Quantum Mechanics in Non-Inertial Frames with a Multi-Temporal Quantization Scheme: II) Non-Relativistic Particles.

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Abstract

The non-relativistic version of the multi-temporal quantization scheme of relativistic particles in a family of non-inertial frames (see hep-th/0502194) is defined. At the classical level the description of a family of non-rigid non-inertial frames, containing the standard rigidly linear accelerated and rotating ones, is given in the framework of parametrized Galilei theories. Then the multi-temporal quantization, in which the gauge variables, describing the non-inertial effects, are not quantized but considered as c-number generalized times, is applied to non relativistic particles. It is shown that with a suitable ordering there is unitary evolution in all times and that, after the separation of the center of mass, it is still possible to identify the inertial bound states. The few existing results of quantization in rigid non-inertial frames are recovered as special cases.

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I. INTRODUCTION

In a preceding paper (referred to as I) [1] a relativistic quantum mechanics for a system of N positive-energy particles in a family of relativistic non-inertial frames was defined in the framework of parametrized Minkowski theories.

In this paper there is the study of the non-relativistic limit of this non-inertial quantum mechanics. A classical parametrized Galilei theory is constructed where a choice of non-inertial coordinates is realized as a gauge choice. This is done without any use of the relativistic theory of reference I, so that knowledge of results of I is not necessary. However it is also shown that the non-relativistic parametrized theory can be obtained by the relativistic one making the exact limit $c \to \infty$. In this approach a very general notion of non-inertial coordinates is used. Indeed, treating the 3-dimensional Newtonian Space as a flat 3-dimensional manifold, non-inertial coordinates are defined applying a time dependent coordinate transformation on the inertial ones. The non-inertial coordinates associated to the traditional accelerated or rotating frames are found as particular cases of linear, rigid, time-dependent coordinate transformations.

The corresponding quantum theory is obtained by means of the multi-temporal quantization scheme for first class constraints, in which only the particle degrees of freedom are quantized, but considered as c-number generalized times. By means of a suitable ordering as many coupled Schroedinger-like equations as first class constraints are obtained satisfying the same algebra as the Poisson bracket algebra of the classical constraints. It is possible to define a Hilbert space, whose wave functions depend on time and on the generalized times as parameters. All the Hamiltonians in the Schroedinger-like equations are self-adjoint operators and the scalar product is independent from all the times, so that the evolution is unitary. By choosing a path in the parametric space of the generalized times (namely a non-inertial frame) the non-inertial, self-adjoint, Hamiltonian for the non-inertial evolution can be found. In the particular case of rigidly linear accelerated or rotating frames known results are reproduced as special cases. Moreover, it is possible to study the separation of the center of mass from the relative variables and to apply it to the definition of bound states in non-inertial frames.

In Section II a classical non-relativistic parametrized theory with first class constraint is constructed and it is also shown that the gauge fixing on the gauge variables are interpretable
as a choice of non-inertial, in general non-rigid, coordinates. In Section III there is the study of the multi-temporal quantization scheme, of the non-relativistic coupled Schroedinger-like equations and of a scalar product independent from all the times so that the evolution is unitary. A path in the parametric space of the generalized times allows to find the self-adjoint, Hamiltonian for the non-inertial evolution. In Section IV, there is the restriction of the general theory to the particular case of the traditional non-inertial rigid frames and it is shown that the previous attempts [10, 11, 12] to define non-inertial quantum mechanics in rigid non-inertial frames are special sub-cases of the multi-temporal approach. In Section V, a study of the separation between center of mass and relative coordinates is done and some observation on the possible definitions of bound states in non-inertial frames both in the rigid and non-rigid cases. In Section VI there are some concluding remarks. In Appendix A the equivalence between the non-relativistic limit of relativistic theory of I and the non-relativistic parametrized theory is shown. In Appendix B there is the study of the spinning particles case. In Appendix C Galilei transformation and Galilei relativity are discussed.
II. CLASSICAL PARAMETRIZED GALILEI THEORIES

A. Some Geometric Definitions

In a non-relativistic theory, each event is identified by its position in a 3-dimensional space $\mathbb{R}^3$ and by its (absolute) time $t$. In the 3-dimensional space there exist a family of preferred reference frames: the inertial reference frames. We choose one of these ones taking a basis of unit vectors $(\hat{i}_1, \hat{i}_2, \hat{i}_3)$ defining three orthogonal directions placed on an arbitrary (fixed) origin: the position in space of each event will be identified by its cartesian coordinates $x^a$ on the cartesian axis so defined. These coordinates are the inertial coordinates.

In some cases, it can be useful to use a four dimensional Newtonian space-time $\mathbb{R} \times \mathbb{R}^3$ where the (absolute) time $t$ and the inertial cartesian spatial coordinates $\vec{x}$ represent a particular choice of 4-dimensional coordinates. From a mathematical point of view, the Newtonian space-time can be regarded as a 4-dimensional manifold where a more general atlas of coordinates can be used such that, for each chart, $\xi^\mu = \xi^\mu(t, \vec{x})$. The Cartan’s approach to Newtonian space-time [2], used for example in Ref.[3], is based on this observation.

Here, following Ref.[4], we use a more simple approach where we admit time dependent coordinates transformations only on the 3-dimensional space. After one of these transformations the 3-dimensional space is parametrized by a set of global, in general non cartesian coordinates $\sigma^r$. This invertible, global coordinates transformation has the form ($a, r = 1, 2, 3$)

$$x^a = A^a(t, \vec{\sigma}),$$ (2.1)

with inverse

$$\sigma^r = S^r(t, \vec{x}).$$ (2.2)

If we define the 3-dimesional Jacobian

$$J^a_r(t, \vec{\sigma}) = \frac{\partial A^a(t, \vec{\sigma})}{\partial \sigma^r},$$ (2.3)

the invertibility conditions are

$$\det J(t, \vec{\sigma}) > 0.$$ (2.4)

In particular we will use the inverse Jacobian
\[ \tilde{J}_a^r(t, \tilde{\sigma}) = \left[ \frac{\partial S^r(t, \tilde{x})}{\partial x^a} \right]_{\tilde{x} = \tilde{s}(t, \tilde{\sigma})}, \] (2.5)

satisfying

\[ J^a_r(t, \tilde{\sigma}) \tilde{J}_b^r(t, \tilde{\sigma}) = \delta^a_b, \quad \tilde{J}_a^s(t, \tilde{\sigma}) J^a_r(t, \tilde{\sigma}) = \delta^s_r. \] (2.6)

The \( \sigma^r \)'s are the non-inertial coordinates of a non-inertial reference frame implicitly defined by Eq.(2.1). As we shall see in Section IV, the traditional linear accelerated and rotating non-inertial frames (the rigid frames obtained by means of rigid time dependent transformations), are particular cases of this more general approach.

In the following subsections we develop a Lagrangian and Hamiltonian theory where the choice of the non-inertial coordinates \( \sigma^r \), that is the choice of the coordinate transformation (2.1) can be interpreted as a gauge-choice. This is done by introducing a singular Lagrangian theory where the gauge parameters, namely the functions \( \mathcal{A}^a(t, \tilde{\sigma}) \), are considered as Lagrangian configuration variables. At the Hamiltonian level this implies the presence of first class Dirac constraints.

**B. Lagrangian Formulation**

If we apply the coordinates transformation (2.1) to a system of \( N \) non-relativistic interacting particles, the particle coordinates \( x^a_i(t) \), \( i = 1, \ldots, N \), in the inertial frame must be parametrized in the form

\[ x^a_i(t) = \mathcal{A}^a(t, \tilde{\eta}_i(t)), \] (2.7)

so that the standard velocities acquire the following form

\[ \dot{x}^a_i(t) = \frac{d\mathcal{A}^a(t, \tilde{\eta}_i(t))}{dt} = \]

\[ = \text{def} \left[ \frac{\partial \mathcal{A}^a(t, \tilde{\eta}_i(t))}{\partial t} + J^a_r(t, \tilde{\eta}_i(t)) \frac{d\eta^r_i(t)}{dt} \right]. \] (2.8)
If we apply these transformations to the usual inertial equations of motion

$$\frac{m_i}{d} \frac{d}{dt} \dot{x}_i^a(t) = -\frac{\partial V}{\partial x_i^a}(t, \vec{x}_1(t), ..., \vec{x}_N(t)), \quad (2.9)$$

we obtain

$$\frac{m_i}{d} \frac{d}{dt} \frac{dA^a(t, \vec{\eta}_i(t))}{dt} = -\frac{\partial V}{\partial x_i^a}|_{\vec{x}_i=\vec{A}(t, \vec{\eta}_i(t))} = -\tilde{J}_a^r(t, \vec{\eta}_i(t)) \sum_j \frac{\partial V}{\partial \eta_i^r}. \quad (2.10)$$

From now on $V$ will denote the following expression for the potential

$$V = V\left(t, \vec{A}(t, \vec{\eta}_1(t)), ..., \vec{A}(t, \vec{\eta}_N(t))\right). \quad (2.11)$$

Let us consider the following action

$$S = \int dt L(t), \quad (2.12)$$

with the Lagrangian

$$L(t) = \int d^3 \sigma \sum_i \delta^3(\vec{\sigma} - \vec{\eta}_i(t)) \frac{1}{2m_i} \left( \frac{\partial \vec{A}(t, \vec{\sigma})}{\partial t} + \bar{J}_a^r(t, \vec{\sigma}) \frac{d\eta_i^r(t)}{dt} \right)^2 - V. \quad (2.13)$$

In the Lagrangian (2.13) the configuration variables are the particles positions $\vec{\eta}_i(t)$ and the functions $A^a(t, \vec{\sigma})$ (with velocities $\partial A^a(t, \vec{\sigma})/\partial t$). The Euler-Lagrange equations

$$\frac{\delta S}{\delta \vec{\eta}_i(t)} = 0, \quad (2.14)$$

generated by a variation of the variables $\vec{\eta}_i(t)$, are just Eqs.(2.10).

However under the local Noether transformations

$$\delta \vec{\eta}_i^a(t) = \varepsilon^a(t, \vec{\eta}_i(t)) \bar{J}_a^r(t, \vec{\eta}_i(t)), \quad (2.15)$$

$$\delta A^a(t, \vec{\sigma}) = \varepsilon^a(t, \vec{\sigma}),$$

We use an interaction potential $V(t, \vec{x}_1, ..., \vec{x}_N)$ which is time-dependence and without rotational or translational symmetries. This is useful for the discussion of the examples of Section VI. As shown in Appendix C, the equations of motion are invariant under Galilei transformations only if the interaction potential is time independent and invariant under rotations and translations.
depending upon the arbitrary functions \( \varepsilon^a(t, \vec{\sigma}) \), the Lagrangian is invariant \( \delta L = 0 \). This implies that the Euler-Lagrange equations for \( \mathcal{A}^a(t, \vec{\sigma}) \), are not independent from the Eqs.(2.10), but satisfy the contracted Bianchi identities

\[
\frac{\delta S}{\delta \mathcal{A}(t, \vec{\sigma})} = \sum_i \delta^3(\vec{\sigma} - \vec{\eta}_i(t)) \tilde{J}^r_a(t, \vec{\eta}_i(t)) \frac{\delta S}{\delta \eta^r_i(t)} = 0. 
\]

(2.16)

This means that the configuration variables \( \mathcal{A}^a(t, \vec{\sigma}) \) are left arbitrary (in other terms they are \textit{gauge variables}) and that at the Hamiltonian level there are \textit{first class Dirac constraints} in the phase space.

