$k$–strings and baryon vertices in SU($N$) gauge theories

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It is pointed out that the sine law for the $k$–string tension emerges as the critical threshold below which the spatial $Z_N$ symmetry of the static baryon potential is spontaneously broken. This result applies not only to SU($N$) gauge theories, but to any gauge system with stable $k$–strings admitting a baryon vertex made with $N$ sources in the fundamental representation. Some simple examples are worked out.

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The linear rising of the interquark potential in SU($N$) gauge theories suggests that the chromo-electric flux tube between static quarks in the fundamental representation is localised in a thin tube joining those charges and it is widely believed that the long-distance physics of such flux tubes is given by an effective string theory.

In addition to charges in the fundamental representation one can consider the potential between static charges in higher representations of the gauge group. This has been the subject of considerable theoretical attention since mid-1970’s, when it was pointed out [1] that the strong coupling limit or dimensional reduction arguments [2] suggest the Casimir scaling law, i.e. the hypothesis that the string tension for a given representation is proportional to the quadratic Casimir operator. This seems to describe accurately the potential between sources in different representations extracted from numerical studies of SU(3) lattice gauge theory at intermediate distances [3,4,5].

Note however that the long-distance properties of the flux tube attached to a charge built up of $j$ copies of the fundamental representation should depend only on its $N$-ality $k \equiv j \mod N$, the reason being that all representations with the same $k$ can be converted into each other by the emission of a suitable number of soft gluons. As a consequence, the heavier strings of given $N$-ality $k$ are expected to decay into the string with smallest string tension $2 \sigma_k$. The corresponding string is usually referred to as a $k$-string. If its tension $\sigma_k$ satisfies the inequality $\sigma_k < k \sigma$, where $\sigma \equiv \sigma_1$ is the tension of the fundamental string, the $k$-string is stable against decay into $k$ 1-strings. Charge conjugation implies $\sigma_k = \sigma_{N-k}$, therefore SU($N$) has only $[N/2]$ distinct $k$-strings.

Stable $k$ string are expected to belong to the antisymmetric representation with $k$ quarks. This is also supported by Casimir scaling, which in this case yields

$$\sigma_k^{(c)} = \sigma \frac{k(N-k)}{N-1}. \quad (1)$$

In the large $N$ limit it has been argued [6] that $\sigma_k = k \sigma + O(1/N^2)$ which seems to exclude Casimir scaling law as an exact formula.

Another competing hypothesis is the sine law:

$$\sigma_k^{(s)} = \sigma \frac{\sin(k\pi/N)}{\sin(\pi/N)}, \quad (2)$$

which has been found in $\mathcal{N} = 2$ supersymmetric SU($N$) gauge theory softly broken to $\mathcal{N} = 1$ [8], in the M theory description of $\mathcal{N} = 1$ supersymmetric SU($N$) gauge theory, called MQCD [9] and, more recently, in the AdS/CFT framework [10]. In some cases this formula is expected to be exact, while in others perturbative corrections have been found [11].

Lattice calculations in pure SU($N$) gauge models for $N = 6$ [12] and $N = 4, 5, 6, 8$ [13,14] in $D = 3+1$ point to the $k$–string tensions lying partway between the Casimir scaling and the sine law; however there is no complete consensus and some dedicated studies favour the sine formula [15]. Enlarging the analysis to other gauge groups, we shall see two instances of $Z_4$ gauge models where the 2-string tension can be exactly evaluated in any dimension in an almost obvious way. In one case, as it turns out, $\sigma_2 = 2 \sigma$ while in the other $\sigma_1 = \sigma$ (or, more generally, $\sigma_k = \sigma$ for a $Z_N$ gauge group) and there are reasons to believe that there exists a set of gauge models which continuously interpolate between these two extremal values. Nonetheless, Eq. (2) plays a special role: in this Letter we show that in whatever gauge theory in which the center of the gauge group is $Z_N$ the $k$–string tension given by the sine law has a simple geometrical meaning: it is the threshold below which the spatial $Z_N$ symmetry of the baryon vertex is spontaneously broken.

