Observables in Extended Percolation Models of Causal Set Cosmology

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Abstract

Classical sequential growth models for causal sets provide an important step towards the formulation of a quantum causal set dynamics. The covariant observables in a class of these models known as generalised percolation have been completely characterised in terms of physically well-defined “stem sets” and yield an insight into the nature of observables in quantum causal set cosmology. We discuss a recent extension of generalised percolation and show that the characterisation of covariant observables in terms of stem sets is also complete in this extension.

1 Introduction

In causal set quantum gravity, a classical stochastic analogue of a quantum dynamics is provided by the classical sequential growth (CSG) models \(^\text{II} \). Such stochastic models provide a useful arena in which to investigate some of the thorny issues, such as general covariance, with which we expect to have to grapple in the quantum theory. In these models, a labelled causal set grows, element by element so that at each stage \(n\), all possible transitions from an \(n\)-element causal set to an \((n + 1)\)-element causal set are assigned probabilities by the specific model. If this growth is (a) Markovian, i.e. obeys Markovian sum rules for the probabilities, and satisfies (b) general covariance (GC) (label independence) and (c) Bell Causality (BC) \(^\text{II} \), then the solution is a class of models we call the Rideout Sorkin (RS) models.

It has been suggested that quantum sequential growth models can be sought along similar lines, constructing a “decoherence functional”, rather than a probability measure, on the sample space of causal sets, satisfying the appropriate quantum analogues of the conditions (a), (b) and (c). However, as discussed in \(^\text{II} \), finding a quantum analogue of the Bell Causality condition is a difficult task and it is perhaps useful to understand as much as possible about the original condition in its CSG setting.

Bell Causality is given in \(^\text{II} \) as a condition on ratios of transition probabilities, and is strictly defined only when all transitions have non-zero probability; RS models
satisfying this condition are termed “generic”. In these models, any probability \( \tau \) for a transition at stage \( n \) in which the new element has \( \varpi \) ancestors, and \( m \) parents, can be expressed in terms of a set of coupling constants \( \{ t_0 = 1, t_1, \ldots, t_i, \ldots \} \):

\[
\tau = \frac{\lambda(\varpi, m)}{\lambda(n, 0)},
\]

where \( \lambda(\varpi, m) = \sum_{i=m}^{\varpi} \binom{\varpi}{i} t_i \). Each \( t_i \) is not a transition probability, but the ratio of the probabilities of the following two transitions: (i) the “timid” transition from the \( i \)-antichain (the causal set of \( i \) unrelated points) in which the \((i + 1)\)th element is added to the future of all elements in the \( i \)-antichain and (ii) the “gregarious” transition from the \( i \)-antichain to the \((i + 1)\)-antichain. Denoting the probability of (ii) by \( q_i \), \( t_i \) can be expressed as

\[
t_i = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \frac{1}{q_k}.
\]

If we specify the dynamics in terms of the free parameters \( \{ t_i \} \), subject only to the conditions \( t_i \geq 0 \) and \( t_0 = 1 \), certain transition probabilities can then vanish. Note however that \( q_k \) cannot vanish for any \( k \geq 1 \) because

\[
q_k^{-1} = \sum_{i=0}^{k} \binom{k}{i} t_i .
\]

These models, defined by the \( \{ t_i \} \), are called “generalised percolation” (GP) dynamics. Thus, while a GP is Markovian and generally covariant [1] the BC condition of [1] is not well defined.

For these reasons, it is important to understand how the vanishing of certain probabilities affects the principle of Bell causality. In [1], Bell causality is defined as

\[
\frac{\alpha}{\bar{\alpha}} = \frac{\beta}{\bar{\beta}},
\]

where \( \alpha, \beta \) are probabilities of two transitions from an \( n \)-element causal set \( C_n \) with a set of common “spectators” \( S \). A spectator of a transition from a causal set \( C_n \) is an element of \( C_n \) which is not in the ancestor set of the newly born element. \( \tilde{\alpha}, \tilde{\beta} \) are the probabilities of corresponding transitions from the \((n - |S|)\)-element causal set \( C_n \setminus S \). For non-vanishing probabilities this condition makes precise the physical requirement that “spectators do not matter”, and that \( \alpha \) and \( \bar{\alpha} \) are proportional to each other with a positive constant of proportionality which is the same for the ratio of any other appropriately defined pair of transition probabilities \( \beta \) and \( \bar{\beta} \).

In [3] it is shown that an extension of GP (which includes GP as a special case) is the general solution of GC and a weaker causality condition, which allows transition amplitudes including the \( q_k \) \((k > 0)\) to vanish. This condition, called Weak Bell Causality (WBC) is that (i) condition [1] holds if all four transitions are non-vanishing; (ii) if \( \tilde{\alpha} = 0 \) then \( \alpha = 0 \); (iii) if \( \alpha = 0 \) and \( \beta \neq 0 \) then \( \bar{\alpha} = 0 \) and
β ≠ 0; (iv) If α and β are both zero, then nothing can be inferred about the $\tilde{\alpha}$ and $\tilde{\beta}$. The causets admitted by this dynamics possess a characteristic feature: the existence of different “eras” of GP, each independent of the others. We will refer to this dynamics as “extended percolation” (EP) dynamics.