The Lagrangian (2.13) defines a \textit{parametrized Galilei theory}. See Appendix C for the study of the Galilei transformations.

C. Hamiltonian Formulation

From the Lagrangian (2.13) we get the following canonical \textit{momenta}

\[
p^r_i(t) = \frac{\partial L}{\partial \left( \frac{d \eta^r_i(t)}{dt} \right)} = m_i \sum_a J^a_r(t, \vec{\eta}_i(t)) \frac{d \mathcal{A}^a(t, \vec{\eta}_i(t))}{dt},
\]

\[
\rho^a(t, \vec{\sigma}) = \frac{\delta L}{\delta \left( \frac{d \mathcal{A}^a(t, \vec{\sigma})}{dt} \right)} = \sum_i m_i \frac{d \mathcal{A}^a(t, \vec{\eta}_i(t))}{dt} \delta^3(\vec{\sigma} - \vec{\eta}_i),
\]

(2.17)

whose Poisson brackets are

\[
\{ \eta^r_i(t), p^s_j(t) \} = \delta^r_s \delta_{ij}
\]

\[
\{ \mathcal{A}^a(t, \vec{\sigma}), \rho^b(t, \vec{\sigma}') \} = \delta^{ab} \delta^3(\vec{\sigma} - \vec{\sigma}').
\]

(2.18)

The momenta satisfy the following Dirac constraints

\[
\mathcal{H}_a(t, \vec{\sigma}) = \rho^a(t, \vec{\sigma}) - \sum_i \delta^3(\vec{\sigma} - \eta_i(t)) \tilde{J}^r_a(t, \vec{\eta}_i(t)) p^r_i(t) \approx 0.
\]

(2.19)

The canonical Hamiltonian is
\[ H_c(t) = + \sum_i p_{ir}(t) \frac{\partial \eta_i^r(t)}{\partial t} + \int d^3\sigma \sum_a \rho^a(t, \sigma) \mathcal{A}_a(t, \vec{\sigma}) - L(t) = \]
\[ = \sum_i \frac{1}{2m_i} \sum_a \left[ \tilde{J}^r_a(t, \vec{\eta}_i(t)) p_{ir}(t) \right] \left[ \tilde{J}^s_a(t, \vec{\eta}_i(t)) p_{is}(t) \right] + V, \quad (2.20) \]

but the motion equations are generated by the Dirac Hamiltonian

\[ H_D(t) = H_c(t) + \int d^3\sigma \lambda^a(t, \vec{\sigma}) \mathcal{H}_a(t, \vec{\sigma}), \quad (2.21) \]

where \( \lambda^a(t, \vec{\sigma}) \) are arbitrary Dirac’s multipliers. The constraints (2.19) are first class Dirac constraints because their Poisson brackets are

\[ \{ H_c(t), \mathcal{H}_a(t, \vec{\sigma}) \} = \{ \mathcal{H}_a(t, \vec{\sigma}), \mathcal{H}_b(t, \vec{\sigma}') \} = 0. \quad (2.22) \]

These constraints are the canonical generators of the Noether gauge transformations, corresponding to a change in the definition of the non inertial coordinates \( \vec{\sigma} \), under which the Lagrangian is invariant.

The equations of motion generated by the Dirac Hamiltonian (2.21) imply

\[ \lambda^a(t, \vec{\sigma}) = \frac{\partial \mathcal{A}_a(t, \vec{\sigma})}{\partial t}. \quad (2.23) \]

A gauge fixing for the first class constraints is a choice of a fixed form of the function \( \mathcal{A}_a(t, \vec{\sigma}) \)

\[ \mathcal{A}_a(t, \vec{\sigma}) = \mathcal{A}_a^F(t, \vec{\sigma}), \quad (2.24) \]

and this is equivalent to fix the arbitrary multipliers:

\[ \lambda^a(t, \vec{\sigma}) = \frac{\partial \mathcal{A}_a^F(t, \vec{\sigma})}{\partial t}. \quad (2.25) \]

After such a choice the only remaining canonical variables are \( \eta_i^r(t), p_{ir}(t) \). The equations of motion for a function \( F(t, \vec{\eta}_i, \vec{p}_i) \) of the particle canonical variables, evaluated in the fixed gauge, are the Hamilton-Dirac equations generated by the Hamiltonian (2.21) and restricted to Eqs. (2.24) and (2.25)

\[^2 \text{ We use the subscript "F" for the geometrical quantities calculated in a fixed gauge.} \]
\[
\left( \frac{d}{dt} F \right)_{\mathcal{A}=A_F} \triangleq \left[ \frac{\partial}{\partial t} F + \{ F, H_c \} - \sum_i \{ F, \lambda^a(t, \vec{\eta}_i) \tilde{J}_{\mathcal{F}a}(t, \vec{\eta}_i(t)) p_{ir} \} \right]_{\mathcal{A}=A_F} . \tag{2.26}
\]

Since the gauge fixings are explicitly time dependent, it can be shown that in the non-inertial frame corresponding to the given gauge these equations are generated by the non-inertial Hamiltonian

\[
H_{ni}(t) = \sum_i \sum_a \left[ \tilde{J}_{\mathcal{F}a}(t, \vec{\eta}_i(t)) p_{ir}(t) \right] \left[ \tilde{J}_{\mathcal{F}a}(t, \vec{\eta}_i(t)) p_{ia}(t) \right] \frac{2m_i}{\tilde{J}_{\mathcal{F}a}(t, \vec{\eta}_i(t)) p_{ir}(t)} + \sum_i V_{\mathcal{F}}(t, \vec{\eta}_i) p_{ir}(t), \tag{2.27}
\]

where we have introduced the velocity field

\[
V_{\mathcal{F}}(t, \vec{\sigma}) = \tilde{J}_{\mathcal{F}a}(t, \vec{\sigma}) \frac{\partial A_{\mathcal{F}a}(t, \vec{\sigma})}{\partial t}. \tag{2.28}
\]

Then Eqs.(2.26) can be written in the form

\[
\left( \frac{d}{dt} F \right)_{\mathcal{A}=A_F} \triangleq \frac{\partial}{\partial t} F + \{ F, H_{ni}(t) \}. \tag{2.29}
\]
III. MULTI-TEMPORAL QUANTIZATION

In the traditional Dirac Quantization scheme of a classical theory with first class constraints \([5]\), all the canonical variables are quantized ignoring the presence of the constraints and a non-physical Hilbert space is constructed. After a suitable choice of the ordering, the constraints are mapped in operators and physical states are defined as the zero eigenvectors of the quantum constraints. The main difficulty of this approach is the definition of a physical Hilbert space, with a physical scalar product, since usually the quantum constraints have zero in their continuum spectrum and then the physical states do not define a subspace of the non-physical Hilbert Space.

When it is possible to identify the gauge variables associated to Dirac constraints (as it happens in the parametrized theory of previous Section, where it is possible to identify the \(A^a(t, \vec{\sigma})\) as the gauge variables), we can avoid the construction of the non-physical Hilbert space defining directly the physical Hilbert space. Indeed, motivated by a classical multi-temporal approach to constrained dynamics \([6]\), \([7]\) and by the use of an analogous many-time quantization scheme for two particles relativistic systems \([8]\), we treat in a different way the gauge variables and their momenta with respect to the other physical variables. These latter are quantized and interpreted as operators and the physical Hilbert space is constructed so to realize an irreducible representation of their canonical commutation relations. Instead the gauge variables are interpreted as generalized times and their momenta are mapped in time derivatives. The contraints are mapped in coupled generalized Schroedinger equations that govern the dependence on the generalized times of the wave function. One has to find an ordering for the quantization of the constraints such that the quantum algebra of the constraints implies the integrability of the generalized Schroedinger equations. The physical scalar product in the Hilbert space has to be independent from all the generalized times.

This scheme of quantization has already been used in I for parametrized Minkowski theories.

A. Quantization: ”Times”, Operators and Hilbert Space.

We go now to apply the multi-temporal quantization scheme to the parametrized Galilei theory of the previous Section.

i) We shall consider the gauge variables \(A^a(\vec{\sigma})\) as c-number generalized times, with the conjugate momenta replaced by the following functional derivatives
\[ \rho^a(\bar{\sigma}) \leftrightarrow -i\hbar \frac{\delta}{\delta \mathcal{A}^a(\bar{\sigma})}. \]  

(3.1)

ii) The positions and momenta \( \eta^r_i, \kappa_i \) of the particles are quantized in the standard way as operators on a Hilbert space satisfying the canonical commutation relations. We choose a representation where the \( \eta^r_i \)'s are multiplicative operators and where

\[ p^r_i \mapsto -i\hbar \frac{\partial}{\partial \eta^r_i}, \]

are derivative operators on the \( Hilbert space \) of square integrable complex functions \( \mathcal{H} = L^2(C, R^{3N}) \) with scalar product \(^3\)

\[ (\Psi_1, \Psi_2) = \int \left( \prod_i d^3 \eta_i \right) \overline{\Psi}_1(\eta_i) \Psi_2(\eta_i). \]

(3.3)

**B. "Generalized" Temporal Evolution**

A state will evolve in the Hilbert State \( \mathcal{H} \) as a functional of the \( \) time \( t \) and of the \( generalized times \ \mathcal{A}^a(\bar{\sigma}) \). The evolution in these \( generalized times \) is determined by the quantization of the classical Dirac contraints in the form \(^4\)

\[ \widehat{\mathcal{H}}_a(\bar{\sigma}) \cdot \Psi(\eta_i; t, \mathcal{A}^a) = 0, \]

(3.4)

\(^3\) In I, it is shown that we can develop the quantum theory using also a scalar product with the \( invariant measure \)

\[ (\Phi_1, \Phi_2)_{inv} = \int \left( \prod_i \det J(t, \eta_i) d^3 \eta_i \right) \overline{\Phi}_1(\eta_i) \Phi_2(\eta_i). \]

The wave functions \( \Phi \) have the following relation with the \( \Psi \)'s

\[ \Psi(\eta_i; t, \mathcal{A}^a) = \sqrt{\prod_i \det J(t, \eta_i)} \Phi(\eta_i; t, \mathcal{A}^a). \]

As discussed in I, in this approach the conservation of probability

\[ \frac{\partial}{\partial t} (\Phi_1, \Phi_2)_{inv} = \frac{\delta}{\delta \mathcal{A}^a(\bar{\sigma})} (\Phi_1, \Phi_2)_{inv} = 0, \]

is implied if the new wave functions satisfy generalized Schoredinger equations with non-self-adjoint Hamiltonians, which are obtained with a \( non symmetrized \) ordering choice in the Eqs.(3.8) and (3.10).

