Generation of Small Neutrino Majorana Masses in a Randall-Sundrum Model

Mu-Chun Chen$^{1}$

$^{1}$High Energy Theory Group, Department of Physics, Brookhaven National Laboratory, Upton, NY 11973-5000

Abstract

We propose a model, in the framework of 5$D$ with warped geometry, in which small neutrino Majorana masses are generated by tree level coupling of lepton doublets to a $SU(2)_L$-triplet scalar field, which is coupled to a bulk SM-singlet. The neutrino mass scale is determined by the bulk mass term ($\alpha_S$) of the singlet as $v e^{-2(\alpha_S-1)\pi kR}$. This suppression is due to a small overlap between the profile of the singlet zero mode and the triplet, which is confined to the TeV brane. The generic form for the neutrino mass matrix due to the overlap between the fermions is not compatible with the LMA solution. This is overcome by imposing a $Z_4$ symmetry, which is softly broken by couplings of the triplet Higgs to the lepton doublets. This model successfully reproduces the observed masses and mixing angles in charged lepton sector as well as in the neutrino sector, in addition to having a prediction of $|U_{e3}| \sim 0(0.01)$. The mass of the triplet is of the order of a TeV, and could be produced at upcoming collider experiments. The doubly charged member of the triplet can decay into two same sign charged leptons yielding the whole triplet coupling matrix which, in turn, gives the mixing matrix in the neutrino sector.
1. INTRODUCTION

Many new ideas aiming to solve the gauge hierarchy problem have emerged in recent years. Randall and Sundrum (RS) [1] have proposed a solution based on non-factorizable geometry in a slice of $AdS_5$ space with warped background metric. The warped metric is a solution to the Einstein equation imposing the 4D Poincare invariant,

$$ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2,$$

with a scale factor

$$\sigma(y) = k|y|,$$

where $y$ is the 5-th coordinate, $\eta_{\mu\nu}$ is the metric in 4D flat space given by diag ($-1,1,1,1$), and $k$ is a parameter related to the Ricci scalar and bulk vacuum energy. It is naturally of the order of the 5D Planck scale, $M_{pl}$. Throughout this paper we adopt the convention that Greek letters ($\mu, \nu, ...$) refer to 4D space-time indices while Latin letters ($A, B, ...$) refer to 5D indices. The space along the 5-th dimension is $S^1/Z_2$. Two 3-branes are located at the orbifold fixed points: the Planck-brane at $y = 0$ and the TeV-brane at $y = \pi R$.

Consider the SM Higgs field confined to the TeV brane, the five-dimensional action for the scalar sector reads,

$$\int d^4x \int dy \sqrt{-g} \delta \left(y - \pi R\right) \left[ g^{\mu\nu} D_\mu H(x)^\dagger D_\nu H(x) - \lambda \left(|H(x)|^2 - v_5^2\right)^2\right],$$

where $\sqrt{-g}$ is defined as $\sqrt{-g} \equiv \sqrt{-\text{det}g_{AB}}$, which is equal to $e^{-4\sigma(y)}$ for the RS metric; $v_5$ is the vacuum expectation value (VEV) of the Higgs, thus the symmetry breaking scale in 5D. After integrating out the 5-th coordinate, the kinetic term for $H(x)$ has a coefficient $e^{-2\pi kR}$. In order to get the canonically normalized kinetic term for $H(x)$, we must perform the following rescaling $H(x) \to e^{\pi kR} \tilde{H}(x)$, and identify $\tilde{H}(x)$ as the physical Higgs field. The resulting effective 4D action with canonically normalized kinetic term then reads

$$\int d^4x \eta^{\mu\nu} D_\mu \tilde{H}(x)^\dagger D_\nu \tilde{H}(x) - \lambda \left(|\tilde{H}(x)|^2 - e^{2\pi kR} v_5^2\right)^2.$$

The symmetry breaking scale in the 4D effective theory, $v$, is therefore related to the 5D symmetry breaking scale by

$$v = e^{-\pi kR} v_5.$$
Note that in the 5D action of Eq. (3), \( v_5 \) has mass dimension 1, thus its natural value is of the order of \( M_{pl} \). The 4D Planck scale, \( M_{pl} \), is obtained after integrating out the fifth coordinate in the gravitational action, and is given by

\[
\frac{M_{pl}^2}{k} = \frac{M_{pl}^3}{k} (1 - e^{-2\pi kR}).
\] (6)