The main focus of this work is to study the question of observables in EP dynamics. In [4] the set of covariant observables is completely characterised for generalised percolation dynamics in terms of physically comprehensible sets called “stem sets”: A stem $b$ in a causal set is analogous to a past set $P = J^-(P)$ in continuum spacetime [5], i.e. $b$ is a stem if and only if $b = \text{Past}(b)$. A causal set $C$ is said to contain a stem $b$ if there exist a labeling of $C$ such that the first $|b|$ elements of $C$ are isomorphic to $b$. A stem set $\text{stem}(b)$ is then the set of completed causets which contain $b$ as a stem. For any GP, the transition probabilities provide a measure on the sample space of all labelled completed causal sets. This measure can then be restricted to a covariant one on the space of all unlabelled completed causets. The main result of [4] is that in generalised percolation any measurable set is an element of the sigma algebra generated by the stem sets, up to sets of measure zero.

A characterisation of covariant observables in terms of stem sets makes physical sense. In the continuum, for example, a black hole in an asymptotically flat spacetime $(M, g)$ is defined in terms of a past set: $J^-(I^+) ≠ M$. The existence of a black hole is clearly a covariant “observable” of the classical theory. Another example of an observable is the number of bounces that a given cosmology has undergone. What the result of [4] shows is that asking stem questions will yield all the information that a GP can provide: the stem questions are physically complete. The hope is that a stem set characterisation of observables will be complete in quantum causal set dynamics as well. In view of this larger goal, it is important to check the robustness of this result for any EP dynamics, which exhausts the class of Markovian dynamics compatible with general covariance and WBC. The main aim of the present work is to show that this indeed is the case.

In Section 2 we describe the key aspects of EP dynamics relevant to the discussion of observables. In Section 3 we prove our main result, namely that any covariant measurable set is an element of the sigma algebra generated by the stem sets, up to sets of measure zero. In Section 4, we discuss why Bell Causality needs to be framed as a condition on ratios of transition probabilities. We also consider an alternative causality condition we call Product Bell Causality (PBC) – $\alpha \tilde{\beta} = \tilde{\alpha} \beta$ – and show that it is strictly weaker than WBC. Both WBC and PBC therefore generalise [4] for the case of vanishing transition probabilities. In the appendix we show that although PBC is strictly weaker than WBC, with the additional requirement of general covariance, the resulting dynamics is the same.

2 EP

An EP model [3] consists of a sequence of GP models run one after the other, each for an “era” consisting of some finite number of stages. The resulting finite causets – which we dub “turtles” – are placed one above the other in sequence to form a
“stack”. (To place a causet a “above” causet b, set every element of a to the future of every element of b.) Moreover, which GP gets run at any era in the sequence, and how long it is to be run for, depends on the realization of the process – i.e. the actual causet generated – in the eras preceding it. There may or may not be a final infinite era in which the GP of that era is run to infinity. If the latter does occur, the infinite causet generated in that final era, which sits on top of a “tower” of turtles, is called a “yertle” [6].

We will elaborate on this description in what follows.

As in [4] Ω(n) is the set of all n-element unlabelled causets, Ω(N) is the set of all finite unlabelled causets, where N is the set of natural numbers and Ω is the set of completed (infinite) causal sets. Ω(n), Ω(N) and Ω are their labelled counterparts. For any c ∈ Ω, let c(n) denote the set of the first n elements of c.

We recall the useful terminology of “break”: we say that a causet c has a “break at rank n with past c(n)” if it contains a stem, c(n), of cardinality n such that every element of the complement of c(n) is above every element of c(n). A causal set c ∈ Ω can have several breaks, but they must be ordered: if the breaks occur at ranks (n1, n2, … nk, …) with ni < ni+1, then c(ni) ⊂ c(ni+1).

Claim 1 If c has a break at rank n with the past c(n), then c(n) is the unique stem in c with cardinality n. Equivalently, any natural labelling of c must label c(n) first.

Proof Recall that a causet c has a stem b iff there is a natural labelling of c in which b is labelled first. Consider e an element in the complement of c(n). It is above every element of c(n) and in any natural labelling, its label must be greater than n. Therefore e cannot be contained in a stem of cardinality n. □

Examples of breaks: An infinite chain has an infinite number of breaks where the pasts of these breaks are the set of finite chains. Another example of causets with breaks are those formed in originary dynamics: all causets generated by this dynamics have a break at rank 1 with the past of the break being the single element causet.

Let a^i ∈ Ω(n_i), i = {1, 2, … k} where n_i ∈ N with k ∈ N ∪ {∞}. The stack a^1 ⊲ a^2 ⊲ … a^k is formed by putting a^2 above a^1, a^3 above a^2 etc. It is an ∑_i n_i element causet with breaks at ranks ∑_i=1 n_i, l ∈ {1, … k} with pasts a^1 ⊲ a^2 ⊲ … a^l. Note that a stack may have other breaks but the ones at ∑_i=1 n_i, l ∈ {1, … k} are specified as part of the definition of the stack. The causets between the specified breaks in a stack are finite by definition. We will say that a causet c ∈ Ω contains the stack a^1 ⊲ a^2 ⊲ … a^k if c has a break at rank ∑_i n_i with past a^1 ⊲ a^2 ⊲ … a^k.