\(^4\) We use the notation \( \Psi(\eta_i; t, \mathcal{A}^a) \) to indicate that the wave function is a function of the positions \( \eta_i \) and of the time \( t \), but a functional of the \( generalized times \ \mathcal{A}^a(\bar{\sigma}) \).
whereas the evolution in the time $t$ is determined by the quantum Dirac Hamiltonian through the Schrödinger equation

$$i \hbar \frac{\partial}{\partial t} \Psi (\vec{\eta}; t, \mathcal{A}^a) = \hat{H}_c \cdot \Psi (\vec{\eta}; t, \mathcal{A}^a).$$  \hspace{1cm} (3.5)$$

The explicit form of Eqs.(3.5) and (3.4) is obtained by using the rules i) and ii). To solve the ordering problems, we define the operators

$$\hat{K}_{ia} = -\frac{i\hbar}{2} \left[ \tilde{J}^r_a(\vec{\eta}), \frac{\partial}{\partial \eta^r_i} \right]_+ = -i\hbar \tilde{J}^r_a(\vec{\eta}) \frac{\partial}{\partial \eta^r_i} - \frac{i\hbar}{2} \frac{\partial \tilde{J}^r_a(\vec{\eta})}{\partial \eta^r_i}. \hspace{1cm} (3.6)$$

They are the self-adjoint observables corresponding to classical functions $\tilde{J}^r_a(\vec{\eta}) p_{ir}$. Then we assume the following ordering inside the canonical Hamiltonian $H_c$

$$\sum_a \tilde{J}^r_a(\vec{\eta}) p_{ir} \tilde{J}^s_a(\vec{\eta}) p_{is} \mapsto -\hbar^2 \Delta'_i = \sum_a \hat{K}_a \cdot \hat{K}_a. \hspace{1cm} (3.7)$$

Instead inside the constraints $\mathcal{H}_a(t, \vec{\sigma})$ we introduce the ordering

$$\sum_i \delta^3(\vec{\sigma} - \vec{\eta}) \tilde{J}^r_a(\vec{\eta}) p_{ir} \mapsto \hat{T}_a(\vec{\sigma}, \mathcal{A}^a) = -\frac{i\hbar}{2} \sum_i \left[ \delta^3(\vec{\sigma} - \vec{\eta}) \tilde{J}^r_a(\vec{\eta}), \frac{\partial}{\partial \eta^r_i} \right]_+. \hspace{1cm} (3.8)$$

With these definition we obtain the \textit{generalized Schrödinger equations}:

$$\left( i \hbar \frac{\partial}{\partial t} - \hat{E}[\mathcal{A}^a] \right) \Psi (\vec{\eta}; t, \mathcal{A}^a) = 0, \hspace{1cm} (3.9)$$

$$\hat{\mathcal{H}}_a(\vec{\sigma}) \cdot \Psi (\vec{\eta}; t, \mathcal{A}^a) = \left( -i \hbar \frac{\delta}{\delta \mathcal{A}^a(\vec{\sigma})} - \hat{T}_a(\vec{\sigma}, \mathcal{A}^a) \right) \Psi (\vec{\eta}; t, \mathcal{A}^a) = 0, \hspace{1cm} (3.10)$$

where we used the notation

$$\hat{H}_c \equiv \hat{E}[\mathcal{A}^a] = -\hbar^2 \sum_i \frac{1}{2m_i} \Delta'_i + \mathcal{V}. \hspace{1cm} (3.11)$$

With this notation we want to emphasize that the operator $\hat{E}[\mathcal{A}^a]$ is the energy defined by the inertial observer adopting the inertial coordinates $x^a$ expressed as function of the coordinates $\vec{\sigma}$ defined by $x^a = \mathcal{A}^a(\vec{\sigma})$.

The chosen ordering is such that the generalized Hamiltonians $\hat{E}[\mathcal{A}^a]$ and $\hat{T}_a(\vec{\sigma}, \mathcal{A}^a)$ are self-adjoint operators and it \textit{guarantees the formal integrability} of Eqs.(3.9) and (3.10), namely
\[
\left[ \hat{E}[A^a], \hat{H}_a(\sigma) \right] = \left[ \hat{H}_a(\sigma), \hat{H}_b(\sigma') \right] = 0. \tag{3.12}
\]

We can formalize the general\textit{ized time evolution} by introducing a space of generalized times, parametrized with the time \( t \) and with the generalized times \( A^a(\sigma) \). Such a space of generalized times is the cartesian product \( \mathcal{M} = \mathbb{R} \times C^\infty(\mathbb{R}^3,\mathbb{R}^3) \), where \( C^\infty(\mathbb{R}^3,\mathbb{R}^3) \) is the space of the differentiable and invertible functions from \( \mathbb{R}^3 \) on \( \mathbb{R}^3 \), whose elements are represented by \((t, A^a) \ [A^a \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)] \). Then, the generalized temporal evolution can be defined as the map

\[
T : \mathcal{M} \times H \mapsto H, \tag{3.13}
\]

\[
T \left[ (t, A^a), \Psi_o(\tilde{\eta}_i) \right] = \Psi \left( \tilde{\eta}_i, t, A^a \right), \tag{3.14}
\]

where \( \Psi_o(\tilde{\eta}_i) \) is the initial condition, namely the value assigned to the state in a point \((t_o, A^a_{in}(\sigma)) \in \mathcal{M} \)

\[
\Psi \left( \tilde{\eta}_i, t_o, A^a_{in} \right) = \Psi_o(\tilde{\eta}_i). \tag{3.15}
\]

Since the generalized Hamiltonians are self adjoint we get

\[
\frac{\partial}{\partial t} (\Psi_1, \Psi_2) = \frac{\delta}{\delta A^a(\sigma)} (\Psi_1, \Psi_2) = 0. \tag{3.16}
\]

This implies that the time evolution \( T \) defines a unitary transformation in Hilbert space \( H \).

To discuss this fact explicitly, it convenient to assign the initial condition by choosing \( A^a_{in}(\sigma) = \sigma^a \). Then the solution of the generalized Schroedinger equations (3.9) and (3.10), satisfying the initial condition (3.15) when evaluated at the generalized times \( t = t_o, A^a(\sigma) = A^a_{in}(\sigma) \), is explicitly given by

\[
\Psi \left( \tilde{\eta}_i, t, A^a \right) = \exp \left[ -\frac{i}{\hbar} (t - t_o) \hat{E}[A^a] \right] \cdot \Psi' \left( \tilde{\eta}_i; A^a \right) = \exp \left[ -\frac{i}{\hbar} (t - t_o) \hat{E}[A^a] \right] \cdot U'[A^a] \cdot \Psi_o(\tilde{\eta}_i) \overset{df}{=} U(t, A^a) \cdot \Psi_o(\tilde{\eta}_i), \tag{3.17}
\]

where
\[
\Psi' (\vec{\eta}; A^a) = \sqrt{\prod_i \det J(\vec{\eta}_i)} \Psi_o \left( \vec{A}(\vec{\eta}_i) \right) \overset{def}{=} U'[A^a] \cdot \Psi_o (\vec{\eta}_i). \quad (3.18)
\]

If \( \Psi'_1, \Psi'_2 \) are obtained from two different initial conditions \( \Psi_{1,o}, \Psi_{2,o} \) as in Eq.(3.18), it can be shown that, by using the change of variables \( \vec{\eta}'_i = \vec{A}(\vec{\eta}_i) \), we get

\[
\int \left( \prod_i d^3 \eta'_i \right) \Psi_{o,1}(\vec{\eta}'_i) \Psi_{o,2}(\vec{\eta}'_i) = \int \left( \prod_i d^3 \eta_i \right) \Psi'_1(\vec{\eta}; A^a) \Psi'_2(\vec{\eta}; A^a). \quad (3.19)
\]

Then Eq.(3.18) defines a unitary transformation on the Hilbert Space \( H \). The inverse of \( U'[A^a] \) is also a self-adjoint operator

\[
U'^+[A^a] \cdot \Psi'(\vec{\eta}'; A^a) = \frac{1}{\sqrt{\prod_i \det J(t, \vec{S}(\vec{\eta}'_i))}} \Psi' \left( \vec{S}(\vec{\eta}'_i), A^a \right). \quad (3.20)
\]

Moreover, since \( \exp \left[ -\frac{i}{\hbar} (t - t_o) \hat{E}[A^a] \right] \) is a unitary operator, also Eq.(3.17) defines a unitary transformation.

By using this result it can be shown that the values taken by a solution (3.17) in two different points \( (t_1, \vec{A}_{1}^a(\vec{\sigma})) \), \( (t_2, \vec{A}_{2}^a(\vec{\sigma})) \), of the space of the generalized times, are connected by a unitary transformation

\[
\Psi (\vec{\eta}, t_1, \vec{A}_1^a) = \exp \left[ -\frac{i}{\hbar} (t_2 - t_1) \hat{E}[A^a] \right] \cdot U'[A_2^a] \cdot U'^+[A_1^a] \cdot \Psi (\vec{\eta}, t_1, \vec{A}_1^a). \quad (3.21)
\]

C. Definition of a Non-Inertial Frame

In the classical theory we select a non-inertial frame by fixing the gauge variables \( \vec{A}(t, \vec{\sigma}) \) as in Eqs.(2.24). At the quantum level, this can be realized by defining a path

\[
\mathcal{P}_F(t) = (t, \vec{A}_F^a(t, \vec{\sigma})), \quad (3.22)
\]
in the space of generalized times \( \mathcal{M} \).

We assume the following point of view. Each physical state is represented by the \textit{generalized wave function} (3.17). The observer adopting a set of non-inertial coordinates \( \vec{\sigma} \), implicitly defined by the coordinates transformation \( \vec{A}_F^a(t, \vec{\sigma}) \), will describe the state by means of the wave function \( \Psi (\vec{\eta}, t, \vec{A}^a) \) evaluated along the path \( \mathcal{P}_F(t) \).
\[ \psi_F(t, \vec{\eta}) = \Psi(\vec{\eta}, t; A_F^a(t)). \] (3.23)

Since we have
\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_F(t, \vec{\eta}) = i \hbar \left[ \frac{\partial \Psi}{\partial t}(\vec{\eta}, t; A_F^a(t)) + \right.

+ i \hbar \int d^3\sigma \frac{\partial A_F^a(t, \vec{\sigma})}{\partial t} \left[ \frac{\delta \Psi}{\delta A^a(\vec{\sigma})} \right](\vec{\eta}, t; A_F^a(t)) \right],
\] (3.24)

we see that Eqs.(3.9) and (3.10) imply the following non-inertial Schrödinger equation along the path \( \mathcal{P}_F(t) \) in the space of generalized times
\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_F(t, \vec{\eta}) = \left[ \hat{H}_{ni} + i \hbar \sum_{i=1}^N \left( V_F^r(t, \vec{\eta}) \frac{\partial}{\partial \eta^i} + \frac{1}{2} \frac{\partial V_F^r(t, \vec{\eta})}{\partial \eta^i} \right) \right] \psi_F(t, \vec{\eta}) =
\]
\[ \stackrel{\text{def}}{=} \hat{H}_{ni} \cdot \psi_F(t, \vec{\eta}). \] (3.25)

The non-inertial Hamiltonian operator \( \hat{H}_{ni} \) is just the quantized version of the non-inertial Hamiltonian of Eq.(2.27).

For each value \( t \), \( \psi_F(t, \vec{\eta}) \) is a state in the Hilbert space \( \mathbf{H} \). The \( t \)-dependent non-inertial Hamiltonian defined in Eq.(3.25) is self-adjoint and the \( t \)-evolution along the path defines a unitary transformation.

Indeed let \( (t_1, A_F^a(\vec{\sigma}) = A_F^a(t_1, \vec{\sigma}), (t_2, A_F^a(\vec{\sigma}) = A_F^a(t_2, \vec{\sigma})) \), be the extremal points of a path in the generalized times space. Then Eq.(3.21) implies that \( \psi_F(t_1, \vec{\eta}) \) and \( \psi_F(t_2, \vec{\eta}) \) are connected by a unitary transformations. Moreover this transformation does not depend on the path but only on its extremal points.

