Infinite number of conditions for local mixed state manipulations

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It is shown that a finite number of conditions are not sufficient to determine the locality of transformations between two probability distributions of pure states as well as the locality of transformations between two $d \times d$ mixed states with $d \geq 4$. As an example, an infinite, but minimal, set of necessary and sufficient conditions for the existence of a local procedure that converts one probability distribution of two pure pair of qubits into another one is found.

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Entanglement is one of the main ingredients of non-intuitive quantum phenomena. Besides of being of interest from a fundamental point of view, entanglement has been identified as a non-local resource for quantum information processing. In particular, shared bipartite entanglement is a crucial resource for many quantum information tasks and therefore its quantification is very important. Pure bipartite entanglement is quantified asymptotically by the entropy of entanglement 2 and deterministically by a set of entanglement monotones 3, 4. Nevertheless, mixed state entanglement is far more rich and lacks a complete quantification despite the enormous efforts that has been made in the last years 5. Since in practice one usually works with mixed states, quantification of mixed state entanglement is extremely important for quantum information processing.

In this paper we show a surprising result on the number of measures of entanglement that are required to quantify the non-local resources of bipartite mixed states. An entangled pair of qudits (each of dimension $d$) is described by a finite dimensional density matrix $\rho$ with $d^4$ elements. Therefore, a finite number of independent measures of entanglement (i.e. entanglement monotones) are required to quantify completely the non-local resources of $\rho$; that is, there must be a finite set of entanglement monotones such that any measure of entanglement can be expressed as a function of these monotones 6. Yet, it is shown below that for $d \geq 4$ a finite number of measures of entanglement is insufficient to determine whether a transformation $T : \rho \rightarrow \sigma$ can be realized by means of local operations and classical communication (LOCC). That is, for any finite number of measures of entanglement, say $\{E_k\}$, there exist $\rho$ and $\sigma$ such that $E_k(\rho) \geq E_k(\sigma)$ (for all $k$) and yet $\rho$ can not be transformed to $\sigma$ by LOCC. As will be shown in the following, similar results hold also for converting one probability distribution of pure states to another.

For a given transformation, $T$, between two finite dimensional bipartite pure entangled states, there is a set of entanglement monotones that provide necessary and sufficient conditions to determine if $T$ is local (i.e. can be realized by LOCC). The family of entanglement monotones $E_k (k = 0, 1, 2, ..., d-1)$ which introduced in 7 were first defined over the set of pure states as

$$E_k (|\psi\rangle) = \sum_{i=k}^{d-1} \lambda_i,$$  \hspace{1cm} (1)

where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1}$ are the Schmidt numbers of the $d \times d$-dimensional bipartite state $|\psi\rangle$, and extended to mixed states by means of the convex roof extension. For a pure state $|\psi\rangle$ these measures of entanglement quantify completely the non-local resources since all the Schmidt coefficients of $|\psi\rangle$ are determined by them. According to Nielsen theorem 8 the transformation between two bipartite states $T : |\psi\rangle \rightarrow |\phi\rangle$ can be performed by LOCC iff

$$E_k (|\psi\rangle) \geq E_k (|\phi\rangle) \ \forall \ k.$$  \hspace{1cm} (2)

This result has been extended by Jonathan and Plenio 9 to the case where the transformation $T$ lead to several possible final states. That is,

$$T : |\psi\rangle \rightarrow D,$$  \hspace{1cm} (3)

where $D = \{p_i, |\phi_i\rangle\}$ is a probability distribution of final states with $p_i$ being the probability that $T$ outputs the state $|\phi_i\rangle$. It has been shown in 10 that $T$ is local iff

$$E_k (|\psi\rangle) \geq E_k (D) \equiv \sum_i p_i E_k (|\phi_i\rangle) \ \forall \ k.$$  \hspace{1cm} (4)

There are other sets of entanglement monotones, such as the concurrence monotones 11, that quantify completely the non-local resources of a pure state. However, the advantage of the entanglement monotones defined in Eq. 10 is that they provide sufficient conditions for the transformation $T$ in Eq. 3 to be local. In the following we show that a finite set of entanglement monotones with this property does not exist (1) if $T$ represents a transformation between two probability distributions of pure bipartite states, $T : D_1 \rightarrow D_2$, and (2) if $T$ represents a transformation between two mixed bipartite states $T : \rho \rightarrow \sigma$.