This is a useful concept because an EP generates a causet with breaks between each of the turtles. The notion of breaks and stacks can be unambiguously extended to labelled causets since they are label invariant concepts. We can think of the dynamics during each era as a GP “relative” to the causet which has already occurred in the previous eras. Let c_n ∈ Ω(n) have a break at stage m < n with past c(m). Consider a transition from c_n → c_{n+1} such that c_{n+1} also has a break at rank m with
past $c(m)$. We will refer to a transition which “preserves the break” as a transition relative to $c(m)$. $c \in \Omega(|a| + k)$ is an $a$-relative $k$-antichain with a break at rank $|a|$ and past $a$ if $c \setminus a$ is a $k$-antichain, i.e. it is a $k$-antichain stacked on top of $a$. Let $q(a)_i$ be the probability for the transition from the $a$-relative $i$-antichain to the $a$-relative $(i + 1)$-antichain for $i > 0$, and $q(a)_0$ the probability of the timid transition from $a$. As in \[^2\] we define the coupling constants relative to $a$:

$$t(a)_i = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \frac{1}{q(a)_k},$$

which is the ratio of the following two transition probabilities: (i) the probability of the timid transition from the $a$-relative $i$-antichain and (ii) $q(a)_i$.

Since an EP is specified iteratively, era by era, it will be useful to introduce a compact notation for the coupling constants for any era, showing the dependence on the causet generated in previous eras. After the $k$th era of an EP, the resulting causet will be a stack, $a \equiv a^1 \bowtie a^2 \bowtie \cdots \bowtie a^k$. The $(k + 1)$th era is given by a number $n(a) \in \mathbb{N} \cup \{\infty\}$ and a set of $n(a)$ coupling constants relative to $a$, \[ \{t(a)_0 = 1, t(a)_1, \ldots, t(a)_{n(a)-1}\}. \]

We will use the notation $\Phi(a, n(a))$ to denote these coupling constants. \[^1\] We can construct each era explicitly as follows:

**First Era:** In this first era, one starts with the empty set $\emptyset$, and the dynamics is given by $\Phi(\emptyset, n(\emptyset))$, $n(\emptyset) \in \mathbb{N} \cup \{\infty\}$. This dynamics is equivalent to a probability distribution on $\Omega(n(\emptyset))$.

If $n(\emptyset) = \infty$ this is the whole EP: it is a single GP.

If $n(\emptyset) < \infty$, we define $E(\emptyset) \subseteq \Omega(n(\emptyset))$ to be the set of causets with non-zero probability, with the equality holding only if none of the transition probabilities up to stage $n(\emptyset) - 1$ vanish.

**Second Era:** For each element, $a^1$ of $E(\emptyset)$, there is an $a^1$-relative GP $\Phi(a^1, n(a^1))$, which generates a probability distribution on the set of $(n(\emptyset) + n(a^1))$-element causets which have a break at rank $n(\emptyset)$ with past $a^1$.

If $n(a^1) = \infty$ the process $\Phi(a^1, n(a^1))$ is run to infinity.

If $n(a^1) < \infty$ define $E(a^1) \subseteq \Omega(n(\emptyset) + n(a^1))$ to be the set of causets with non-zero probability. The inclusion is strict, since the growth only allows causets which have a break at rank $n(\emptyset)$ with past $a^1$.

Notice that different $a^1 \in E(\emptyset)$ give different second-era dynamics that are independent of each other, hence the need for the labeling $(a^1)$.

**Third Era:** For each $a^1 \bowtie a^2 \in E(a^1)$ there is an $(a^1 \bowtie a^2)$-relative GP $\Phi(a^1 \bowtie a^2, n(a^1 \bowtie a^2))$. If $n(a^2) < \infty$, let $E(a^1 \bowtie a^2) \subseteq \Omega(n(\emptyset) + n(a^1) + n(a^2))$ be the set of causets with non-zero probability. And so on.

A turtle is the set of all elements born during a single finite era. A yertle is the set of all elements born during an infinite era (necessarily the last).

\[^1\]Note that the couplings for the $k + 1$th era are not only relative to $a$ but also depend on $a$. To be strict we’d need to add a further explicit dependence on $a$ to the $t$’s. However, this would encumber the already heavy notation and we retain the current form for simplicity.
Define $\Xi(N)$ to be the set of finite stacks with non-zero probability in the EP, i.e.,
$$\Xi(N) = \bigcup_k E(a^1 \triangleleft a^2 \triangleleft \ldots \triangleleft a^k).$$  \hfill (6)

Define $\Xi(\infty)$ to be the set of infinite stacks $a^1 \triangleleft a^2 \triangleleft \ldots \triangleleft a^\infty$ such that all their finite substacks are in $\Xi$. Define $\Xi = \Xi(N) \cup \Xi(\infty) \cup \{\emptyset\}$. We will say that a stack is admitted by the dynamics if it is an element of $\Xi$.

**Special Cases:**

1. Generalised percolation is a special case of EP with a single era: $\Xi = \{\emptyset\}$, $n(\emptyset) = \infty$.

2. An “originary” generalised percolation is specified by taking a GP and putting the causet generated above a single minimal element (the “origin”). Since $q(\emptyset)1 = q_1 = 0$ it is not a GP, but an EP with $n(\emptyset) = 1$ in the first era and $n(\{\cdot\}) = \infty$ in the second, where $\{\cdot\}$ is the single element causal set.