We want show that the non-inertial wave function \( \psi_F(t, \vec{\eta}) \) can be obtained by a time-dependent unitary transformation from the wave function \( \psi_{in}(t, \vec{\eta}) \) defined in an inertial frame. To show it, we choose another path \( \mathcal{P}_{in}(t) = (t, A_{in}^a(t, \vec{\sigma})) \), where \( A_{in}^a(t, \vec{\sigma}) \equiv \sigma^a \). This path defines the inertial frame with inertial coordinates \( x^a = \sigma^a \). The wave function restricted to this path is the wave function in the inertial reference frame
\[
\psi_{in}(t, \vec{\eta}) = \Psi(\vec{\eta}, t, A_{in}^a). \] (3.26)

Moreover in this case Eq.(3.25) becomes the usual inertial Schrödinger equation.
\[ i\hbar \frac{\partial \psi_{in}}{\partial t}(t, \vec{\eta}_i) = \hat{E}_{in} \cdot \psi_{in}(t, \vec{\eta}_i), \]

\[ \hat{E}_{in} = \hat{E}[\mathcal{A}_{in}^a] = -\hbar^2 \sum_{i,a} \frac{1}{2m_i} \left( \frac{\partial}{\partial \eta_{i}^a} \right)^2 + V(t, \vec{\eta}_1, ..., \vec{\eta}_N), \quad (3.27) \]

and then we get

\[ \psi_{in}(t, \vec{\eta}_i) = \exp \left[ -\frac{i}{\hbar} (t - t_o) \hat{E}_{in} \right] \cdot \psi_{o,in}(t, \vec{\eta}_i). \quad (3.28) \]

The two different observers using different sets of coordinates, the inertial one defined by the path \( \mathcal{P}_{in}(t) \) and the non-inertial one defined by a path \( \mathcal{P}_F(t) \), describe the same physical system by evaluating the same \( \Psi(\vec{\eta}_i, T, \mathcal{A}^a) \) on different paths \( \mathcal{P}_{in}(t), \mathcal{P}_F(t) \), so that they obtain two different wave functions \( \psi_{in}(t, \vec{\eta}_i), \psi_F(t, \vec{\eta}_i) \) in \( \mathbf{H} \). However, there exist a time dependent unitary transformation mapping \( \psi_{in} \) to \( \psi_F \). To show this we can use the general solution of Eqs.(3.18) and (3.17) to write \( \psi_F \) in terms of \( \psi_{in} \).

1) In Eq.(3.15) let it be \( \Psi_o(\vec{\eta}_i) = \psi_{o,in}(\vec{\eta}_i) \). Then, using the definitions (3.11) and (3.27) and Eq.(3.18), it is only matter of tedious calculation to show that, for each \( \mathcal{A}^a(\vec{\sigma}) \), we get

\[ \hat{E}[\mathcal{A}^a] \cdot \Psi'(\vec{\eta}_i; \mathcal{A}^a) = \sqrt{\prod_i \det J(\vec{\eta}_i)} \left[ \hat{E}_{in} \cdot \psi_{o,in}(\vec{\eta}_i) \right] \left( \vec{\mathcal{A}}(\vec{\eta}_i) \right) = U[\mathcal{A}^a] \cdot \left[ \hat{E}_{in} \cdot \Psi_{o,in}(\vec{\eta}_i) \right]. \quad (3.29) \]

Then, using the result (3.29) and Eq.(3.28), Eqs.(3.17) become

\[ \Psi(\vec{\eta}_i, t, \mathcal{A}^a) = \sqrt{\prod_i \det J(\vec{\eta}_i) \psi_{in}(t, \vec{\mathcal{A}}(\vec{\eta}_i))}. \quad (3.30) \]

2) Finally, we can evaluate this general solution on the path \( \mathcal{P}_F(t) \) and we obtain the searched result

\[ \psi_F(t, \vec{\eta}_i) = \sqrt{\prod_i \det J_F(t, \vec{\eta}_i) \psi_{in}(t, \vec{\mathcal{A}}_F(t, \vec{\eta}_i))} \overset{\text{def}}{=} U_F(t) \cdot \psi_{in}(t, \vec{\eta}_i). \quad (3.31) \]

This relation defines the time dependent unitary transformation \( U_F(t) \). The unitarity can be checked by making the change of variables \( \vec{\eta}_i' = \vec{\mathcal{A}}_F(t, \vec{\eta}_i) \) as in the transformation (3.18)

\[ \int \left( \prod_i d^3\eta_i \right) \bar{\psi}_{1,in}(t, \vec{\eta}_i) \psi_{2,in}(t, \vec{\eta}_i') = \int \left( \prod_i d^3\eta_i \right) \bar{\psi}_{1,F}(t, \vec{\eta}_i) \psi_{2,F}(t, \vec{\eta}_i). \quad (3.32) \]
Using these results we can discuss Eq.(3.25) from a new point of view. By using the form (3.31) for $\psi_F$ we can rewrite

$$\hat{H}_{ni}(t) = U_F(t) \cdot \hat{E}_{in} \cdot U_F^+(t) - U_F(t) \frac{dU_F^+(t)}{dt},$$

(3.33)

where, as a consequence of Eq.(3.29), $\hat{E}[A_F^r] = U_F(t) \cdot \hat{E}_{in} \cdot U_F^+(t)$ is the energy in a inertial reference frame written in non-inertial coordinates and where

$$-U_F(t) \frac{dU_F^+(t)}{dt} = +i\hbar \sum_{i=1}^{N} \left( V_F^r(t, \vec{\eta}_i) \frac{\partial}{\partial \eta^r_i} + \frac{1}{2} \frac{\partial V_F^r(t, \vec{\eta}_i)}{\partial \eta^r_i} \right),$$

(3.34)

is the quantum inertial potential.
IV. RIGID NON-INERTIAL FRAMES

In the previous Sections we worked with a very general definition of non-inertial coordinates for non-relativistic mechanics. Nevertheless, in a non-relativistic context rigidly linear accelerated or rotating frames are usually used. The traditional approach defines these non-inertial rigid frames by introducing a time-dependent basis of unit vectors \( \mathbf{b}_1(t), \mathbf{b}_2(t), \mathbf{b}_3(t) \) connected by a time-dependent rotation to the fixed basis \( \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3 \)

\[
\mathbf{b}_r(t) = R_{ra}(t) \mathbf{i}_a,
\]

which is placed on a moving origin, whose inertial coordinates are \( y^a(t) \). The corresponding cartesian coordinates are the non-inertial (rigid) coordinates \( \sigma^r \). This class of reference frames is obtained in our approach with the choice

\[
\mathcal{A}_F^a(t, \sigma) = y^a(t) + \sigma^r R_{ra}(t).
\]

The time dependent rotation \( R(t) \) can be expressed in terms of time dependent Euler’s angles \( \alpha(t), \beta(t), \gamma(t) \). Following the convention of Ref.[9] the explicit form of the rotation is

\[
R(\alpha, \beta, \gamma) =
\begin{pmatrix}
\cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & - \sin \beta \cos \gamma \\
- \cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\
\cos \alpha \sin \beta & - \sin \alpha \sin \beta & \cos \beta
\end{pmatrix}.
\]

Since we have

\[
\Omega_{rs}(t) = \left( \frac{dR(t)}{dt} R^T(t) \right)_{rs} = -\Omega_{sr}(t),
\]

we can define the angular velocity

\[
\omega^a(t) = \frac{1}{2} \varepsilon^{rsu} \Omega_{su}(t).
\]

Moreover by defining

\[
v^r(t) = R_{ra}(t) \frac{dy^a(t)}{dt},
\]
we obtain

\[ V_{F}^{r}(t, \vec{\sigma}) = v^{r}(t) + \left[ \vec{\omega}(t) \times \vec{\sigma} \right]^{r}. \]  

(4.6)

**A. Rigid Non-Inertial Frames: the Classical Case**

Using the previous results in classical Hamiltonian (2.27) and defining the components of the total momentum and of the total angular momentum along the axis of the rotating frames

\[ P^{r}(t) = \sum_{i=1}^{N} p_{i}^{r}(t), \quad J^{r}(t) = \sum_{i=1}^{N} \left( \vec{\eta}_{i}(t) \times \vec{p}_{i}(t) \right)^{r}, \]  

(4.7)

we obtain the classical non-inertial Hamiltonian for rigid non-inertial frames

\[ H_{ni}(t) = \sum_{i} \frac{\hat{p}_{i}^{\sigma}(t)}{2m_{i}} + V - \vec{v}(t) \cdot \vec{P}(t) - \vec{\omega}(t) \cdot \vec{J}(t). \]  

(4.8)

It can be shown that the Hamilton equations obtained using the Hamiltonian (4.8) imply

\[ \frac{d^{2} \vec{\eta}_{i}(t)}{dt^{2}} = - \left( \frac{d\vec{v}(t)}{dt} + \vec{\omega}(t) \times \vec{v}(t) \right) - \frac{d\vec{\omega}(t)}{dt} \times \vec{\eta}_{i}(t) + \]

\[ - 2\vec{\omega}(t) \times \frac{d\vec{\eta}_{i}(t)}{dt} - \left( \vec{\omega}(t) \times (\vec{\omega}(t) \times \vec{\eta}_{i}(t)) \right) - \frac{1}{m_{i}} \nabla_{\eta_{i}} V. \]  

(4.9)

These are the standard equations of motion of particles in a non-inertial rigid reference frame: the four terms in the second member of the second equation are the standard Euler, Jacobi, Coriolis and centrifugal forces, respectively.

**B. Rigid Non-Inertial Frames: the Quantum Case**

Using the results (4.6) in the quantum non-inertial Hamiltonian (3.25), we obtain the quantum Hamiltonian for rigid non-inertial frames

\[ \hat{H}_{ni}(t) = \hat{E}[A_{F}^{a}] + \frac{\hbar}{2} \left[ \vec{v}(t) \cdot \sum_{i} \nabla_{m_{i}} + \vec{\omega}(t) \cdot \sum_{i} (\vec{\eta}_{i} \times \nabla_{m_{i}}) \right]. \]  

(4.10)

In this rigid case it is useful to show that the explicit form of \( \hat{E}[A_{F}^{a}] \) is
\[ \hat{E}[\mathcal{A}_F^a] = \sum_{i,r} \frac{\hbar^2}{2m_i} \left( \frac{\partial}{\partial \eta_i^r} \right) + \mathbf{V} \left( t, \mathcal{A}_F^a(t, \vec{\eta}), \ldots, \mathcal{A}_F^a(t, \vec{\eta}_N) \right). \]  

Each solution \( \psi_F(t, \vec{\eta}_i) \) of the non-inertial Schroedinger equation with Hamiltonian (4.10) can be obtained by using Eq.(4.2) in the general solution (3.31). If we observe that in this case we have \( \det J(t, \vec{\sigma}) = \det R(t) = 1 \), we obtain

\[ \psi_F(t, \vec{\eta}_i) = \psi_{in}\left( t, y^a(t) + \eta_i^r R_{ra}(t) \right) = \mathcal{U}_T(t) \cdot \mathcal{U}_R(t) \cdot \Psi_{in}(t, \vec{\eta}). \]  

Here we used the following time-dependent translations and rotations

\[ \mathcal{U}_T(t) = \exp \left( \frac{i}{\hbar} \sum_{a,r} y^a(t) \hat{P}^r R_{ra}(t) \right), \]

\[ \mathcal{U}_R(t) = \exp \left( \frac{i}{\hbar} \gamma(t) \hat{J}^3 \right) \exp \left( \frac{i}{\hbar} \beta(t) \hat{J}^2 \right) \exp \left( \frac{i}{\hbar} \alpha(t) \hat{J}^1 \right), \]

where

\[ \hat{P}^r = -i\hbar \sum_i \frac{\partial}{\partial \eta_i^r}, \quad \hat{J}^r = -i\hbar (\vec{\eta} \times \nabla)\eta_i^r, \]

are the quantum total momentum and total angular momentum.

