\circ$; the black hole’s spin $a = 0.9M$. We generically find that the flux sums converge exponentially with $l$.

Mino’s adiabatic self force and $Q$’s evolution. Recent work by Mino [12] makes it possible to compute the time average $\langle Q \rangle$ of $Q$, allowing us to compute $\Phi_{0}(t)$ in Eq. (1) for generic orbits. The regularized self force $\tilde{f}^{b}$ involves an integral over the orbiting body’s past worldline [11, 12]. Mino shows that this integral can be replaced in the adiabatic limit by a relatively simple expression involving the difference between “retarded” and “advanced” forces: $f^{b} = (f^{b}_{\text{ret}} - f^{b}_{\text{adv}})/2$. The “retarded” force depends on events on the orbiting body’s past lightcone; the “advanced” force depends on the future lightcone. Their difference removes the divergent point-particle self interaction; the remaining force drives the inspiral. This result is strikingly similar to Dirac’s result [10], and indeed reproduces the rule posited (without proof) by Gal’tsov [33].

The Carter constant is given by $Q = Q_{ab}p^{a}p^{b}$, where $Q_{ab}$ is a Killing tensor and $p^{b}$ is the small body’s 4-momentum. Taking a time derivative yields $\dot{Q} = 2Q_{ab}p^{a}p^{b} = 2Q_{ab}^{\mu}j^{\mu}$. We express the self force in terms of the radiative Green’s function using Mino’s result, and use the expansion of the radiative Green’s function in terms of modes [35]. Time averaging yields an expression of the form [16]

$$\langle \dot{Q} \rangle = \sum_{r=\infty} W[Q_{lmkn}(r), R_{lmkn}(r)].$$  \hspace{1cm} (7)

The quantities $R_{lmkn}(r)$ come from $R_{lm\omega}(r)$ by writing $R_{lm\omega}(r) = \sum_{kn} R_{lmkn}(r)\delta(\omega - \omega_{kn})$. The first sum in Eq. (7) means that the expression is evaluated near the horizon and near infinity. The quantities $Q_{lmkn}(r)$ are computed via an integral similar to that in Eq. (6), but with the source $T_{lm\omega}$ replaced by a new source built from the Killing tensor $Q_{ab}$ and the orbiting body’s 4-velocity, and evaluated at $\omega = \omega_{kn}$. Finally $W$ is the Wronskian which is independent of $r$ near the horizon and near infinity as both $R_{lmkn}(r)$ and $Q_{lmkn}(r)$ satisfy the homogeneous Teukolsky equation at those locations. Details of this calculation will be presented elsewhere [16]. Using this result it will be as straightforward to compute $\langle Q \rangle$ as it is currently to compute $\langle \dot{E} \rangle$ and $\langle L_{z} \rangle$.

Applications and future directions. Building inspiral waveforms requires us to compute backreaction effects upon a dense “grid” of orbits in parameter space. Each orbit is represented by a coordinate $(p, e, i)$; backreaction gives the tangent $(\dot{p}, \dot{e}, \dot{i})$ to the inspiral trajectory. We flow along this tangent vector, building the trajectory $[p(t), e(t), i(t)]$. Choosing initial conditions, we likewise build the inspiral worldline $s^{\mu}(t) = [t, r(t), \theta(t), \phi(t)]$. To date, only circular ($e = 0$) [36] and equatorial ($i = 0, 180^\circ$) [37] inspirals have been computed; the general case is under development [34], though we can present waveform “snapshots” (Fig. 2).

**FIG. 2:** Gravitational waveform for several strong field orbits; all are for a hole with spin $a = 0.9M$, and have $p = 4$. Top panel: eccentricity $e = 0.5$, inclination $i = 45^\circ$. Middle: $e = 0.5, i = 45^\circ$. Bottom: $e = 0.5, i = 0^\circ$. The orbital parameters, particularly eccentricity, richly influence the wave’s harmonic structure.

The cases we have studied in detail demonstrate the rich physics encoded by these events. Three particularly interesting features are:

- **Tidal coupling to the event horizon**: The orbiting body raises a tidal bulge on the black hole. This bulge interacts with the
orbit, transferring the hole’s spin to the orbit, just as planetary tides can transfer angular momentum to a satellite. If the hole rotates rapidly and the orbit has shallow inclination, the tide can significantly prolong the inspiral [26].

- **Strong field precessions**: The time to oscillate through $\pi$ radians of $\phi$. The mismatch between these timescales gives perihelion precession, a classic GR test. With black holes, this effect can be large if the hole “whirls” many times near the hole before “zooming” out to large radius. This “zoom-whirl” behavior leaves a distinctive strong field precession.

Future work will develop ARR and waveforms with a focus upon templates for future GW searches. One goal is to use spectral methods for solving many of the formalism’s equations. Such methods are typically exponentially convergent in the number of basis functions. Since ARR requires many multipoles, each must be as accurate as possible. We are very encouraged by the work of Fujita and Tagoshi [38], who demonstrate that a particular set of basis functions allows modal solutions with essentially double precision accuracy.

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[17] For example, for a $10^3 M_\odot$ circular inspiral into a $10^9 M_\odot$, $a = 0.999 M$ black hole, the $\mu/M$ corrections in the post-3.5-Newtonian phase of the Fourier transform of the waveform that correspond to $\Phi_1(t)$ give a phase error between 0.003 Hz and 0.03 Hz (the last year of inspiral [18]) of $\leq 0.8$ cycles (after optimizing time-of-arrival and overall phase). For detection purposes, one needs phase coherence for only $\sim 3$ weeks [5]; the phase error over this timescale will be even smaller.
[29] It may be possible to combine a frequency-domain inspiral calculation with a time-domain approach to the emitted radiation. The averaging needed for ARR is naturally done in the frequency domain; time domain codes are superior for radiation generation and propagation. A hybrid approach could combine the best features of both toolkits.