### 3 Main Result

Following [4], for each EP we have $\Omega$ the sample space of all unlabelled, past finite, completed causets, $\mathcal{R}$ the collection of physical measurable sets and $\mathcal{R}(\mathcal{S})$ the sigma algebra generated by $\mathcal{S}$ the family of all stem sets. The identification of $\mathcal{R}(\mathcal{S})$ as the complete set of physical covariant questions for GP is the main result of [4]. We want to prove this same result for EP:

**Proposition 1** In an EP, the family of stem sets, $\mathcal{S}$, generates the sigma algebra, $\mathcal{R}$, of covariant measurable sets up to sets of measure zero.

As in [4] a crucial concept is that of a rogue which is a causet which has at least one “clone,” where a clone of a causet $c$ is a non-isomorphic causet which has the same stems as $c$. The set of all rogues is written $\Theta$. An important kinematical result of [4] is a characterisation of the set of rogues $\Theta$: $c \in \Theta$ iff $c$ contains a level with infinitely many non-maximal elements.

An example of an infinite level is the first (and only level) of an infinite antichain. However, all the elements in this level are maximal and hence the infinite antichain is not a rogue causal set. It is shown in [4] that $\Theta$ can be built by performing countable set operations on stem sets so $\Theta \in \mathcal{R}(\mathcal{S})$. Since rogues are not specified by their stems, their occurrence would be an obstacle to proving Proposition 1. We need to understand how rogues can arise in EP dynamics.

We define $\mathcal{T}$, the set of towers
$$\mathcal{T} = \{ x \in \Xi : x \subset y, \ y \in \Xi \implies x = y \}. \hfill (7)$$

$\mathcal{T}$ contains all the infinite admitted stacks and those finite admitted stacks such that the next era is infinite. We first notice that
Lemma 1 An admitted stack $a^1 < a^2 < \ldots < a^k \in \Xi$ is a stem of a causet $c$ generated by the dynamics iff $c$ has a break with past $a^1 < a^2 < \ldots < a^k$. Moreover, any causet with this stack as a stem can be generated by only one sequence of GPs for the first $k$ eras and the $(k + 1)$th era will be the GP $\Phi(a^1 < a^2 < \ldots < a^k, n(a^1 < a^2 < \ldots < a^k))$.

Proof $a^1 < a^2 < \ldots < a^k \in \Xi$ and so there’s a sequence of GPs that generate it: $\{\Phi(a^1 < a^2 < \ldots < a^l, n(a^1 < a^2 < \ldots < a^l))\}, l = 0, 1, \ldots, k$, with $a^0 = \emptyset$.

Assume $a^1 < a^2 < \ldots < a^k$ is a stem of a causet, $c$, generated by the EP. The initial era is fixed to be the GP $\Phi(\emptyset, n(\emptyset))$. Since $|a_1| = n(\emptyset)$, either all elements of $a_1$ are born in the first era, or there is some element $x \notin a_1$ which is born in the first era and some $y \in a_1$ born in a subsequent era. Since this is an EP, this implies $x < y$, which means $a$ is not a stem which is a contradiction. Therefore $a^1$ is generated in the first era and the second era must be $\Phi(a^1, n(a^1))$. $|a^2| = n(a^1)$ and either all elements of $a^2$ are born in the second era or not. The latter possibility leads to a contradiction as before, so that the third era is $\Phi(a^1 < a^2, n(a^1 < a^2))$. And so on through the $k$th era. So $c$ has a break with past $a^1 < a^2 < \ldots < a^k$.

Therefore if $a^1 < a^2 < \ldots < a^k$ is a stem of $c$, then it is generated by the GPs $\{\Phi(a^1 < a^2 < \ldots < a^l, n(a^1 < a^2 < \ldots < a^l))\}, l \in \{0, 1, \ldots, k\}$.

Assume that $a^1 < a^2 < \ldots < a^k$ is a stack in $c$, i.e. $c$ has a break with past $a^1 < a^2 < \ldots < a^k$.

Then by Claim 1, it is a stem in $c$. □

Corollary 1 Let $\tau, \tau' \in T$. If $\tau$ and $\tau'$ are both towers in the same causet $c$ generated by the dynamics then $\tau = \tau'$.

It is therefore possible to divide the set of causets that can occur into disjoint sets based on a classification of complete towers. Since no tower contains an infinite level, causets which are infinite towers cannot be rogues. A causal set that is generated by the dynamics and contains a finite tower $\tau \in T$ will have a yertle stacked on top of the tower and will be a rogue iff the yertle is a rogue. Any finite $\tau \in T$ can be classified according to the GP $\Phi(\tau, n(\tau) = \infty)$:

(a) $t_{(\tau)}0 = 1$, $t_{(\tau)}i = 0, \forall i > 0$. This is a deterministic “dust dynamics” which almost surely produces a yertle which is the infinite antichain and hence not a rogue.

(b) $t_{(\tau)}0 = 1$, $t_{(\tau)}1 \neq 0$, $t_{(\tau)}i = 0, \forall i > 1$. This is also deterministic, the “Forest dynamics”, and almost surely produces the “Forest yertle” which consists of infinitely many trees in which every element has infinitely many descendants and every element, except the minimal elements, has exactly one ancestor. The Forest yertle is a rogue.