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In order to prove the main result of this paper (Theorem 1) we first consider a transformation between two probability distributions. Suppose Alice and Bob share a pure state $|\psi\rangle$ and then perform quantum operations (not necessarily local) represented by a transformation, $T$, that outputs the state $|\psi_i\rangle$ ($i = 1, 2, ..., n_2$) with probability $p_i$. We then say that Alice and Bob share a probability distribution denoted by $D_1 = \{p_i, |\psi_i\rangle\}_{i=1}^{n_2}$. Suppose now that for each $i$, if the transformation $T$ outputs the state $|\psi_i\rangle$, Alice and Bob perform again quantum operations which are represented by the transformation $T_i$. We denote by $\{|\phi_j\rangle\}$ ($j = 1, 2, ..., n_2$) all the possible output states such that each transformation $T_i$ outputs the state $|\phi_j\rangle$ with conditional probability $q_{ij}$. Thus, the transformation $T \equiv (T_1, T_2, ..., T_{n_2})$ between the probability distributions $D_1$ and $D_2 \equiv \{|\phi_j\rangle\}_{j=1}^{n_2}$, outputs the state $|\phi_j\rangle$ with probability $q_j = \sum_i p_i q_{ij}$. We start with the most simple, but non-trivial, case in which $n_1 = 2$ and both $|\psi_1\rangle$ and $|\psi_2\rangle$ are $2 \times 2$-dimensional pure states. We also take $n_2 = 1$; that is, the distribution $D_2 = |\phi\rangle$, where $|\phi\rangle$ is also a $2 \times 2$-dimensional pure state.

For any entangled pair of qubits with Schmidt numbers $\lambda_0$ and $\lambda_1$ (i.e. $|\psi\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$) we define the parameter $x = 2 \min\{\lambda_0, \lambda_1\}$. Note that $0 \leq x \leq 1$, and from Eq. (11) $x = 2E_1(|\psi\rangle)$; i.e. $x$ measure the entanglement of $|\psi\rangle$. Moreover, any measure of entanglement, $E$, for a pure pair of qubits can be written as a function of $x$; that is, $E(|\psi\rangle) = f(x)$. The function $f$ is not necessarily continuous (for example, take $E(|\psi\rangle)$ to measure the Schmidt number of $|\psi\rangle$), but it must satisfy the following conditions:

(i) Monotonicity: for $x_1 \geq x_2$, $f(x_1) \geq f(x_2)$

(ii) Concavity: $f(\sum p_i x_i) \geq \sum p_i f(x_i)$

(iii) Normalization: $f(0) = 0$, $f(1) = 1$.

Conditions (i) and (ii) are necessary for entanglement monotones \[3, 4, 7\], whereas condition (iii) is useful in comparison between different measures of entanglement so that for all measures of entanglement the Bell state has one unit of entanglement. Moreover, these three conditions are sufficient to prove the following simple two lemmas, which will be useful for later:

**Lemma 1:** $f(x) \geq x$.

*Proof:* $f(x) = f(x \cdot 1 + (1-x) \cdot 0)$ and from conditions (ii) and (iii) we have $f(x) \geq x f(1) + (1-x)f(0) = x$.

**Lemma 2:** If $f(x_1) = f(x_2)$ for $x_1 \neq x_2$ then $f(x_1) = f(x_2)$.

*Proof:* We assume that $x_1 < x_2 < 1$; thus, there is $0 < t < 1$ such that $x_2 = x_1 + 1 - t$. From condition (ii) we have $f(x_2) \geq f(x_1) + 1 - t$, and since $f(x_1) = f(x_2)$ we get $f(x_1) \geq 1$. From (i) and (iii) we have $f(x_1) \leq 1$ and therefore $f(x_1) = f(x_2) = 1$.