A baryonic vertex is a gauge-invariant coupling of $N$ multillets in the fundamental representation which gives rise to finite energy configurations with $N$ external quarks [16,17]. When the separations among these quarks is large, one expects that $N$ strings of chromo-electric flux form, which meet at a common junction; their world-sheet forms a $N$–bladed surface with a common intersection. Due to its shape in the case of SU(3), this description of the baryonic potential is known as the Y-Ansatz. A different description, known as $\Delta$–Ansatz, follows from the assumption [18] that the two-body interaction is the relevant one for any SU($N$). Its name comes from the linear rise of the potential with the perimeter of the triangle in the SU(3) case. Lattice data for SU(3)
one has proportional to the total length of the strings, therefore the $\Delta$- Ansatz breaks down and there is a gradual transition, The thin lines denote the fundamental strings, while the thick lines are 2- and 3-strings.

[19, 20, 21] seem to support the $\Delta$- Ansatz at short distances. At interquark separations larger than $\sim 0.8$ fm the $\Delta$- Ansatz breaks down and there is a gradual transition to the Y-Ansatz.

In the mapping from four-dimensional gauge theories to string theory in AdS space [22] a $SU(N)$ baryon vertex has been explicitly obtained by wrapping a fivebrane over $S^5$ [22].

When $N > 3$ one has to envisage the possibility that $k$ neighbouring strings of the baryon vertex coalesce into a single $k-$string [19]. Note that the external quarks belong to a fully antisymmetric combination, which is the most favourable condition to $k-$string formation. What is the cost in energy of such a configuration? To answer this question, we place external quarks at the $N$ corners of a regular polygon inscribed in a circle of radius $R$. When $R$ is much larger than the string formation scale the gauge flux is squeezed into 1-strings attached to the quarks. Preparing the external charges in a spatially $Z_N$-symmetric configuration does not necessarily imply that the structure of the gauge flux preserves $Z_N$: allowing strings to coalesce requires breaking such a symmetry. Assuming $k$-strings do not form leads to a gauge flux concentrated in a $N$-bladed string worldsheet bounded by the parallel world-lines of the static quarks. In this case the configuration of minimal energy is $Z_N$-symmetric. The baryon potential $V_N$ is roughly proportional to the total length of the strings, therefore one has $V_N = N \sigma R + O(1/R)$, where the leading $O(1/R)$ correction is due to the Casimir energy [24, 25].

The formation of $k-$strings breaks the spatial $Z_N$ symmetry. Some examples of these configurations are drawn in Fig.1. More general symmetry-breaking schemes can be encoded in an arbitrary partition of $N = k_1 + k_2 + \cdots + k_m$, where $k_a$ is the number of neighbouring strings which coalesce into a single $k_a$-string. The associated string configuration is generated by iterating the basic motif depicted in Fig.2. Denoting by $Re^{i\frac{2\pi a}{N}}$ $(j = 1, 2 \ldots N)$ the position of the vertices in the polygon, we can generalise the static potential to

$$V_{(k_1, k_2, \ldots, k_m)} = \sigma R \sum_{a=1}^{m} f_{k_a}(\rho_a) + O\left(\frac{1}{R}\right),$$

where $\rho_a R$ is the length of the $k_a$-string and

$$f_k(\rho) = \rho \frac{\sigma_k}{\sigma} + \sum_{j=0}^{k-1} \left|\rho - e^{i\frac{2\pi a}{N}}\right|,$$

with $\alpha = \frac{k-1}{N} \pi$ (see Fig.2). This choice of $\alpha$ eliminates any dependence on the orientation of the $k-$string. The Ansatz [23] is based on the assumption that the common junction of the strings (which is the symmetry axis of the configuration) is not displaced by $k-$string formation. This is obviously true as long as the system preserves a residual symmetry, but it is also justified for more general string breaking schemes, owing to the fact that we are interested in the threshold of the symmetry breaking.