(c) $t_{(\tau)}0 = 1$, $t_{(\tau)}1 \neq 0$ and $t_{(\tau)}i \neq 0$ for some $i \geq 2$. It was proved in [4] that such dynamics cannot produce rogues.

For a pure GP dynamics, the Forest dynamics is the only one that can produce a rogue, but it is deterministic and that is enough to prove that that the stem sets
generate the sigma algebra of covariant measurable sets up to sets of measure zero. An EP is made of pieces of GP’s and so all three types can occur in a single EP. Although a Forest yertle dynamics is deterministic, an EP dynamics which contains it may not be and one requires a proof different from that of [4]. We will henceforth refer to causets with yertles of type \( \square \) as forested towers. A clone of a forested tower \( c \) is a causet non-isomorphic to \( c \) which has the same stems as \( c \). If \( \tau \) is the tower in \( c \), then every clone of \( c \) is a causal set with a tower \( \tau \) with a clone of the Forest on top of it.

We are now in a position to prove our main theorem, restated in the following way.

**Proposition 2** In any EP, for every set \( A \in \mathcal{R} \), there is a set \( B \in \mathcal{R}(S) \) such that \( \mu(A \triangle B) = 0 \).

**Proof**
Consider a measurable set \( A \in \mathcal{R} \). Consider the set, \( T \), of finite towers corresponding to all the forested towers in \( A \). \( T \) is countable so we can list the elements of \( T = \{ \tau_1, \tau_2, \ldots \} \). Since each \( \tau_k \) is a tower, lemma \( \square \) implies that \( n(\tau_k) = \infty \) and \( \Phi(\tau_k, n(\tau_k)) \) is the Forest dynamics.
Define
\[
F \equiv \bigcup_i \text{stem}(\tau_i)
\] (8)
and let \( B = A \cap \Theta^c \sqcup \Theta \cap F \). In [4] it was proved that \( A \cap \Theta^c \) is an element of \( \mathcal{R}(S) \) and the proof is independent of the measure and so holds here also. \( F \in R(S) \) and so \( B \in \mathcal{R}(S) \) also.
We have
\[
B \Delta A = A \cap \Theta \cap F^c \sqcup A^c \cap \Theta \cap F
\] (9)
and
\[
\mu(B \Delta A) = \mu(A \cap \Theta \cap F^c) + \mu(A^c \cap \Theta \cap F).
\] (10)
\( A \cap \Theta \cap F^c \) contains only rogues in \( A \) which do not have any \( \tau_k \) as a stem. So these rogues are not forested towers. Rogues which are not forested towers almost surely do not happen (proof below). \( A^c \cap \Theta \cap F \) is the set of rogues, not in \( A \), which have at least one \( \tau_k \) as a stem. By lemma \( \square \) a causet \( c \) generated by the dynamics which has \( \tau_k \) as a stem must have a break with past \( \tau_k \). And the subsequent GP must be the Forest dynamics. Since an element of \( A^c \cap \Theta \cap F \) cannot be a tower \( \tau_k \) with the Forest above (because that is an element of \( A \)) it must be a tower \( \tau_k \) with a clone of the Forest above, and these almost surely do not occur (see below).
So \( \mu(B \Delta A) = 0 \).
\( \square \)

**Claim 2**: Rogues that are not forested towers almost surely do not happen.
Proof  Rogues must have a level containing infinitely many non-maximal elements. The only place an infinite level can occur in a causet generated by an EP is in a yertle. The only yertle dynamics which can generate a causet with a level with infinitely many non-maximal elements is the Forest dynamics which almost surely generates the Forest. □

4 Generalisations of the Bell Causality Condition

4.1 Why Bell Causality involves ratios

Let us reflect on the reason that the alternative condition $\alpha = \tilde{\alpha}$ is not the appropriate one for Bell causality. Let us call this condition Very Strong Bell Causality (VSBC). In CSG models there is no background causal structure, rather the events themselves are the (growth of the) causal structure. When there is a background causal structure, the two conditions BC and VSBC (or rather their natural analogues) are equivalent, but in CSG models where there’s no background, and where the conditions are expressed in terms of the unphysical parameter time labelling stages, they are very different as explained below.

VSBC is extremely strong in the context of CSG, indeed it would imply that the only possible dynamics are the one in which, almost surely, the infinite antichain grows and the one in which, almost surely, the causal set which is the infinite antichain above a single minimal element grows.

First, to see that there is a problem consider all possible transitions from a finite causal set $B$, $B \rightarrow B_i$, $i = 1, 2, \ldots n$. The associated transition probabilities $P(B \rightarrow B_i)$ sum to 1. Now, consider a causal set $C$ which contains $B$ as a stem. For each transition $B \rightarrow B_i$ there is a corresponding transition $C \rightarrow C_i$ for which all the elements in $C$ that are not in $B$ are spectators. Therefore, we must have $\sum_i P(C \rightarrow C_i) = 1$ also. But these will not exhaust the possible transitions from $C$, and this condition will then force all the other ones to have zero probability.

Indeed, starting with the transition from a single element to the two-chain (with prob $p$) or the two-antichain (with prob $1 - p$), you can quickly show that you run into contradictions unless $p = 1$ or 0. In figure 1, some transitions are shown with probabilities deduced from VSBC. Consider the possible transitions from the 3-element causet “L”, which is the union of a two chain and a single unrelated element. Three are shown, two of which have probability $p$ and one has probability $1 - p$. This implies that either $p = 0$ or the “L” has zero probability which can only happen if $p = 0$ or $p = 1$. If $p = 0$ then only the infinite antichain can be produced. If $p = 1$ then the “infinite antichain over one minimal element” is produced.