We are now ready to prove the following observation (Proposition 1). For simplicity we will restrict our attention to probability distributions $D_1$, $D_2$ of initial and final $2 \times 2$ dimensional states with $n_1 = 2$ and $n_2 = 1$. The proposition is trivially applicable for higher dimensions and/or for higher $n_1$ and $n_2$.

**Proposition 1** For any finite number of measures of entanglement $\{E_k\}_{k=1}^s$ ($s < \infty$), there are probability distributions $D_1$ (with $n_1 = 2$) and $D_2$ (with $n_2 = 1$) such that

$$E_k(D_1) \geq E_k(D_2) \quad \forall \quad k = 1, 2, ..., s$$

although the transformation $T : D_1 \rightarrow D_2$ cannot be realized by LOCC with certainty.

*Proof:* Let us denote by $x_1$, $x_2$ and $y$ twice of the minimal Schmidt numbers of $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\phi\rangle$, respectively. With these notations, Eq. (4) can be rewritten as

$$p_1 f_k(x_1) + p_2 f_k(x_2) \geq f_k(y) \quad \forall \quad k = 1, 2, ..., s$$

where $\{f_k\}_{k=1}^s$ is a finite set of functions that satisfy conditions (i)-(iii) above. Thus, we would like first to find $x_1$, $x_2$, $y$ and $p_1$ ($p_2 = 1 - p_1$) such that Eq. (6) is satisfied for all $k$. Now, the set $\{f_k\}$ may include the Schmidt function (i.e. $f(x) = 0$ and $f(x > 0) = 1$). Besides the Schmidt function, for all the other functions there exist $x > 0$ such that $f_k(x) < 1$. Therefore, since the set $\{f_k\}$ is finite, there exist $y > 0$ such that besides the Schmidt function, $f_k(y) < 1$ for all $k$. We can also find $0 < p_1 < 1$ such that $f_k(y) < p_1$ for all $k$ (besides the Schmidt function). Thus, by taking $x_1 = 1$, the inequality in Eq. (6) is satisfied for any value of $0 < x_2 \leq 2$. Note that even for the Schmidt function the inequality is satisfied. However, the transformation $T : D_1 \rightarrow D_2$ cannot be realized by LOCC if $x_2 < y$. That is, the state $|\psi_2\rangle$ occurs with probability $p_2 = 1 - p_1 > 0$ and according to Nielsen theorem the transformation $|\psi_2\rangle \rightarrow |\phi\rangle$ can be realized by LOCC if and only if $y \geq x_2$. Thus, by taking $0 < x_2 < y$ we prove proposition 1.

In the above observation we see that finite number of conditions are insufficient to determine if a transformation between two probability distributions, $T : D_1 \rightarrow D_2$, can be realized locally. A natural question that presents itself is then: what are the sufficient conditions for $T$ to be local? In the general case, the answer to this question appears to be complicated especially since it involves infinite number of conditions. Nevertheless, in the following proposition we provide sufficient conditions for the case in which both $D_1$ and $D_2$ denotes probability distributions of two $2 \times 2$ dimensional pure states.

**Proposition 2** Let two distant parties share a probability distribution, $D_1$, of $2$ (initial) $2 \times 2$-dimensional pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ with corresponding probabilities $p_1$ and $p_2$, respectively. Let also $D_2$ be a probability distribution of $2$ (final) $2 \times 2$-dimensional pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ with corresponding probabilities $q_1$ and $q_2$, respectively. Then the transformation $T : D_1 \rightarrow D_2$ can be realized by LOCC if, and only if,

$$E_\mu(D_1) \geq E_\mu(D_2) \quad \forall \quad 0 \leq \mu \leq 1 ,$$

where

$$E_\mu(|\psi\rangle) \equiv f_\mu(x) \equiv \{ \begin{array}{cl} x/\mu & \text{for } x \leq \mu \\ 1 & \text{for } x > \mu \end{array} \}$$

(8)
Note that $f_{μ=0}$ is the Schmidt function and $f_{μ=1}(x) = x$.