In order not to break the $Z_N$ symmetry the configuration of minimal energy should be characterised by $\rho_{k_a} = 0$ for all $a = 1, \ldots, m$.

First, from the observation that $g(\rho) = \rho + |\rho - e^{i\theta}|$ is always increasing and $g''(\rho) > 0$ for any $\rho$ and $\theta$, one shows at once that the function $f_k(\rho)$ has only one minimum in the whole $\rho$ range. Then, Taylor expanding $f_k(\rho)$ around $\rho = 0$ yields

$$f_k(\rho) = k + \rho \frac{\sigma_k - \sigma(s)}{\sigma} + \frac{\rho^2}{4} \left(k - \sin \frac{2\pi k}{N}\right) + \cdots$$

where the crucial sine ratio $\frac{\sigma(s)}{\sigma}$ in the linear term arises from the geometric sum

$$\sum_{l=0}^{k-1} e^{i\frac{2\pi l}{N}} = e^{-i\alpha} - e^{-i\alpha+i\frac{2\pi}{N}} = \sin \frac{\pi k}{N} = \frac{\Sigma_k}{\sigma}.$$  

Assuming $\sigma_k = \sigma(s)$ for all allowed values of $k$ yields the three relationships

$$f_k'(0) = 0, \quad f_k''(0) > 0, \quad V_{(k_1, k_2, \ldots, k_m)} = V_N,$$
which tell us that for \( \sigma_k \geq \sigma_k^{(s)} \) the \( Z_N \)-symmetric baryon vertex should be stable against the formation of \( k \)-strings, while for \( \sigma_k < \sigma_k^{(s)} \) this is no longer true and the system breaks up into less symmetric configurations made with \( k \)-strings. In other terms, the sine law is the critical threshold below which the \( Z_N \) symmetry of the baryon vertex is spontaneously broken.

In 3 + 1 dimensions one can place the static quarks in non-planar configurations, but only in the special cases \( N = 4, 6, 8, 12, 20 \) one can arrange them in a fully symmetric configuration, corresponding to the vertices of the platonic solids. The thresholds of symmetry breaking are in these cases lower than the corresponding \( Z_N \) values. For instance, for the tetrahedron we get \( \frac{\sigma_4}{\sigma_4^{(s)}} = \frac{\sqrt{3}}{3} \).

Notice that if \( N/k \) is an integer, there is a new baryon vertex coupling \( N/k \) external charges lying in the fully antisymmetric representation made with \( k \) quarks. It follows at once that the subset \( \sigma_k^{(s)} \) constitutes the set of critical thresholds for the spontaneous breaking of the \( Z_N \) symmetry of this kind of baryon.

In contradistinction to what happens in MQCD or in other supersymmetric gauge theories, lattice calculations in pure \( SU(N) \) with \( N = 4, 5, 6, 8, 12, 13, 14 \) put all these gauge models in the broken symmetry phase. The scheme of spontaneous symmetry breaking depends on the spectrum of the \( k \)-string tensions. For instance, in 3D \( SU(6) \) it turns out that the pattern \( Z_6 \rightarrow Z_3 \) is preferred to \( Z_6 \rightarrow Z_2 \).

Numerical methods for revealing the distribution of gauge fields within the static baryonic potential are now available \cite{26, 27, 28}. The question is whether such a spontaneous symmetry breaking effect is accessible in realistic simulations: if the size of the system is not large enough, we expect that the \( SU(N) \) baryonic vertex undergoes back-and-forth tunnelling among the different vacua, obscuring this effect.