The reason that VSBC is so restrictive here is that the causal structure is itself dynamical. In the context of a theory with a fixed background causal structure the two conditions VSBC and BC (with the transition probabilities interpreted as conditional probabilities for events occurring in specified spacetime regions) are actually equivalent because the events one is considering occur in fixed spacelike separated
regions $A$ and $B$, say. Let $B$ be the region of the spectator events, then one can enumerate an exhaustive set of events in region $A$. The conditional probabilities of these sum to one (something must occur in region $A$) and so equality of all their ratios implies equality of the conditional probabilities themselves.

In CSG there is no fixed causal structure and the Bell causality condition is imposed on the dynamics at the level of the labelled (non-covariant) process. In the labelled process, the transition probabilities are for the “next birth” but there’s no reason that the next element has to be born “in region $A$” – and if there are very many spectators around, it is very likely not to be. This ”draining of probability from region $A$ births” conflicts with VSBC, and is the underlying reason why VSBC forces a lot of transition probabilities to be zero – too many, as pointed out above, leaving the dynamics trivial.

Condition BC however remains “ok”, and we believe it retains a large degree of plausibility. It has the good property that its closest analog in the standard (background causal structure) situation is equivalent to the usual causality condition (the one that gives rise to the Bell inequalities and goes by a variety of names, see [2]). And it is very restrictive while still allowing an interesting family of dynamics. Whether or not it is the unique condition that one can legitimately identify as physical Bell causality remains unsettled. Whether, in particular, its implications for the covariant measure can be discovered/understood in covariant terms – in terms of stem predicates – is an interesting open question.

4.2 PBC is weaker than WBC

Product Bell Causality (PBC) satisfies conditions (i) and (iv) of WBC, but not conditions (ii) and (iii). Namely, if $\tilde{\alpha} = 0$, then either $\alpha$, or $\tilde{\beta}$ or both must vanish. If $\alpha \neq 0$ and $\tilde{\beta} = 0$, then this violates condition (ii) of WBC. Condition (iii) is then similarly violated: If $\alpha = 0$, $\beta \neq 0$ then while this implies $\tilde{\alpha} = 0$, it does not necessarily imply the non-vanishing of $\tilde{\beta}$ (required by WBC for consistency with (ii)).

We show that violations of conditions (ii) and (iii) are compatible with PBC. Let Figure 1: Transition probabilities for VSBC

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us consider the transitions \( \alpha : C_{n-1} \rightarrow C_n, \beta : C_{n-1} \rightarrow C'_n \), with a non-empty set of common spectators \( S \). Let \( \tilde{C}_{m-1} = C_{n-1} \setminus S, m = n - |S| \). Let \( \tilde{\alpha} : \tilde{C}_{m-1} \rightarrow \tilde{C}_m, \tilde{\beta} : \tilde{C}_{m-1} \rightarrow \tilde{C}'_m \) be the corresponding transitions without \( S \). Define the following sets of paths in the space of causets \( \gamma \equiv (C_1, C_2 \ldots C_n), \gamma' \equiv (C_1, \tilde{C}_2 \ldots \tilde{C}_{m-1}, \tilde{C}_m, \ldots C_n), \rho \equiv (C_1, C_2' \ldots C_{n-1}', C_n'), \rho' \equiv (C_1, C_2' \ldots \tilde{C}_{m-1}', \tilde{C}'_m, \ldots C_n'), \) where \( C_1 \) is the one element causal set. While \( \gamma \) and \( \gamma' \) intersect at \( C_1 \) and at \( C_n \), we also require that they do not intersect at \( \tilde{C}_m \). One can always find such \( \gamma, \gamma' \) in the space of causets. Similarly, \( \rho \) and \( \rho' \) are required not to intersect at \( C'_m \). We show an example in Fig. 2. Now, if \( \tilde{\alpha} = 0 \), \( \text{Prob}_{\gamma}(\tilde{C}_m) = 0 \) and therefore \( \text{Prob}_{\gamma'}(C_n) = 0 \).

$$\text{Figure 2: Example of } \tilde{\alpha} = 0, \alpha \neq 0, \text{ illustrating the difference between WBC and PBC.}$$

If \( \alpha \neq 0 \), then PBC requires that \( \tilde{\beta} = 0 \), so that \( \text{Prob}_{\rho'}(\tilde{C}_m') = 0 \) and hence \( \text{Prob}_{\rho'}(C_n') = 0 \). However, if only PBC is imposed, it is possible for \( \text{Prob}_{\gamma}(C_n) \neq 0 \), and \( \text{Prob}_{\rho}(C'_n) \neq 0 \), which implies that \( \alpha \) and \( \beta \) need not vanish. In particular, if \( \alpha \neq 0 \), then PBC implies that \( \tilde{\beta} = 0 \) and one cannot deduce anything about \( \beta \). Thus, there is no contradiction with the condition of PBC in taking \( \alpha \neq 0 \).