**Proof:** It is easy to see that the functions $f_{μ}$ satisfy conditions (i)-(iii); that is, they are entanglement monotones. As such, the inequalities in Eq. (7) are necessary conditions that any local transformation $T$ must satisfy. We will now show that they are also sufficient.

Denoting by $x_1$ and $x_2$ twice the minimal Schmidt number of $|ψ_1⟩$ and $|ψ_2⟩$, respectively, we first note that according to Eq. (7) if $x_1 = x_2$ the condition $E_{μ=1}(D_1) > E_{μ=1}(D_2)$ is a sufficient condition for $T$ to be local. Thus, without loss of generality we take $x_1 > x_2$. We also denote by $y_1$ and $y_2$ twice the minimal Schmidt number of $|φ_1⟩$ and $|φ_2⟩$, respectively, and assume that $y_1 ≥ y_2$. Now, if $y_2 > x_2$ we get $E_{μ=1}(D_1) < E_{μ=1}(D_2) = 1$; that is, the condition in Eq. (7) is not satisfied for $μ = y_2$. Thus we have $x_2 ≥ y_2$.

The most general transformation $T : D_1 → D_2$ consist of two transformations which we denote by $T_1$ and $T_2$. The transformation $T_1 (T_2)$ on $|ψ_1⟩$ ($|ψ_2⟩$) outputs the states $|φ_1⟩$, $|φ_2⟩$ with conditional probabilities $q_{11}$, $q_{21} = 1 - q_{11}$ ($q_{12}, q_{22} = 1 - q_{12}$), respectively. Thus, the probabilities $q_{11}$ and $q_{12}$ must satisfy $q_1 = p_1 q_{11} + p_2 q_{12}$ (or equivalently $q_2 = p_1 q_{21} + p_2 q_{22}$). According to Eq. (9), $T_1$ and $T_2$ can be realized by LOCC (i.e. $T$ can be realized by LOCC) if, and only if,

\[
\begin{align*}
  x_1 &≥ q_{11} y_1 + (1 - q_{11}) y_2 \\
  x_2 &≥ q_{12} y_1 + (1 - q_{12}) y_2 .
\end{align*}
\]

We now consider the three possible options:

(a) $x_1 > x_2 ≥ y_2$: in this case we take $q_{11} = q_{12} = q_1$. According to Eq. (9), the transformations $T_1$ and $T_2$ can be realized by LOCC. Hence, the transformation $T = (T_1, T_2) : D_1 → D_2$ can also be realized by LOCC.

(b) $x_1 ≥ y_1 > x_2 ≥ y_2$: in this case, according to Eq. (9), the transformation $T_1$ can be realized by LOCC for any value of $q_{11}$. On the other hand, the transformation $T_2$ can be realized by LOCC only if $q_{12} ≤ q_{12}^{max} = (x_2 - y_2)/(y_1 - y_2)$. Thus, the transformation $T = (T_1, T_2) : D_1 → D_2$ can be realized by LOCC only if $p_1 + q_{12}^{max} p_2 ≥ q_1$.

Now, it can be shown that this condition is satisfied by taking $μ = y_1$ in Eq. (7).

(c) $y_1 ≥ x_1 > x_2 ≥ y_2$: in this case, according to Eq. (9), the transformations $T_1$ and $T_2$ can be realized by LOCC only if $q_{11} ≤ q_{11}^{max} = (x_2 - y_2)/(y_1 - y_2)$ and $q_{12} ≤ q_{12}^{max}$. Thus, the transformation $T$ can be realized by LOCC only if $q_{11}^{max} p_1 + q_{12}^{max} p_2 ≥ q_1$.

Now, following the same lines of proposition 1, there exist values of $p_1$, $λ$ and $η$ such that $E_k(μ) ≥ E_k(φ)$ for all $k$ and (2) $η > λ > 0$. We would like to show now that Alice and Bob can not convert $ρ$ to $φ$ by LOCC. For this purpose, we define a set of entanglement monotones similar to the one defined in Eq. (7).