Much larger volumes can be reached studying gauge systems with discrete gauge groups. Indeed it has to be emphasised that the above considerations apply to whatever confining gauge theory admitting a baryonic vertex. Let us restrict attention to two particularly simple gauge models. The partition function of the first model is

\[
Z_{Z_N}(\beta) = \sum_{\phi_\ell \in \{0, \beta/2\}} \prod_\ell e^{\beta \cos \phi_\ell}, \quad \phi_\ell = \sum_{\ell \in \gamma} \phi_\ell, \quad k \text{ is the number of units of the fundamental charge of } Z_N.
\]

It has been shown long ago \cite{29} that in the case \( N = 4 \) the above gauge theory is fully equivalent to a \( Z_2 \times Z_2 \) theory in any space dimension, namely,

\[
Z_{Z_4}(\beta) = Z_{Z_2 \times Z_2}(\beta/2) = Z_{Z_2}^2(\beta/2).
\]

All the \( Z_4 \) quantities can be expressed through the \( Z_2 \) quantities. In particular

\[
\langle W_2(\gamma) \rangle_{Z_4, \beta} = \langle W_1(\gamma) \rangle_{Z_2, \beta} = \langle W_1(\gamma) \rangle_{Z_2, \beta}^2,
\]

The confining phase shows up in an area law decay of the vacuum expectation value of large Wilson loops, therefore comparing left-hand side and right-hand side of Eq.\cite{10} yields \( \sigma_2 = 2\sigma \). Thus, there is no stable 2-string and
the baryonic vertex keeps its spatial symmetry in any dimension.

The other class of illustrative models is defined through a simple modification of Eq. (3):

$$Z_{N}^{'}(\beta) = \sum_{\phi_{\ell} \in \{\pm 1\}} \prod_{P} e^{\beta \delta_{\phi_{P},0}},$$  \hspace{1cm} (11)

where now the plaquette variable is given by the Kronecker delta $\delta_{\phi,0}$, which is 1 only if $\phi \equiv 0 \text{ modulo } 2\pi$. Sewing together a proper number of plaquettes one can generate any Wilson loop, which inherits the same property, i.e. $W_{k}(\gamma) = \delta_{k\phi,0}$. Since all the configurations with $W_{1}(\gamma) = 1$ imply $W_{k>1}(\gamma) = 1$, it follows that $\langle W_{k}(\gamma) \rangle \geq \langle W_{1}(\gamma) \rangle$ with $k > 1$. Arguing as above, taking into account that $\sigma_{k} \geq \sigma$, now we get $\sigma_{k} = \sigma$, so that $\sigma_{k} < \sigma^{(s)}$ for any $N$ and in any dimension. As a consequence, the spatial $Z_{N}$ symmetry of the baryonic vertex is spontaneously broken. As $N$ increases, the pattern of symmetry breaking may become rather involved. As an example, using Eqs. (3) and (4) one can see that a tern of symmetry breaking may become rather involved.

In order to see the distribution of the gauge field inside a baryon vertex we simulated the $Z_{4}$ gauge models defined in Eqs. (3) and (4) in $D = 2 + 1$. We used the plaquette as a probe in the vacuum modified by the presence of four static sources placed at the corners of a large square of side $L$. They are represented by four parallel Wilson lines wrapped around a periodic direction. To reach the required large distances, the Monte Carlo simulations were actually performed in the dual versions of these systems, which are simple spin models: the system (3) can be exactly mapped into the clock $Z_{4}$ model and the system (4) in the 4-state Potts model. This choice allows one to use efficient nonlocal cluster simulation algorithms [31]. The resulting flux-tube profile is shown in Fig. (3) and in Fig. (4). In the former the spatial $Z_{4}$ symmetry of the baryon vertex is preserved, while in the latter the formation of a 2-string is clearly visible. Its length is about 40% of the side $L$, according to the configuration which minimises Eq. (4), yielding, in this case, $L - \ell = L/\sqrt{3}$.

In conclusion, in this work it has been found a simple physical interpretation of a behaviour of $k$-strings -the sine law- which so far was regarded as a mathematical consequence of supersymmetry. Here it has been instead related to the marginal stability of $Z_{N}$-symmetric baryon vertices. It would be very interesting to discover some relation between these two approaches.

[30] F. Bissey et al. [hep-lat/0501004].