When GC is imposed, then \( \text{Prob}_{\gamma}(C_n) = \text{Prob}_{\gamma'}(C_n) = \text{Prob}(C_n) \) and similarly \( \text{Prob}_{\rho}(C'_n) = \text{Prob}_{\rho'}(C'_n) = \text{Prob}(C'_n) \). Note however, that even without imposing GC, WBC requires specifically that \( \alpha = 0 \) which then implies that GC is satisfied for the pairs of paths \( (\gamma, \gamma') \) and \( (\rho, \rho') \). Indeed, the differences between the two generalised Bell causality conditions remain even after imposing GC, since for PBC \( \text{Prob}_{\gamma}(C_n) = \text{Prob}_{\gamma'}(C_n) = 0 \) does not imply specifically that \( \alpha = 0 \).

Nevertheless, the resulting dynamics in both cases is EP, as we will show in the appendix. This can be traced to the fact that if \( \text{Prob}(C_{n-1}) = 0 \), the transitions from \( C_{n-1} \) become irrelevant to the dynamics.

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References


Appendix: PBC + GC ⇒ EP

**Lemma A-1:** (A) Let \( C_n \) be an \( n \)-element causal set with \( \text{Prob}(C_n) \neq 0 \). Let \( q_i \neq 0 \) for \( 0 \leq i \leq n - 1 \). Then the gregarious transition \( g \) from \( C_n \) is given by the \( n \)-antichain to \((n + 1)\)-antichain transition \( q_n \). (B) Let \( C_{(a)n} \) be an \((n + |a|)\)-element causal set with break at rank \( |a| \) with past \( a \), such that \( \text{Prob}(C_{(a)n}) \neq 0 \). Let \( q_{(a)i} \neq 0 \) for \( 0 \leq i \leq (n(a) - 1) \). Then the \( a \)-relative gregarious transition \( g_{(a)} \) from \( C_{(a)n} \) is given by the \( a \)-relative \( n \)-antichain to \( a \)-relative \((n + 1)\)-antichain transition \( q_{(a)n} \).

**Proof:** Define an *atomisation* of \( C_{(0)n} \) as follows. For any \( C_{(0)n} \) there exists a \( k \leq n \) such that \( C_{(0)n} \) can be grown from a \( k \)-antichain \( C_{(0)k} \) along some path \( \gamma \). Let \( C_{(k+i)}^{(0)} \) represent the \( i \)th element in this growth, with \( 0 \leq i \leq n - k \). An atomisation of \( C_{(0)n} \) is then the set of \( n \)-element causets \( C_{(0)n}^{(i)} \equiv C_{(0)n}^{(0)} \sqcup i \)-antichain. Such an atomisation plays a crucial role in the proof of Lemma 2 in [1]. We will also need to consider the set of causets labelled by \( i, j \), \( C_{(k+i)}^{(j)} \equiv C_{(k+i-j)}^{(0)} \sqcup j \)-antichain, with \( 0 \leq i \leq n - k \) and \( 0 \leq j \leq i \). The \( C_{(k+i)}^{(j)} \) are then \((k + i)\)-antichains.

Define the transitions

\[
\alpha_i^{(j)} : C_{(k+i)}^{(j)} \rightarrow C_{(k+i+1)}^{(j)} \quad \text{(A-1)}
\]

\[
\beta_i^{(j)} : C_{(k+i)}^{(j)} \rightarrow C_{(k+i+1)}^{(j+1)} \quad \text{(A-2)}
\]

so that \( \beta_i^{(i)} = q_{(k+i)} \). Each \( \beta_i^{(j)} \) is therefore a gregarious transition with a spectator set \( S_\beta = C_{(k+i)}^{(j)} \). The decomposition \( C_{(k+i)}^{(j)} \equiv C_{(k+i-j)}^{(0)} \sqcup j \)-antichain, then tells us that \( \alpha_i^{(j)} \) is a bold transition with a spectator set \( S_\alpha \supset j - \text{antichain} \), if \( j \neq 0 \).
Therefore, the common spectator set for \( \alpha_i^{(j)} \) and \( \beta_i^{(j)} \) includes the \( j \)-antichain. The transitions without the \( j \)-antichain spectators are then \( \alpha_i^{(0)} \) and \( \beta_i^{(0)} \), from \( C_{k+i-j}^{(0)} \). PBC then tells us that

\[
\beta_i^{(j)} \alpha_i^{(i-j)} = \beta_i^{(0)} \alpha_i^{(j)}.
\]  

(A-3)

PBC and GC can be combined to give

\[
\beta_i^{(j)} \alpha_i^{(i-j-1)} = \beta_i^{(j-1)} \alpha_i^{(j-1)} \Rightarrow \beta_i^{(j)} = \beta_i^{(j-1)}, \quad \text{if} \quad \alpha_i^{(j-1)} \neq 0.
\]  

(A-4)

Moreover, since \( \text{Prob}(C_n^{(0)}) \neq 0 \),

\[
\alpha_i^{(0)} \neq 0, \quad 0 \leq i \leq n - k, \quad q_l \neq 0 \forall l < k.
\]  

(A-5)