C. Arbitrary Phase Factor and Comparison with other Approaches

In the non-inertial wave function we can add an arbitrary phase factor. In other terms a non-inertial observer can choose to represent a physical state with a wave function

\[ \psi_F'(t, \vec{\eta}) = \exp \left( \frac{i}{\hbar} \Lambda(t, \vec{\eta}) \right) \psi_F(t, \vec{\eta}). \]  

At the classical level this correspond to a time-dependent canonical transformation

\[ p_{ir} \longrightarrow p'_{ir} = p_{ir} + \frac{\partial \Lambda(t, \vec{\eta})}{\partial \eta_i^r}. \]  

This freedom is useful especially in the rigid case (4.2). As discussed in Appendix A of Ref.[10], in this case a convenient choice is \( (\vec{X} = \sum_i m_i \vec{\eta}_i / M, \ M = \sum_i m_i) \)

\[ \frac{i}{\hbar} \Lambda(t, \vec{\eta}) = -\frac{i}{\hbar} M \vec{X} \cdot \vec{v}(t) + F(t). \]  

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To add the phase factor is equivalent to do a new time-dependent unitary transformation. For example the phase (4.17) is equivalent to the Galileo boost

\[ u_B(t, F) = \exp \left( -\frac{i}{\hbar} M \vec{X} \cdot \vec{v}(t) + F(t) \right). \]  

(4.18)

At the classical level, the corresponding canonical transformation maps the momenta \( p_{ir}' \)'s in boosted ones

\[ p_{ir}' = p_{ir} - m_i v^r(t). \]  

(4.19)

In the rigid case (4.2), the wave function (4.15), with the choice (4.17), satisfies the Schrödinger equation with Hamiltonian

\[ \hat{H}_{ni}'(t) = \hat{E}[A^a_a] + \vec{a}(t) \cdot \sum_i m_i \vec{\eta}_i + i\hbar \vec{\omega}(t) \cdot \sum_i \vec{\eta}_i \times \nabla_{\vec{\eta}} + \left( i\hbar \dot{F}(t) - \frac{1}{2} M \vec{v}^2(t) \right), \]  

(4.20)

where now the acceleration

\[ a^r(t) = R_{ra}(t) \frac{d^2 y^a(t)}{dt^2}, \]  

(4.21)

gives a momentum-independent potential \( +\vec{a}(t) \cdot \sum_i m_i \vec{\eta}_i \). Using Eqs.(4.12), a solution for \( \psi'_F \) can be expressed in the form

\[ \psi'_F(t, \vec{\eta}) = e^{i\Lambda(t, \vec{\eta})} u_T(t) \cdot u_R(t) \cdot \psi_{in}(t, \vec{\eta}) \overset{def}{=} \tilde{u}_T[y^a(t), F] \cdot u_R(t) \cdot \psi_{in}(t, \vec{\eta}), \]  

(4.22)

where we have defined

\[ \tilde{u}_T[y^a(t), F] \overset{def}{=} e^{i\Lambda(t, \vec{\eta})} u_T(t). \]  

(4.23)

If in Eq.(4.17) we choose \( F(t) = M \vec{y}^a(t) \cdot \dot{\vec{y}}(t) \), the result (4.22) reproduces the approach of Ref.[11]. In fact the choice done for \( F \) is such that the \( \tilde{u}_T[y^a(t), F]'s \) give the projective representation of the acceleration group used in Ref.[11]

\[ \tilde{u}_T[y^a_1(t), F] \cdot \tilde{u}_T[y^a_2(t), F] = \exp \left( -\frac{i}{\hbar} M \vec{y}_1(t) \cdot \vec{y}_2(t) \right) \tilde{u}_T[y^a_1(t) + y^a_2(t), F]. \]  

(4.24)

Instead in Ref.[12] the sequence of time-dependent unitary transformations \( u_B(t) \cdot u_T(t) \cdot u_R(t) \) is used to construct quantum mechanics in non-inertial (rigid) frames in the Heisenberg picture. It can be shown that the result (4.22) is equivalent to the approach of Ref.[12] with a mapping from the Heisenberg picture to the Schrödinger picture.
V. CENTER OF MASS, RELATIVE VARIABLES AND BOUND STATES

In this Section we discuss the case of two particles mutually interacting with time-independent, rotationally invariant, potential, whose form in an inertial frame is

\[ V = V(|\vec{x}_1 - \vec{x}_2|). \] (5.1)

A. Center of Mass and Relative Variables

Using Eq.(4.2), the interaction potential (5.1) can be expressed in a rigid non-inertial frame in the simple form

\[ V = V(|\vec{A}_F(t, \eta_1) - \vec{A}_F(t, \eta_2)|) = V(|\eta_1 - \eta_2|). \] (5.2)

In the rigid non-inertial frame defined by Eq.(4.2) we can use a standard separation between center of mass and relative coordinates

\[ X^r = \frac{m_1}{M} \eta^r_1 + \frac{m_2}{M} \eta^r_2, \quad \rho^r = \eta^r_1 - \eta^r_2, \] (5.3)

where we introduced the total mass \( M = m_1 + m_2 \). The reduced mass is \( \mu = m_1 m_2 / M \). With standard methods we can obtain the following expressions

\[ \hat{P}^r = -i\hbar \left( \frac{\partial}{\partial \eta^r_1} + \frac{\partial}{\partial \eta^r_2} \right) = -i\hbar \frac{\partial}{\partial X^r} = -i\hbar \nabla^r_X, \] (5.4)

\[ \hat{J}^r = \hat{L}^r + \hat{S}^r, \]

\[ \hat{L}^r = -i\hbar (\vec{X} \times \nabla_X)^r, \quad \hat{S}^r = -i\hbar (\vec{\rho} \times \nabla_\rho)^r. \] (5.5)

The Hamiltonian (4.10) can be written as the sum of a center of mass Hamiltonian and a relative Hamiltonian.
\[ \hat{H}_{ni} = \hat{H}_{ni,cm} + \hat{H}_{ni,rel}, \]

\[ \hat{H}_{ni,cm} = -\frac{\hbar^2}{2M} \nabla_X^2 + i\hbar \vec{v}(t) \cdot \nabla X + i\hbar \vec{\omega}(t) \cdot (\vec{X} \times \nabla X), \quad (5.6) \]

\[ \hat{H}_{ni,rel} = -\frac{\hbar^2}{2\mu} \nabla_\varrho^2 + V(\varrho) + i\hbar \vec{\omega}(t) \cdot (\vec{\varrho} \times \nabla \varrho). \quad (5.7) \]

As a consequence, there exist solutions of the non-inertial Schroedinger equation with Hamiltonian (4.10) factorized in center of mass and relative parts \( \psi_F(t, \vec{\eta}_1, \vec{\eta}_2) = \psi_{F,cm}(t, \vec{X}) \psi_{F,rel}(t, \vec{\varrho}) \). If we make the same separation in the inertial energy of Eq.(3.27), we obtain

\[ \hat{E}_{in} = \hat{E}_{in,cm} + \hat{E}_{in,rel}, \]

\[ \hat{E}_{in,cm} = -\frac{\hbar^2}{2M} \nabla_X^2, \quad (5.8) \]

\[ \hat{E}_{in,rel} = -\frac{\hbar^2}{2\mu} \nabla_\varrho^2 + V(\varrho). \quad (5.9) \]

The corresponding inertial wave function is \( \psi_{in}(t, \vec{\eta}_1, \vec{\eta}_2) = \tilde{\psi}(t, \vec{X}, \vec{\varrho}) = \psi_{in,cm}(t, \vec{X}) \psi_{in,rel}(t, \vec{\varrho}) \). We can observe that Eqs.(4.12) can be rewritten in the factorized form

\[ \psi_{F,cm}(t, \vec{X}) = \mathcal{U}_T(t) \cdot \mathcal{U}_L(t) \psi_{in,cm}(t, \vec{X}), \quad (5.10) \]

\[ \psi_{F,rel}(t, \vec{\varrho}) = \mathcal{U}_S(t) \psi_{in,rel}(t, \vec{\varrho}), \quad (5.11) \]

where we used the following definitions

\[ \mathcal{U}_L(t) = \exp \left( \frac{i}{\hbar} \gamma(t) \hat{L}^3 \right) \exp \left( \frac{i}{\hbar} \beta(t) \hat{L}^2 \right) \exp \left( \frac{i}{\hbar} \alpha(t) \hat{L}^3 \right), \quad (5.12) \]

\[ \mathcal{U}_S(t) = \exp \left( \frac{i}{\hbar} \gamma(t) \hat{S}^3 \right) \exp \left( \frac{i}{\hbar} \beta(t) \hat{S}^2 \right) \exp \left( \frac{i}{\hbar} \alpha(t) \hat{S}^3 \right). \quad (5.13) \]
We can observe that also the Hamiltonian (4.20) can be rewritten as the sum of center of mass part and relative parts

\[ \hat{H}'_{ni}(t) = \hat{H}'_{ni,cm}(t) + \hat{H}_{ni,rel}(t), \]  

where

\[ \hat{H}'_{ni,cm}(t) = -\frac{\hbar^2}{2M} \nabla_X^2 + M \vec{a} \cdot \vec{X} + i\hbar \vec{X} \times \nabla_X + \left( i\hbar \vec{F}(t) - \frac{1}{2}M \vec{v}^2(t) \right), \]

and where \( \hat{H}_{ni,rel}(t) \) is the same of Eq.(5.7).

On the contrary, we cannot obtain the center of mass and relative variable factorization if we use non-rigid non-inertial coordinates. At the classical level the best we can do is to observe that we can return to coordinates in the inertial frame by a point canonical transformation

\[ \eta'^a_i = A^a_F(t, \vec{\eta}_i), \quad p'^i_{ia} = \tilde{J}^r_{Fa}(t, \vec{\eta}_i) p^{ir}, \]

and we can apply center of mass and relative variable transformation to these inertial coordinates by a second canonical transformation

\[ \bar{\rho} = \vec{\eta}'_1 - \vec{\eta}'_2, \quad \bar{X} = \frac{m_1}{M} \vec{\eta}'_1 + \frac{m_2}{M} \vec{\eta}'_2, \]

\[ \bar{\pi} = \frac{m_2}{M} \vec{p}'_1 - \frac{m_1}{M} \vec{p}'_2, \quad \bar{P} = \vec{p}'_1 + \vec{p}'_2. \]

The inverse total canonical transformation allows to define non-inertial non-rigid notion of center of mass and relative variables.

Contrary to I, where a quantum implementation of this classical approach is very complex, in this non-relativistic case the inverse total canonical transformation can be implemented with simple observations. Indeed the inverse of Eq.(5.17) is implemented by a coordinates transformation on the inertial wave function written in terms of the relative and center of mass coordinates \( \psi_{in}(t, \vec{\eta}_1, \vec{\eta}_2) = \tilde{\psi}_{in} \left( t, \bar{X}(\vec{\eta}_1, \vec{\eta}_2), \bar{\rho}(\vec{\eta}_1, \vec{\eta}_2) \right) \). This implies that the energy operators have to be rewritten as
\[ E_{in,cm} = -\frac{\hbar^2}{2M} \sum_{a,i} \left( \frac{\partial}{\partial \eta_1^a} + \frac{\partial}{\partial \eta_2^a} \right)^2, \]
\[ E_{in,rel} = -\frac{\hbar^2}{2\mu} \sum_{a,i} \left( \frac{m_2}{M} \frac{\partial}{\partial \eta_1^a} - \frac{m_1}{M} \frac{\partial}{\partial \eta_2^a} \right)^2 + V \left( | \vec{\eta}_1^a - \vec{\eta}_2^a | \right). \] (5.18)

Then we can apply the time dependent transformation \( U_F(t) \) implementing the inverse of the canonical transformation (5.16). In particular we obtain the following form of the non-inertial Hamiltonian

\[ \hat{H}_{ni} = \hat{E}_{cm}[A_F^a] + \hat{E}_{rel}[A_F^a] - U_F(t) \frac{dU_F^+(t)}{dt}, \]
\[ \hat{E}_{cm}[A_F^a] \overset{def}{=} U_F(t) \cdot \hat{E}_{in,cm} \cdot U_F^+(t), \] (5.19)
\[ \hat{E}_{rel}[A_F^a] \overset{def}{=} U_F(t) \cdot \hat{E}_{in,rel} \cdot U_F^+(t), \] (5.20)

where in the quantum inertial potentials the center of mass and relative variables remain mixed. As a consequence the non-inertial wave function cannot be factorized in general non-rigid non-inertial frames.