Thus the electroweak scale, \( v \), and the 4D Planck scale are related by

\[
\frac{v}{M_{pl}} = e^{-\pi kR} \sqrt{\frac{k}{M_{pl}}} \left( \frac{1}{(1 - e^{-2\pi kR})^{1/2}} \right) \sim e^{-\pi kR} \sqrt{\frac{k}{M_{pl}}}. \] (7)

Assuming \( k = M_{pl} \), with the choice of \( kR \approx 11 \), we see that the TeV electroweak symmetry breaking scale, \( v \), can be derived from the warped factor \( e^{-\pi kR} \). Thus the gauge hierarchy between 4D electroweak symmetry breaking scale and the 4D Planck scale is resolved. In what follows, the SM Higgs doublet will be the only field confined to the TeV-brane; all other fields are allowed to propagate in the 5-th dimension, unless otherwise stated [2, 3].

The conventional way to generate neutrino masses in 4D is the see-saw mechanism. In this mechanism, the smallness of the neutrino masses is related to the large mass scale (typically of the order of the grand unification scale) of the right-handed neutrinos. A natural framework to accommodate the see-saw mechanism is \( SO(10) \). For reviews, see for example, Ref. [4, 5]. In the Randall-Sundrum scenario, the warped geometry provides new ways to generate the fermion mass hierarchy. Models with RS geometry have been constructed to naturally accommodate small Dirac neutrino masses [3, 6, 7]. To see how it works, let us first consider the Yukawa interaction in 5D,

\[
\frac{Y_r}{M_{pl}} \int d^4x \int dy \sqrt{-g} \Psi_R(x, y)\Psi_L(x, y)H(x)\delta(y - \pi R),
\] (8)

where \( r = e, \nu, ... \) etc. The 4D effective Yukawa coupling is obtained after integrating out the 5-th coordinate [2],

\[
\tilde{Y}_r = \frac{Y_r}{M_{pl}} \left( \frac{1 - 2c_{L,r}}{e^{(1-2c_{L,r})\pi kR} - 1} \right)^{1/2} \left( \frac{1 - 2c_{L,r}}{e^{(1-2c_{L,r})\pi kR} - 1} \right)^{1/2} e^{(1-c_{L,r}-c_{R,r})\pi kR},
\] (9)

where \( c_{L,r} \) and \( c_{R,r} \) parameterize the bulk mass terms of the fermions. Detailed derivations of the above results can be found in the following sections. (Throughout the paper, the 5D coupling constants are un-tilded, while the 4D effective coupling constants are tilded.) It is clear that to generate small Dirac masses for neutrinos requires

\[
\frac{m_\nu}{m_e} = \frac{\tilde{Y}_\nu \langle H(x) \rangle}{\tilde{Y}_e \langle H(x) \rangle} \sim \left( \frac{1 - 2c_{R,\nu}}{e^{(1-2c_{R,\nu})\pi kR} - 1} \right)^{1/2} \left( \frac{1 - 2c_{R,\nu}}{e^{(1-2c_{R,\nu})\pi kR} - 1} \right)^{1/2} e^{-c_{R,\nu}\pi kR} \ll 1.
\] (10)
This inequality can be satisfied if $c_{R,e} < c_{R,\nu}$. Therefore, if the right-handed neutrino is localized closer to the Planck brane than the right-handed charged lepton is, the smallness of the neutrino masses compared to the charged lepton masses can be explained.

If the neutrinoless double beta decay is observed, neutrino masses must be of the Majorana type. Nevertheless, it is difficult to generate small Majorana masses for the neutrinos through the conventional see-saw mechanism with warped geometry. To see how this comes about, consider the following operator

$$\int d^4x \int dy \sqrt{-g} \frac{\lambda_{ij}}{M_{pl}^2} H(x)^2 \Psi_{L,i}^T(x,y) \overline{C}\Psi_{L,j}(x,y) \delta(y - \pi R)$$

$$\equiv \int d^4x \tilde{M}_{ij} \Psi_{L,i}^{(0)}(x) \overline{C}\Psi_{L,j}^{(0)}(x) + \cdots,$$

where $C$ is the charge conjugation operator, $\cdots$ denotes terms involving higher level KK modes (throughout this paper), and the four-dimensional effective Majorana mass matrix is given by

$$\tilde{M}_{ij} = \frac{1}{2\pi R M_{pl}^2} \left[ \frac{(1 - 2c_{L,i})\pi kR}{e(1 - 2c_{L,i})\pi kR - 1} \right]^{1/2} \left[ \frac{(1 - 2c_{L,j})\pi kR}{e(1 - 2c_{L,j})\pi kR - 1} \right]^{1/2}$$