(A) We need to prove that \( g = \beta_{n-k}^{(0)} = \beta_{n-k}^{(n-k)} \) when \( q_i \neq 0, \forall i < n \). If \( n = k \) then we’re done, since \( g = q_k = \beta_0^{(0)} \). Let us therefore assume that \( n > k \). We use a proof by induction. Let \( j = i \) in (A-3). Since \( \alpha_0^{(0)} \) and \( \beta_0^{(0)} \) are non-zero, \( \alpha_i^{(i)} \neq 0 \) iff \( \beta_i^{(i)} \neq 0 \). Since the latter is non-vanishing for all \( 0 \leq i < n - k \), \( \alpha_i^{(i)} \neq 0 \forall 0 \leq i < n - k \). Putting \( j = i - 1 \), and using these results we can deduce that \( \alpha_i^{(i-1)} \neq 0 \forall 0 \leq i < n - k \) and \( \beta_i^{(i-1)} = \beta_i^{(i-2)} = q_{k+i} \forall 0 \leq i < n - k \). Let us assume this to be true for \( j = i - s, s < n - k \), i.e. that \( \beta_i^{(i-s)} \neq 0 \forall 0 \leq i < n - k \). Then \( \alpha_i^{(i-s)} \neq 0 \) from (A-3) and \( \beta_i^{(i-s-1)} = \beta_i^{(i-s)} \neq 0 \). Thus, by induction we see that \( \beta_i^{(i-s)} = q_{n+i} \neq 0 \forall 0 \leq i < n - k, s \leq i \), and \( \alpha_i^{(i-s)} \neq 0 \forall 0 \leq i < n - k, s \leq i \).

Finally, putting \( i = n - k \) in (A-4), we see that \( \beta_{n-k}^{(j)} = \beta_{n-k}^{(j-1)} \forall 0 \leq j < n - k \), since \( \alpha_{n-k-1}^{(j-1)} \neq 0 \). Thus, \( g = \beta_{n-k}^{(0)} = \beta_{n-k}^{(n-k)} = q_n \).

We note that this final step can be replaced by the proof for Lemma 2 in \([1]\), if we replace BC with PBC and use the fact that \( \text{Prob}(C_n^{(i)}) \neq 0 \forall 0 \leq i < n-k \): starting from the \( (k+i) \)-antichain, consider the growth \( C_{k+i}^{(i)} \rightarrow C_{k+i+1}^{(i)} \rightarrow \cdots \rightarrow C_n^{(i)} \), with the transition probabilities \( \alpha_i^{(i)} : C_{k+i}^{(i)} \rightarrow C_{k+i+s}^{(i)} \). Since none of these transitions vanish, \( \text{Prob}(C_n^{(i)}) \neq 0 \forall 0 \leq i < n-k \).

(B) The above proof allows a simple generalisation to this case. Namely replace \( \alpha_i^{(j)}, \beta_i^{(j)} \) with \( \alpha_i^{(a)}, \beta_i^{(a)} \), i.e. transitions relative to the causet \( a \), and \( C_{k+i}^{(j)} \) with \( C_{k+i+k}^{(j)} \), i.e. causets with a break at rank \(|a|\) and past \( a \).

\( \square \)

PBC then tells us that

**Claim A-2:** Let \( q_n \) be the first antichain transition to vanish, i.e. \( q_l \neq 0, l < n \). Then the only non-vanishing transitions at stage \( n \) are the timid transitions. More generally, let \( q_{(a)n(a)} \) be the first \( a \)-relative antichain transition to vanish, i.e. \( q_{(a)l} \neq 0, l < n(a) \). Then the only non-vanishing transitions at stage \( n + n(a) \) are timid transitions.

**Proof:** Let \( g \) be the gregarious transition from some \( C_n \), \( \text{Prob}(C_n) \neq 0 \). From the above Lemma, \( g = q_n = 0 \). If \( \beta \) is a bold transition from \( C_n \), then it shares a set
of common spectators $S$. Removing these spectators gives us the pair of transitions $\tilde{\beta}, \tilde{g}$, from some $C_m = C_n \setminus S, m < n$, where $\tilde{g} = q_m$. Thus, $g\tilde{\beta} = \tilde{g}\beta \implies \beta = 0$. The generalisation is straightforward.

**Claim A-3:** Let $q_n$ be the first antichain transition to vanish. Any causal set $\tilde{c} \in \tilde{\Omega}$ which is admitted has a break at rank $n$. Similarly, if $q_{|a|}n(a)$ is the first $a$-relative antichain transition to vanish, then any causet $\tilde{c} \in \tilde{\Omega}$ which is admitted and has a break at rank $|a|$ with past $a$ also has a break at rank $n(a)$.

**Proof:** Assume otherwise. Let $\tilde{c}(n)$ be the $n$-subcauset of $\tilde{c}$ and $e_m$ the first element in $\tilde{c}$, $m > n$ such that $\tilde{c}(n)$ does not belong to the past of $e_m$. Consider a relabeling $\tilde{c} \to \tilde{c}'$ so that $\tilde{c}'(n) = \tilde{c}(n)$ and $e_m \to e'_{n+1}$. Then adding $e'_{n+1}$ at stage $n$ corresponds to a bold transition at stage $n$ which must vanish. Hence $\text{Prob}(\tilde{c}(n + 1)) = 0$ and hence $\tilde{c}$ does not occur. This proof extends simply to the case of a relative causets. 

As in [3] we see the resulting dynamics is characterised by the occurrence of “eras”. The arguments of [1] can then be simply carried over to show that within each era, the dynamics is GP. In particular, one can use PBC to prove Lemma 3 of [1] within each GP era. The resulting dynamics thus has the form of extended percolation as described.