**B. Bound States**

In inertial frames bounds states are defined looking for stationary solutions of the Schrodinger equation for the relative wave function

\[ i\hbar \frac{d}{dt} \psi_{in,rel}(t, \vec{\varphi}) = \hat{E}_{in,rel} \cdot \psi_{in,rel}(t, \vec{\varphi}), \] (5.21)

whereas the center of mass wave function is a solution of the equation

\[ i\hbar \frac{d}{dt} \psi_{in,cm}(t, \vec{X}) = \hat{E}_{in,cm} \cdot \psi_{in,cm}(t, \vec{X}). \] (5.22)

The stationary solutions of Eq.(5.21) have the form

\[ \psi_{in,rel}^{(n)}(t, \vec{\varphi}) = \exp \left( \frac{i}{\hbar} B_n t \right) \phi_{in}^{(n)}(\vec{\varphi}), \] (5.23)
where the $\phi^{(n)}_\text{in}(\vec{\vartheta})$ are a complete solution of the eigenvalue problem

$$\hat{E}_{\text{in},\text{rel}} \cdot \phi^{(n)}_\text{in}(\vec{\vartheta}) = -B_n \phi^{(n)}_\text{in}(\vec{\vartheta}).$$

(5.24)

If $-B_n$ is an eigenvalue in the discrete spectrum ($B_n > 0$), this solution defines a bound state.

Using Eq. (3.31), the inertial wave function

$$\psi^{(n)}_\text{in}(t, \vec{\eta}_1, \vec{\eta}_2) = \psi^{(n)}_\text{in,cm}(t, \vec{X}) \psi^{(n)}_\text{in,rel}(t, \vec{\vartheta}),$$

(5.25)

can be mapped in the corresponding non-inertial wave function

$$\psi^{(n)}_F(t, \vec{\eta}_1, \vec{\eta}_2) = \mathcal{U}_F(t) \cdot \psi^{(n)}_\text{in}(t, \vec{\eta}_1, \vec{\eta}_2),$$

(5.26)

so that Eq. (5.20) implies

$$\hat{E}_{\text{rel}}[\mathcal{A}^a] \cdot \psi^{(n)}_F(t, \vec{\eta}_1, \vec{\eta}_2) = -B_n \psi^{(n)}_F(t, \vec{\eta}_1, \vec{\eta}_2).$$

(5.27)

Then, there exist solutions of the non-inertial Schrödinger equations that are eigenfunctions of the discrete spectrum of the operator corresponding to the internal energy. These solutions correspond to bound states defined in inertial frames and they can be interpreted naturally as the bound states in every non-inertial (non-rigid) frame.

The previous observations are valid in every non-rigid non-inertial frame, but only for rigid non-inertial frames, the non-inertial wave function can still be factorized in center of mass and relative parts. In fact only in this particular case we can map the factorized inertial solutions $\psi^{(n)}_\text{in,cm}(t, \vec{X})$ and $\phi^{(n)}_\text{in}(\vec{\vartheta})$ in the non-inertial ones using Eqs. (5.11). In this case we get

$$\psi^{(n)}_F(t, \vec{\vartheta}) = \mathcal{U}_S(t) \cdot \psi^{(n)}_\text{in,rel}(t, \vec{\vartheta}) \Rightarrow \hat{E}_{\text{in,rel}} \cdot \psi^{(n)}_F(t, \vec{\vartheta}) = -B_n \psi^{(n)}_F(t, \vec{\vartheta}).$$

(5.28)

In rigid frames, where a relative non-inertial Hamiltonian $\hat{H}_{\text{ni,rel}}(t)$, different from the relative energy $\hat{E}_{\text{rel}}[\mathcal{A}^a]$, exists, we could look for a non-inertial definition of bound states, independent by the inertial one. For instance we could look for a stationary solution of the non-inertial Schrödinger equation for the relative non-inertial wave function

$$i\hbar \frac{d}{dt} \tilde{\psi}_{F,\text{rel}}(t, \vec{\vartheta}) = \hat{H}_{\text{ni,rel}}(t) \cdot \tilde{\psi}_{F,\text{rel}}(t, \vec{\vartheta}),$$

(5.29)
whereas the center of mass wave function is a solution of one of the non-inertial Schrödinger equations with one of the center of mass non-inertial Hamiltonians defined in the previous Sections. Stationary solutions must have the form

$$\tilde{\psi}^{(n)}_{F,\text{rel}}(t, \vec{\varrho}) = \exp \left( + \frac{i}{\hbar} \int_{t_1}^{t} dt_1 h_n(t_1) \right) \tilde{\phi}^{(n)}_{F,\text{rel}}(\vec{\varrho}),$$

(5.30)

where the $\tilde{\phi}^{(n)}_{F,\text{rel}}(\vec{\varrho})$ are solutions of the eigenvalue problem

$$\hat{H}_{ni,\text{rel}}(t) \cdot \tilde{\phi}^{(n)}_{F,\text{rel}}(\vec{\varrho}) = -h_n(t) \tilde{\phi}^{(n)}_{F,\text{rel}}(\vec{\varrho}).$$

(5.31)

Eq.(5.31) is equivalent to a system of infinite eigenvalues problems (one for each instant $"t"$). But we can show that, in general, these eigenvalue problems are not simultaneously solvable. In fact if we take the eigenvalue problems for two different times $t_1$ and $t_2$, we have

$$[\hat{H}_{ni,\text{rel}}(t_1), \hat{H}_{ni,\text{rel}}(t_2)] = -i\hbar \vec{\omega}(t_1) \times \vec{\omega}(t_2) \cdot (\vec{\varrho} \times \nabla_{\varrho}) \neq 0.$$  

(5.32)

Then we cannot define non-inertial bound states looking for stationary solution of the relative Schrödinger equation in general cases. However these stationary solutions can exist in some particular cases. The most important is the case where the rotating frame rotates around a fixed axis $\hat{n}$, with angular velocity $\vec{\omega}(t) = \omega(t)\hat{n}$. In this case, if we use the set of maximal observables $\hat{E}_{\text{rel}}, \hat{S}^2, \hat{S} \cdot \hat{n}$ to label the solutions of Eq.(5.28), they are also solutions of Eq.(5.31). In other terms we have $\tilde{\psi}^{(n)}_{F,\text{rel}} = \psi^{(n)}_{F,\text{rel}}$. Therefore we cannot obtain a really different definition of bound states.
VI. TWO EXAMPLES

In this Section we re-discuss two known cases where the use of a non-inertial frame is useful. In both of these cases the interaction potential has an explicit time-dependence in the inertial reference frame, so that it is more simple to study the non-inertial Schrödinger equation in a non-inertial frame where the interaction potential appears time-independent.

A. Cranking Model

The Cranking Model [13], [14], [15], [16] is a model used in nuclear physics to study the properties of rapidly rotating non-spherical nuclei. Following Ref. [15], in this model it is assumed that the motion of each nucleon in a non-spherical (non rotating) nucleus is determined by an average potential $V_M(\vec{x})$ without spherical symmetries. Rapid rotations are described in a semiclassical manner introducing an active rotation of the potential $V_M(\vec{x})$ around an axis $\hat{n}$ that is not a symmetry axis of the potential. In the following we assume $\hat{n} = \hat{i}_3$. The Hamiltonian in the inertial (laboratory) reference frame of a rotating nucleus described with this model is then

$$\hat{E}_{in} = \sum_i \left[ -\frac{\hbar^2}{2m_i} \sum_a \left( \frac{\partial}{\partial \eta_i^a} \right)^2 + V_M(\vec{R}_{ra}(t)\eta_i^a) \right], \quad R(t) = R(0, 0, \omega t).$$

This means to assume an explicit time-dependent potential in the inertial frame

$$V(t, \vec{x}_1, ..., \vec{x}_N) = \sum_i V_M(\vec{R}_{ra}(t)x_i^a).$$

It is convenient to study the system described by the Hamiltonian (6.1) in a rotating reference frame defined in our notation by the relation

$$A_F^a(t, \vec{\sigma}) = \sigma^r R_{ra}(t).$$

In this frame we must use the non-inertial Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi_F(t, \vec{\eta}_1, ..., \vec{\eta}_N) = \hat{H}_{ni}(\omega) \cdot \dot{\psi}_F(t, \vec{\eta}_1, ..., \vec{\eta}_N),$$

where $\hat{H}_{ni}(\omega)$ is the time-independent non-inertial Hamiltonian

$$\hat{H}_{ni}(\omega) = \sum_i \left[ -\frac{\hbar^2}{2m_i} \sum_r \left( \frac{\partial}{\partial \eta_i^r} \right)^2 + V_M(\vec{\eta}_i) \right] - \omega \hat{J}^3,$$
where, in accord with Eq.(2.11), we have

\[ V(t, \vec{A}_1(t, \vec{\eta}_1), ..., \vec{A}_{N}(t, \vec{\eta}_N)) = \sum_i V_M(\vec{\eta}_i). \]  

(6.6)

As stressed in Ref.[15], the useful quantity is the average value of energy in the inertial laboratory frame. In our notation (see the comment on the definition (3.11)), this means the following evaluation

\[ \langle E \rangle = \left( \psi_F, \hat{E} \left[ A^a_F \right] \cdot \psi_F \right) = \left( \psi_F, \hat{H}_{nt}(\omega) \cdot \psi_F \right) + \omega \left( \psi_F, \hat{J}^3 \cdot \psi_F \right). \]  

(6.7)

**B. Non Inertial Effects in Interferometry with Material Waves**

The experimental results of an accelerated or rotating interferometer for material waves (usually neutrons) compared with the results of the same interferometer ”fixed” in an inertial (laboratory) frame can be interpreted with the presence of a phase shift (see Ref.[17] and its references).

Following the suggestion of Ref.[18] a particle in the fixed interferometer is described by the Hamiltonian

\[ \hat{E}_{in, fixed} = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial \eta^a} \right)^2 + V_I(\vec{\eta}). \]  

(6.8)

where \( V_I \) is the potential defined by the cristal in the interferometer. Since, in general, \( V_I \) has no symmetry, a particle in the same accelerated or rotating interferometer is described in the inertial (laboratory) frame by the Hamiltonian

\[ \hat{E}_{in} = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial \eta^a} \right)^2 + V_I \left( R_{ra}(t) \eta^a - \frac{1}{2} a^r t^2 \right). \]  

(6.9)

Again we have a time-dependent interaction potential in the inertial frame

\[ V(t, \vec{x}) = V_I \left( R_{ra}(t) x^a - \frac{1}{2} a^r t^2 \right). \]  

(6.10)

Then it is useful to study this system in the non-inertial frame

\[ \mathcal{A}^a(t, \vec{\sigma}) = \sigma^r R_{ra}(t) + \frac{a^r}{2} R_{ra}(t) t^2, \]  

(6.11)

where we can use the non-inertial Hamiltonian (4.20) with \( F(t) = (1/6) a^2 t^3 \).
\[ \hat{H}'_{ni}(a, \omega) = \hat{E}_{in, fixed} + m\vec{a} \cdot \vec{\eta} + i\hbar \vec{\omega}(t) \cdot (\vec{\eta} \times \nabla_{\eta}). \] (6.12)

The inertial potentials that make the Hamiltonian (6.12) different from the Hamiltonian (6.8) are used to calculate the observed phase shift [19].
VII. CONCLUSIONS.