$$\cdot \int_{-\pi R}^{\pi R} dy e^{-4\sigma(y)} e^{(2 - c_{L,i})\sigma} e^{(2 - c_{L,j})\sigma} e^{2\sigma(y)} \delta(y - \pi R) \cdot v^2$$

$$= \frac{1}{2\pi R M_{pl}^2} \left[ \frac{(1 - 2c_{L,i})\pi kR}{e(1 - 2c_{L,i})\pi kR - 1} \right]^{1/2} \left[ \frac{(1 - 2c_{L,j})\pi kR}{e(1 - 2c_{L,j})\pi kR - 1} \right]^{1/2} e^{(2 - c_{L,i}, -c_{L,j})\pi kR} \cdot v^2.$$

The only way to have $\tilde{M}_{ij}$ suppressed and generate the correct mass scale for the neutrinos is when $(c_{L,i} + c_{L,j})$ is close to 2. However, this condition leads to unrealistically small charged fermion masses \cite{8}. Therefore, $\lambda_{ij}$ must be extremely small in order to give small neutrino Majorana masses. Even with various mechanisms, including lowering $k/M_{pl}$ to $O(0.01)$ and having very strong thus non-perturbative 5D Yukawa coupling $(Y/g \sim O(10)$ where $g$ is the 5D gauge coupling constant) for the charged fermions, as mentioned in Ref. \cite{8}, to bring down the Majorana masses, a tiny value as small as $10^{-4}$ in natural units for the coupling constant is still needed. It is the aim of our paper to provide a new mechanism that gives rise to small Majorana masses in a more natural way, using parameters all of order unity in natural units. We will show that this is possible if the Majorana masses are generated by coupling the lepton doublets to a $SU(2)$-triplet Higgs.

The next section contains preliminaries concerning the equations of motion of bulk fields and their zero-mode solutions. Our model with a $SU(2)$-triplet is described in Section III. Numerical results are in Section IV and conclusions in Section V.
II. LOCALIZATION OF BULK FIELDS

The equations of motion for various bulk fields are given in the following compact form \[2, 6\]

\[
(e^{2\sigma} \eta^\mu\nu \partial_\mu \partial_\nu + e^{s\sigma} \partial_5 (e^{-s\sigma} \partial_5) - M_\Phi^2) \Phi(x^\mu, y) = 0 ,
\]

where for \( \Phi = (\phi, e^{-2\sigma} \Psi_{L,R}) \) we have \( M_\Phi^2 = (ak^2 + br''(y), C(C \pm 1)k^2 \pm C\sigma''(y)) \) and \( s = (4, 1) \), where \( a \) and \( C \) are bulk mass terms of the scalar and fermionic fields, and \( b \) is a boundary mass term for the scalar field. Due to the presence of these bulk mass terms, components of the bulk field in 4D can develop profiles that depend on the fifth coordinate \( y \). Decompose the field \( \Phi(x^\mu, y) \) into an infinite sum of Kaluza-Kline (KK) modes

\[
\Phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_n \Phi(n)(x^\mu) f_n(y) .
\]

The profile of the \( n \)-th mode, \( f_n(y) \), satisfies

\[
(-e^{s\sigma} \partial_5 (e^{-s\sigma} \partial_5) + \hat{M}_\Phi^2) f_n(y) = e^{2\sigma} m_n^2 f_n(y) ,
\]

where \( \hat{M}_\Phi^2 = (ak^2, C(C \pm 1)k^2) \). \( m_n \) is the mass of the \( n \)-th KK mode. Solutions for the zero modes of various bulk fields have been found previously \[2, 6\]. We summarize the results in the following:

**Spin-0 fields**: If boundary mass terms with \( b = (2 - \alpha) \) are present, a zero mode solution \( (m_0 = 0) \) for the scalar field can exist. The relation between these two parameters \( b \) and \( \alpha \) can be justified in the SUSY limit \[2\]. The zero mode solution is given by

\[
f_0(y) = \frac{1}{N_0} e^{(2-\alpha)\sigma} ,
\]

where \( \alpha = \sqrt{4 + a} \) and \( a \) is the bulk scalar mass. The normalization constant \( 1/N_0 \) is given by

\[
\frac{1}{N_0^2} = \frac{(2 - 2\alpha)\pi k R}{e^{(2-2\alpha)\pi k R} - 1} .
\]