For any given normalized measure of entanglement $E$, we define a set of entanglement monotones as follows

\[
E_μ(|ψ⟩) ≡ \begin{cases} μ^{-1} E(|ψ⟩) & \text{if } E(|ψ⟩) ≤ μ \\ 1 & \text{if } E(|ψ⟩) > μ \end{cases}
\]
where \( \mu \in [0,1] \) and the definition of \( E \) and \( E_\mu \) for mixed state are given in terms of the convex roof extension. Since we assume that \( E \) is an entanglement monotone, it follows from Theorem 2 in [4] that the \( \{E_\mu\} \) are indeed entanglement monotones for all \( 0 \leq \mu \leq 1 \). Let us now take the measure of entanglement, \( E \), given in Eq. (10) to be

\[
E(|\psi\rangle) = \frac{4}{3}(1 - \lambda_{\text{max}}),
\]

(17)

where \( \lambda_{\text{max}} \) is the largest Schmidt number of \( |\psi\rangle \). Thus, for \( \mu = \eta \), \( E_\mu \) as defined in Eqs. (16,17) satisfies

\[
E_{\mu=\eta}(\rho) = \frac{2}{3}p_1 + \frac{4}{3}(1-\lambda)p_2 < \frac{2}{3} = E_{\mu=\eta}(|\phi\rangle).
\]

(18)

That is, there exist entanglement monotone that quantify \( |\phi\rangle \) with more entanglement than \( \rho \). Thus, Alice and Bob can not convert \( \rho \) into \( \sigma \) by LOCC □

Any finite dimensional density matrix, \( \rho \), consists of a finite number of parameters, \( \rho_{ij} \). This means that any measure of entanglement is a function of these parameters. Thus, there must be a finite number of entanglement monotones, say \( \{E_k\}_{k=1}^s \), that quantify completely the non-local resources of \( \rho \). That is, all measures of entanglement can be written as a function of these \( s \) entanglement monotones. Nevertheless, according to theorem 3, these measures of entanglement are not sufficient to determine whether \( \rho \) can be converted into \( \sigma \) by LOCC. A natural question is then arise: is there an infinite number of entanglement monotones that do provide the sufficient conditions for a transformation \( T: \rho \rightarrow \sigma \) to be local? We generalize this question to include probability distributions of mixed states in the following conjecture:

Conjecture: A transformation \( T : D_1 \rightarrow D_2 \) between two probability distributions of mixed states can be realized by LOCC if \( E(D_1) \geq E(D_2) \) for all entanglement monotones \( E \).

The necessity of these conditions is trivial to prove though the sufficiency appears to be complicated to prove due to the complexity of mixed state entanglement. The conjecture above has been proved for the case where \( D_1 \) is a single pure bipartite state and in proposition 2 we have proved this conjecture for the case where \( D_1 \) and \( D_2 \) each is a probability distribution of two pure pair of qubits. If the conjecture above is incorrect for the general case, then entanglement would be insufficient to quantify non-locality in quantum mechanics. Since we have some examples with the flavor of “quantum non-locality without entanglement” [4], it would not be too surprising (though very interesting) if the conjecture above turns out to be incorrect.

To summarize: we have shown that infinite number of conditions (based on entanglement monotones) are required for determining the locality of transformations on a given \( d \times d \)-dimensional mixed bipartite state (with \( d \geq 4 \)) and on a given probability distribution of pure bipartite states. We have also presented a minimal set of infinite number of conditions that are required for determining the locality of transformations between two probability distributions of two pure pair of qubits. We believe that our results will also prove fruitful in further developments on mixed state entanglement.

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[10] For example, the set of entanglement monotones given in Eq. (14) (see [4]), or the set of concurrence monotones given in [3], are both quantifying completely the non-local resources of a pure bipartite state, in the sense that any measure of entanglement can be expressed in terms of these entanglement monotones.
[11] The measure of entanglement, \( E \), is normalized such that \( E = 1 \) for a maximally entangled state.