The main result of this paper is the definition of a quantization scheme for non-relativistic particle systems in a sufficiently general class of non-rigid non-inertial frames, including the usual non-relativistic rigid ones with constant linear acceleration and angular velocity.

This quantization scheme includes as particular cases many of the previous results on quantization in non-inertial frames (usually limited to rigid cases). As a consequence also the phenomenological or experimental results based on these non-inertial quantum theories can be reformulated in the approach of this paper.

Moreover an original analysis of the definition of bound states in non-inertial frames is been presented. It turns out that non-inertial bound states are characterized by the same quantum numbers of the inertial ones being eigenstates of the inertial relative energy, rewritten in terms of non-inertial coordinates.
APPENDIX A: THE NON-RELATIVISTIC THEORY AS EXACT LIMIT OF THE RELATIVISTIC ONE.

In I, using Dirac’s approach [20], where a manifestly covariant Hamiltonian theory is obtained introducing an *admissible foliation* on Minkowski space-time, we have given a canonical description of a system of \( N \) free relativistic particles on a foliation of parallel hyper-planes.

To describe here this approach, first we have to introduce a *external* inertial frame in Minkowski space-time whose pseudo-cartesian coordinates are \( z^\mu \)'s. In such frame we have to introduce a tetrad of orthonormal four vectors, parametrized with a 3-vector \( \vec{\beta} \)

\[
U^\mu(\vec{\beta}) = \left(1; \frac{\beta^i}{\sqrt{1 - \vec{\beta}^2}}\right),
\]

\[
\epsilon^\mu_a(\vec{\beta}) = \left(\frac{\beta^a}{\sqrt{1 - \vec{\beta}^2}}; \delta^a_{i\beta} + \frac{\beta^i \beta^a}{\beta^2} \left(1 - \sqrt{1 - \vec{\beta}^2}\right)\right), \tag{A1}
\]

such that

\[
U_\mu(\vec{\beta}) U^\mu(\vec{\beta}) = 1, \quad U_\mu(\vec{\beta}) \epsilon^\mu_a(\vec{\beta}) = 0, \quad \epsilon^\mu_a(\vec{\beta}) \epsilon_{ab}(\vec{\beta}) = \eta_{ab}. \tag{A2}
\]

Then we define a foliation with parallel hyper-planes by introducing the embeddings

\[
z^\mu(\tau, \vec{\sigma}) = \theta(\tau) U^\mu(\vec{\beta}) + \epsilon^\mu_a(\vec{\beta}) A^a(\tau, \vec{\sigma}). \tag{A3}
\]

Each hyperplane is defined at \( \tau = \text{constant} \) and its points are identified by the curvilinear coordinates \( \sigma^* \), implicitly defined by the invertible coordinates transformation \( A^a(\tau, \vec{\sigma}) \). The parameter \( \tau \) takes the role of *mathematical time* and the function \( \theta(\tau) \) describes the freedom in the choice of this time.

In I it is shown how to construct a canonical theory where a system of \( N \) relativistic particles on the hyperplane (A3) is described on a phase space whose canonical pairs are:

i) The particles coordinates \( \eta^i_\tau(\tau) \) on the hyperplane at \( \tau \), such that the particle world-lines are

\[
x^\mu_i(\tau) = \theta(\tau) U^\mu(\vec{\beta}) + \epsilon^\mu_a(\vec{\beta}) A^a(\tau, \vec{\eta}_i(\tau)), \tag{A4}
\]

and their momenta \( \kappa_{ir}(\tau) \) such that
\[ \{ \eta_i^r(\tau), \kappa_{js}(\tau) \} = -\delta_{ij} \delta_r^s. \]  

(A5)

ii) The degrees of freedom, parametrizing the hyper-planes, \( \theta(\tau), A^a(\tau, \vec{\sigma}) \) and their momenta, \( M_U(\tau), \rho_a(\tau, \vec{\sigma}) \) such that

\[ \{ \theta(\tau), M_U(\tau) \} = -1, \quad \{ A^a(\tau, \vec{\sigma}), \rho_b(\tau, \vec{\sigma}^\prime) \} = -\delta^a_b \delta^3(\vec{\sigma} - \vec{\sigma}^\prime). \]  

(A6)

iii) The momentum-like parameter

\[ k^i = \frac{\beta^i}{\sqrt{1 - \vec{\beta}^2}}, \]

and their position-like conjugate canonical coordinate \( z^i \), such that

\[ \{ z^i, k^j \} = -\delta^{ij}. \]  

(A7)

On this phase space the dynamics is given by the Dirac Hamiltonian

\[ H_D(\tau) = \mu(\tau) H_U(\tau) + \int d^3\sigma \lambda^a(\tau) \mathcal{H}_a(\tau, \vec{\sigma}), \]  

(A8)

where we have used Dirac’s constraints

\[ H_U(\tau) = M_U(\tau) - c \sum_{i=1}^{N} \sqrt{m_i^2 c^2 + \sum_a \tilde{J}_a^i(\tau, \vec{\eta}_i) \kappa_{ir}(\tau) \tilde{J}_a^s(\tau, \vec{\eta}_i) \kappa_{is}(\tau) \approx 0, \]

\[ \mathcal{H}_a(\tau, \vec{\sigma}) = \rho_a(\tau, \vec{\sigma}) - \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \tilde{J}_a^i(\tau, \vec{\eta}_i) \kappa_{ir}(\tau) \approx 0. \]  

(A9)

These constraints tell us that the canonical variables \( \theta(\tau), A^a(\tau, \vec{\sigma}) \) are gauge variables. As shown in I and in Ref.[21] also in the relativistic case they are interpreted in terms of non-inertial frames.

We can see that the canonical coordinates \( k^i, z^i \) are trivially constant on the equations of motion generated by the Dirac Hamiltonian (A8). In the relativistic approach they are useful to have a manifestly Lorentz covariant canonical theory on hyper-planes. Since we want to find the non-relativistic limit, here we have not interested in manifest Lorentz covariance, and we can eliminate these canonical variables adding by hand a pair of second class Dirac constraints, which enforce these variables to take a constant value.
\[ k^i \approx \frac{\beta_o^i}{\sqrt{1 - \beta_o^2}} = \text{contant}, \]

\[ z^i \approx z_o^i = \text{constant}. \]  

(A10)

As discussed in I this step breaks the manifest covariance.

The exact non-relativistic limits \( c \to \infty \) is done by observing that we have

\[ c \beta_o = \bar{u} + \mathcal{O}(1/c), \]  

(A11)

so that we get

\[ U^\mu = \delta^\mu_o + \frac{1}{c} (0; \bar{u}) + \mathcal{O}(1/c^2), \]

\[ \epsilon_a^\mu = \delta_a^\mu + \frac{1}{c} \delta^\mu_o u^a + \mathcal{O}(1/c^2). \]  

(A12)

By defining \( T(\tau) = \theta(\tau)/c \), we arrive at the following expansion of the embedding

\[ z^o(\tau, \vec{\sigma}) = c T(\tau) + \mathcal{O}(1/c), \]

\[ z^i(\tau, \vec{\sigma}) = A^a_i(\tau, \vec{\sigma}) + u^i T(\tau) + \mathcal{O}(1/c). \]  

(A13)

By re-scaling the first equation with a \( c \)-factor \([t(\tau, \vec{\sigma}) = z^o(\tau, \vec{\sigma})/c]\) we obtain in the non-relativistic limit

\[ t(\tau, \vec{\sigma}) = T(\tau), \quad z^i(\tau, \vec{\sigma}) = A^i(\tau, \vec{\sigma}) + u^i T(\tau). \]  

(A14)

This result has the following interpretation: \( T(\tau) \) is the absolute Newtonian time, while the \( y^i = z^i(\tau, \vec{\sigma}) \)'s and the \( x^a = A^a(\tau, \vec{\sigma}) \)'s are the Cartesian orthogonal coordinates of two non-relativistic inertial systems with relative velocity \( \bar{u} = \text{constant} \) in the 3-dimensional absolute Newtonian space.

Let us define the limit of the other variables. The pairs \( A^a(\tau, \vec{\sigma}), \rho_a(\tau, \vec{\sigma}) \) and \( \eta^i(\tau), \kappa_{ir}(\tau) \) are left unchanged by the non-relativistic limit, while the pair \( \theta(\tau), M_U(\tau) \) is replaced by the pair \( T(\tau), K_U(\tau) \) with
\begin{equation}
K_U(\tau) = c M_U(\tau) - \sum_{i=1}^{N} m_i c^2, \quad (A15)
\end{equation}

such that

\begin{equation}
\{ K_U(\tau), T(\tau) \} = 1. \quad (A16)
\end{equation}

The form of the constraints \( \mathcal{H}_a(\tau, \vec{\sigma}) \approx 0 \) remains unchanged in the non-relativistic limit. Instead the constraint \( H_U(\tau) \approx 0 \) has the following expansion

\begin{equation}
H_U(\tau) = \frac{1}{c} \left[ K_U(\tau) - \sum_i \frac{1}{2m_i} \sum_a \tilde{J}_a^r(\tau, \vec{\eta}_i) \kappa_{ir}(\tau) \tilde{J}_a^s(\tau, \vec{\eta}_i) \kappa_{is}(\tau) \right] + \mathcal{O}(1/c^2) \approx 0, \quad (A17)
\end{equation}

and therefore it is replaced by the non-relativistic constraint

\begin{equation}
H_{U nr}(\tau) = K_U(\tau) - \sum_i \frac{1}{2m_i} \sum_a \tilde{J}_a^r(\tau, \vec{\eta}_i) \kappa_{ir}(\tau) \tilde{J}_a^s(\tau, \vec{\eta}_i) \kappa_{is}(\tau) \approx 0. \quad (A18)
\end{equation}

Moreover, we must make the following expansion of the Dirac multiplier \( \mu(\tau) \) in the Dirac Hamiltonian of Eq.(A8)

\begin{equation}
\mu(\tau) = c \rho(\tau) + \mathcal{O}(1/c), \quad (A19)
\end{equation}

if we want to get consistently

\begin{equation}
\frac{d\theta(\tau)}{d\tau} \overset{\circ}{=} -\mu(\tau), \quad \Rightarrow \quad \frac{dT(\tau)}{d\tau} \overset{\circ}{=} -\rho(\tau). \quad (A20)
\end{equation}

Therefore the non-relativistic Dirac Hamiltonian becomes

\begin{equation}
H_{D nr}(\tau) = \rho(\tau) H_{U nr}(\tau) + \int d^3\sigma \, \lambda^a(\tau, \vec{\sigma}) \mathcal{H}_a(\tau, \vec{\sigma}). \quad (A21)
\end{equation}

Finally, if we add the gauge fixing \( T(\tau) = \tau = t \) (implying \( \rho(\tau) = -1 \)) we have to substitute the Dirac Hamiltonian (A21) with the Dirac Hamiltonian (2.21). We have only to observe that in Subsection IIIB we have used the canonical momenta

\begin{equation}
p_{is} = -\kappa_{is}, \quad \rho^a(\vec{\sigma}) = -\rho_a(\vec{\sigma}),
\end{equation}

to have a non-relativistic sign conventions.
Following Ref. [22], we describe classical spin with Grassmann degrees of freedom \( \xi^a_i(t) \). We must add to the Lagrangian (2.13) the spin term

\[
L_{\text{spin}}(\tau) = i \sum_{i,a} \xi^a_i(t) \dot{\xi}^a_i(t). \tag{B1}
\]

The only consequence of the presence of the spin Grassmann degrees of freedom on the Hamiltonian formulation is the presence of the \( \xi^a_i(t) \)'s as canonical variables

\[
\{\xi^a_i(t), \xi^b_j(t)\} = i \delta_{ij} \delta^{ab}. \tag{B2}
\]

Moreover, the potential \( V \) can depend on the spin degrees of freedom. The canonical and Dirac Hamiltonians of Section IIIB are formally unchanged. We must observe that in our construction the \( \xi^a_i(t) \)'s represent the components of the spin of the \( i \)-th particle along the fixed axes of the inertial frame \((\hat{i}_1, \hat{i}_2, \hat{i}_3)\).