Thus the bulk scalar field can be decomposed into

\[
\phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R} N_0} e^{(2-\alpha)\sigma} \phi(0)(x^\mu) + \cdots .
\]

For \( \alpha = 1 \), the normalization factor \( 1/N_0 = 1 \) and the zero mode profile is \( e^{-\sigma(y)} f_0 = 1 \), where the factor \( e^{-\sigma(y)} \) accounts for the non-trivial measure due to the warped geometry. In other words, the zero mode is de-localized, and we recover the conformal limit. For \( \alpha > 1 \) (\( \alpha < 1 \)), the normalization factor is of order \( 1 \) \( (e^{-(1-\alpha)\pi k R}) \), and the zero mode is localized toward the Planck brane (TeV brane).
Spin-1/2 fields: The solution in this case is
\[ f_0(y) = \frac{1}{N_0} e^{-c\sigma}, \tag{19} \]
where \( c = C \) for left-handed fermions and \( c = -C \) for right-handed fermions. The normalization constant \( 1/N_0 \) is given by
\[ \frac{1}{N_0} = \frac{(1 - 2c)\pi kR}{e^{(1-2c)\pi kR} - 1}. \tag{20} \]
Thus the bulk fermion can be decomposed into
\[ \Psi_L(x^\mu, y) = e^{2\sigma} \Phi(x^\mu) = \frac{1}{\sqrt{2\pi R}} \frac{1}{N_0} e^{(2-c)\sigma} \Phi_L(0)(x^\mu) + \cdots. \tag{21} \]
For \( c = 1/2 \), the normalization factor \( 1/N_0 = 1 \) and the profile of the zero mode \( e^{-2\sigma(y)}(e^{2\sigma(y)} f_0(y)) = 1 \) taking into account the measure \( e^{-\frac{3}{2}\sigma(y)} \) due to warped geometry. For \( c > 1/2 \ (c < 1/2) \), the normalization factor is of order 1 \( (e^{-(1/2-c)\pi kR}) \), and the zero mode is localized toward the Planck brane (TeV brane).

III. SMALL MAJORANA MASSES

Now we introduce a \( SU(2) \)-triplet Higgs, \( T \), which carries hypercharge \( Y = 2 \) and lepton number \( L = -2 \), to generate Majorana masses for the left-handed neutrinos. \( T \) can be written in terms of three component fields
\[ T = \begin{pmatrix} \xi^+ / \sqrt{2} & \xi^{++} \\ \xi^0 & -\xi^+ / \sqrt{2} \end{pmatrix}. \tag{22} \]
Its coupling to the lepton doublets is given by
\[ \lambda_{ij}(L_i T \epsilon^{-1} i \tau_2 T L_j) = \lambda_{ij} \left[ -\xi^0 \nu_{Li}^T \epsilon^{-1} \nu_{Lj} + \frac{\xi^+}{\sqrt{2}} \left( \nu_{Li}^T \epsilon^{-1} l_{Li} + l_{Li}^T \epsilon^{-1} \nu_{Lj} \right) + \xi^{++} l_{Li}^T \epsilon^{-1} l_{Lj} \right]. \tag{23} \]
If \( T \) acquires VEV along the \( \xi^0 \) direction, the first term in the above equation then generates Majorana masses for \( \nu_L \),
\[ \lambda_{ij} \xi^0 > \nu_{Li} \nu_{Lj}. \tag{24} \]
To have small neutrino masses compared to the charged lepton masses thus requires either (\( i \)) \( \lambda_{ij} \) is much smaller than the Yukawa coupling of the charged leptons, or (\( ii \)) \( \xi^0 > \) is much smaller than the electroweak symmetry breaking scale, or a combination of both. Nevertheless, the first possibility is constrained by the experimentally well-measured \( \rho \equiv \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} \simeq 1 \) relation. In the
SM, \( \rho = 1 \) is predicted. In the presence of one \( SU(2) \) doublet and one \( SU(2) \) complex triplet with \( Y = 2 \), this relation is modified to be

\[
\rho = \frac{v^2 + 2v_\xi^2}{v^2 + 4v_\xi^2} = 1 - \frac{2v_\xi^2}{v^2 + 4v_\xi^2} \equiv 1 + \Delta \rho ,
\]

where \( < \xi^0 > \equiv v_\xi / \sqrt{2} \). Experimentally, the 2\( \sigma \) level limits are,

\[-1.7 \times 10^{-3} < \Delta \rho < 2.7 \times 10^{-3} . \]

This translates into the following bound on \( v_\xi \),

\[ v_\xi < v / (24\sqrt{2}) . \]

A more detailed study utilizing a consistent renormalization scheme in the presence of a \( SU(2)_L \) has been discussed in Ref. [10]. We have found that this bound can be satisfied with mild fine-tuning of various parameters about 1 part in 100. However, because our philosophy is to restrict most parameters to be strictly of the order unity, we utilize the second possibility in our model. The second possibility has been utilized to generate small neutrino masses in 4\( D \) (See for example, Ref. [11]) and in 5\( D \) with a large extra dimension [12]. It has not been considered before in 5\( D \) Randall-Sundrum model.