Canonical quantization maps Poisson bracket of Grassmann variables into anti-commutators. In our case this means to map the Grassmann variables \( \xi^a_i \) to Pauli matrices

\[
\xi^a_i(t) \mapsto \sqrt{\frac{\hbar}{2}} \sigma^a_i. \tag{B3}
\]

Now the wave functions live in the tensor product space of two-component spinors.

On a fixed non-inertial (non-rigid) frame, the components of the average value of the spin on the fixed axis \((\hat{i}_1, \hat{i}_2, \hat{i}_3)\) of the inertial frame is given by

\[
\langle s^a_i \rangle = \frac{1}{2} (\psi_F, \sigma^a_i \cdot \psi_F). \tag{B4}
\]

If we choose a rigid non-inertial frame, we can also project the spin of each particles along the axis of the rotating frame \((\hat{b}_1(t), \hat{b}_2(t), \hat{b}_3(t))\). To study this situation, we define first the time-dependent unitary operator

\[
U_{R,\text{spin}}(t) = \bigotimes_i \exp \left( \frac{i}{2} \gamma(t) \sigma^3_i \right) \exp \left( \frac{i}{2} \beta(t) \sigma^2_i \right) \exp \left( \frac{i}{2} \alpha(t) \sigma^3_i \right), \tag{B5}
\]

\(^5\) The construction of these Poisson brackets is done by using Dirac brackets to eliminate second class constraints on the conjugate momenta of the \( \xi^a_i \)'s.
and the wave function

\[ \psi_F'(t, \vec{\eta}_i) = U_{R,\text{spin}}(t) \cdot \psi_F(t, \vec{\eta}_i). \]  

(B6)

Then the components of the average value of the spin along the rotating axis \((\hat{b}_1(t), \hat{b}_2(t), \hat{b}_3(t))\) are

\[ \langle \vec{s}_i^r \rangle = \frac{1}{2} R_{ra}(t) (\psi_F, \sigma_i^a \cdot \psi_F) = \frac{1}{2} (\psi_F', \sigma_i^r \cdot \psi_F'). \]  

(B7)

This means that we can represent the components of the spin of the \(i\)-th particles along the axis of the rotating frame with the operator

\[ \vec{s}_i^r = \sigma_i^r, \]  

(B8)

only if we use the wave function \(\psi_F(t, \vec{\eta}_i)\). Since the wave function \(\psi_F(t, \vec{\eta}_i)\) satisfies the non-inertial Schroedinger equation with Hamiltonian (4.10) and since we have

\[ U_{R,\text{spin}}(t) \frac{dU_{R,\text{spin}}^+(t)}{dt} = \sum_i \vec{\omega}(t) \cdot \vec{\sigma}_i \ U_{R,\text{spin}}(t), \]  

(B9)

we get

\[ i\hbar \frac{\partial}{\partial t} \psi_F'(t, \vec{\eta}_i) = \hat{H}_{ni}^s(t) \cdot \psi_F'(t, \vec{\eta}_i), \]  

(B10)

with

\[ \hat{H}_{ni}^s(t) = \hat{E}_{\text{in}}[A^a] + i\hbar \left[ \vec{v}(t) \cdot \sum_i \nabla_{\eta_i} + \vec{\omega}(t) \cdot \sum_i (\vec{\eta}_i \times \nabla_{\eta_i} + 2\vec{\sigma}_i) \right]. \]  

(B11)

The presence of the term \(\vec{\omega} \cdot \vec{\sigma}_i\) is discussed for experimental tests in Ref.[23]
APPENDIX C: GALILEI TRANSFORMATIONS

In Subsection IIA we started choosing an inertial frame with inertial coordinates $x^a$ in Newtonian space-time. However we could have started with another inertial frame whose inertial coordinates $x'^a$ are obtained by means of a Galilei transformation

$$x'^a = R^a_b x^b + v^a t + b^a.$$  \hfill (C1)

At the Lagrangian level of Subsection IIB this implies the following Lagrangian transformations of the functions $\mathcal{A}(t, \sigma)$

$$\mathcal{A}{'a}(t, \sigma) = R^a_b \mathcal{A}^b(t, \sigma) + v^a t + b^a.$$  \hfill (C2)

At the Hamiltonian level the point transformation (C2) is completed with the transformation properties of the canonical momenta

$$\rho{'a}(t, \sigma) = R^a_b \rho^b(t, \sigma) + \sum_i m_i v^a \delta^3(\sigma - \eta_i),$$

$$p'_{ir}(t) = p_{ir} + J^{'a}_r(t, \eta_i) v_a.$$  \hfill (C3)

Eqs.(C2) and (C3) define a time dependent canonical transformation that leaves unchanged the structure of the Dirac Hamiltonian since we have $(\lambda' = \lambda + v^a)$

$$H_D \mapsto H'_{D} = H'_c + \int d^3\sigma \lambda'{}^a(t, \sigma) \mathcal{H}'_a(t, \sigma) - \frac{1}{2} \sum_i m_i \bar{v}^2(t),$$  \hfill (C4)

where $H'_c$ and $\mathcal{H}'_a(t, \sigma)$ are the canonical Hamiltonian and the constraints expressed in terms of $\mathcal{A}'{}^a(t, \sigma)$, $\rho'{}^a(t, \sigma)$, $p'_r(t)$ by inversions of Eqs. (C2)(C3) and where $\frac{1}{2} \sum_i m_i \bar{v}^2(t)$ is a ignorable function only of time. In particular, the form of the constraints is left unchanged by the canonical transformation

$$\mathcal{H}'_a(t, \sigma) = \rho'{}^a(t, \sigma) - \sum_i \delta^3(\sigma - \eta_i(t)) \tilde{J}'_{ir}{}^a(t, \eta_i(t)) p'_{ir}(t) \approx 0.$$  \hfill (C5)

Instead the canonical Hamiltonian becomes
\[
H_c(t) = \sum_i \frac{1}{2m_i} \sum_a \left[ \tilde{J}^{ir}_a(t, \tilde{\eta}_i(t)) p_{ir}(t) \right] \left[ \tilde{J}^{is}_a(t, \tilde{\eta}_i(t)) p_{is}(t) \right] + \\
+ \tilde{V} \left( t, \tilde{A}'(t, \tilde{\eta}_1), ..., \tilde{A}'(t, \tilde{\eta}_N) \right),
\]

(C6)

with

\[
\tilde{V} \left( t, \tilde{A}'(t, \tilde{\eta}_1), ..., \tilde{A}'(t, \tilde{\eta}_N) \right) = V \left( t, \tilde{A}(t, \tilde{\eta}_1), ..., \tilde{A}(t, \tilde{\eta}_N) \right).
\]

(C7)

In the rest of this Appendix we assume that the interaction potential \(V(t, \tilde{x}_1, ..., \tilde{x}_N)\) is time-independent and invariant under rotations and translations, namely that we have

\[
\tilde{V} \left( \tilde{A}'(t, \tilde{\eta}_1), ..., \tilde{A}'(t, \tilde{\eta}_N) \right) = V \left( \tilde{A}'(t, \tilde{\eta}_1), ..., \tilde{A}'(t, \tilde{\eta}_N) \right).
\]

(C8)

Then the form of the canonical Hamiltonian is left unchanged by the Galileo canonical transformation. In this way Galilei relativity principle is implemented in our parametrized Galilei theory.

The canonical generators of the transformation (C2) and (C3) are (the Galilei boosts are \(\tilde{K} = +t \tilde{P}\))

\[
\tilde{J}(t) = \int d^3\sigma \tilde{A}(t, \tilde{\sigma}) \times \tilde{\rho}(t, \tilde{\sigma}),
\]

(C9)

\[
\tilde{P}(t) = \int d^3\sigma \tilde{\rho}(t, \tilde{\sigma}),
\]

(C10)

\[
\tilde{K}(t) = -\sum_i m_i \tilde{A}(t, \tilde{\eta}_i),
\]

(C11)

and an infinitesimal Galilei transformation is given by

\[
\delta F = \{F, G\}, \quad \text{with} \quad G = \delta \tilde{\omega} \cdot \tilde{J}(t) + \delta \tilde{v} \cdot \tilde{P}(t) + \delta \tilde{v} \cdot \left( \tilde{K}(t) + t \tilde{P}(t) \right).
\]

(C12)

When \(\{V, \tilde{J}(t)\} = \{V, \tilde{P}(t)\} = 0\) we get \(\{H_c, \tilde{J}(t)\} = \{H_c, \tilde{P}(t)\} = 0\). Then Eqs. (C9), (C10) and (C11) and the hamiltonian \(H_c\) are the generators of a realization of the Galilei Lie algebra on phase space.
At the quantum level the rules of Subsection IIIA would map the momenta $\rho^a(t, \vec{\sigma})$, appearing in the infinitesimal generators (C9) and (C10), in the functional derivatives $\delta/\delta A^a(\vec{\sigma})$. As noted in I for the canonical generators of the Poincaré group, these functional derivatives are not operators in the Hilbert space $\mathbf{H}$, so that we would not obtain a representation of the Galileo algebra on the Hilbert space. However, since we are interested only in the transformation properties of the physical states, solutions of the generalized Schroedinger equations, we can substitute the functional derivative with the generalized hamiltonian $\hat{T}(\vec{\sigma}, A^a)$. In this way we have (see Eq.(3.8) for the definition of $\hat{K}_ia = \hat{K}_i^a$)

$$\hat{J}^a(t) = \sum_i \epsilon^{abc} A^b(t, \vec{\eta}_i) \hat{K}_c^i, \quad (C13)$$

$$\hat{P}^a(t) = \sum_i \hat{K}_i^a, \quad (C14)$$

$$\hat{K}^a(t) = - \sum_i m_i A^a(t, \vec{\eta}_i). \quad (C15)$$

Now these are self adjoint operators. When the interaction potential is invariant under rotations and translations, that is when $[V, \hat{J}^a(t)] = [V, \hat{P}^a(t)] = 0$ implies $[\hat{E}[A^a], \hat{J}^a(t)] = [\hat{E}[A^a], \hat{P}^a(t)] = 0$, Eqs. (C13), (C14), (C15) and the energy $\hat{E}[A^a]$ become the generators of a commutator projective realization the Galilei Lie algebra on the Hilbert space $\mathbf{H}$. 


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