In what follows, we propose a model that generates small neutrino Majorana masses. Consider the following potential for a scalar field \( \phi \), \( \phi^4 + am_\phi^2 \phi^2 + e\phi \). In the case of a \( SU(2) \)-triplet, \( e \) is induced by the coupling to the SM Higgs doublet, \( e = e' < H >^2 \), where \( e' \) is the coupling constant. It has been shown that \( e \) must be non-zero in order to avoid the Majoron problem [11]. If the square mass term, \( am_\phi^2 \) is positive and large (compared to \( |e/am_\phi^2| \)), the VEV for the scalar field is determined by the quadratic and the linear terms of the potential. For \( am_\phi^2 \gg e' v \), a small VEV for \( \phi \) is generated. There are two possible ways to satisfy this inequality: (i) \( am_\phi^2 \) is much higher than the weak scale while \( e \) scales as \( v^3 \); (ii) \( am_\phi^2 \) is of the weak scale, \( v^2 \), while \( e \) is highly suppressed with respect to \( v^3 \); that is, the coupling \( e' \ll v \). Due to the warped geometry, the relevant mass scale in 4\( D \) is \( v \) rather than \( M_{pl} \). Thus the scalar mass term \( m_\phi \) has to be of the order of the weak scale. This leaves us with the second choice, \( e' \ll v \), which can be naturally achieved utilizing the warped geometry.

As it turns out, having a bulk triplet scalar does not work, as the induced triplet VEV in this case is of the order of the weak scale. As we show below, by confining the triplet scalar to the TeV brane and by introducing a bulk SM singlet scalar field, \( S(x, y) \), naturally small Majorana masses for the neutrinos can indeed arise.
**The Model:** As we have mentioned above, the triplet is confined to the TeV brane along with the SM Higgs doublet. All other fields propagate in the bulk. The bulk Lagrangian for the free singlet scalar field is given by,

\[
\int d^4x \int dy \sqrt{-g} \left[ g^{\mu\nu} (D_\mu S(x, y))^\dagger (D_\nu S(x, y)) - a_S k^2 S(x, y)^2 \right. \\
\left. - b_S k (\delta(y - 0) - \delta(y - \pi R)) S(x, y)^2 \right].
\]  
(28)

For non-zero boundary mass terms with \( b = (2 - \alpha_S) \), where \( \alpha_S = \sqrt{4 + a_S} \), the singlet field has a massless zero mode, whose profile along the 5th dimension is determined by the parameter \( \alpha_S \). We then add the interactions of the singlet scalar with the SM Higgs and the triplet on the TeV brane, described by the scalar potential, \( V_{vis} \). As the couplings of these interactions are highly suppressed compared to the boundary mass terms given in Eq. (28), their effects on the profile of the singlet are negligible. The most general renormalizable (from the 4D point of view) potential which respects the \( SU(2)_L \) symmetry is the following.

\[
V_{vis} = \delta(y - \pi R) \left[ \lambda_H H^4(x) + \mu_H k^2 H^2(x) + \lambda_T T(x)^4 + \mu_T k^2 T(x)^2 + \lambda_S S(x)^4 \\
+ \eta T(x)^2 H(x)^2 + \frac{\chi_1}{M_{pl}} T(x)^2 S(x, y)^2 + \frac{\chi_2}{M_{pl}} H(x)^2 S(x, y)^2 \\
+ \gamma_1 \sqrt{M_{pl}} T^2(x) S(x, y) + \gamma_2 \sqrt{M_{pl}} H^2(x) S(x, y) + \zeta M_{pl} H(x) T(x) H(x) \\
+ \frac{\xi}{\sqrt{M_{pl}}} H(x) T(x) H(x) S(x, y) \right] + h.c.,
\]  
(29)

where \( \lambda_S, \lambda_i, \mu_i (i = T, H) \), \( \eta, \zeta, \chi_i, \gamma_i \) and \( \xi \) are dimensionless \( O(1) \) parameters. Here we have used the following short-hand notations: \( H^4 = (H^\dagger H)^2 \), \( H^2 = H^\dagger H \), \( T^4 = (T^\dagger T + T T^\dagger)^2 \), \( T^2 = T T^\dagger + T^\dagger T \), \( HTH = H^\dagger T H^c \) where \( H^c = i \gamma_2 H^* \). For the quartic term, \( T^2 H^2 \), there are two possible \( SU(2) \) contractions: \( H^\dagger H T^\dagger T \) and \( i \epsilon_{ijk} H^\dagger \sigma^j H T^\dagger \sigma^k T \). Because these two terms are not important for our discussion, we do not distinguish them in the potential. Note that the couplings \( HTH \) and \( HTHS \) explicitly breaks the lepton number. After integrating out the fifth coordinate, we obtain the following effective coupling constants,

\[
\bar{\lambda}_i = \lambda_i \int_{-\pi R}^{\pi R} dy \ e^{-4\sigma} e^{4\sigma} \delta(y - \pi R) = \lambda_i
\]

(similarly for the quartic coupling \( \eta \))  
(30)

\[
\bar{\mu}_i = \mu_i v_5^2 \int_{-\pi R}^{\pi R} dy \ e^{-4\sigma} e^{2\sigma} \delta(y - \pi R) = \mu_i v^2
\]  
(31)
\[ \tilde{\zeta} = \zeta M_{pl} \int_{-\pi R}^{\pi R} dy e^{-4\sigma} e^{3\sigma} \delta(y - \pi R) = \zeta M_{pl} e^{-\pi k R} = \zeta v \]  

\[ \tilde{\chi}_i = \frac{\chi_i}{M_{pl}} \frac{1}{2\pi R} \left[ \frac{(2 - 2\alpha_S)\pi k R}{e^{(2 - 2\alpha_S)\pi k R - 1}} \right]^{1/2} \int_{-\pi R}^{\pi R} dy e^{-4\sigma} e^{2(2 - \alpha_S)\sigma} e^{2\sigma} \delta(y - \pi R) \]

\[ = \frac{\chi_i}{M_{pl}} \frac{1}{2\pi R} \left[ \frac{(2 - 2\alpha_S)\pi k R}{e^{(2 - 2\alpha_S)\pi k R - 1}} \right]^{1/2} e^{-\alpha_S \pi k R} \quad (i = 1, 2) \]  

\[ \tilde{\gamma}_i = \gamma_i \sqrt{M_{pl}} \frac{1}{\sqrt{2\pi R}} \left[ \frac{(2 - 2\alpha_S)\pi k R}{e^{(2 - 2\alpha_S)\pi k R - 1}} \right]^{1/2} \int_{-\pi R}^{\pi R} dy e^{-4\sigma} e^{2(2 - \alpha_S)\sigma} e^{3\sigma} \delta(y - \pi R) \]

\[ = \gamma_i \sqrt{M_{pl}} \frac{1}{\sqrt{2\pi R}} \left[ \frac{(2 - 2\alpha_S)\pi k R}{e^{(2 - 2\alpha_S)\pi k R - 1}} \right]^{1/2} e^{(1 - \alpha_S) \pi k R} . \]  

As the effective coupling \( \tilde{\zeta} \sim v \) induces a weak scale triplet VEV, it is necessary to turn off \( \zeta \). This can be done by imposing a \( Z_4 \) symmetry under which the fields transform as,

\[ H \overset{Z_4}{\rightarrow} H \]

\[ (T, S) \overset{Z_4}{\rightarrow} -(T, S) \]

\[ \Phi_L \overset{Z_4}{\rightarrow} i\Phi_L \]

\[ \Phi_R \overset{Z_4}{\rightarrow} -i\Phi_R \]

This \( Z_4 \) symmetry forbids all the potentially dangerous trilinear couplings \( (\gamma_1, \gamma_2, \zeta) \) in Eq. (29), and the 4D effective scalar potential is then given by,

\[ V_{eff} = \tilde{\lambda}_H H^4(x) + \tilde{\mu}_H H^2(x) + \tilde{\lambda}_T T^4(x) + \tilde{\mu}_T T^2(x)^2 + \tilde{\lambda}_S S^{(0)}(x)^4 \]

\[ + \tilde{\eta} T(x)^2 \tilde{H}(x)^2 + \tilde{\chi}_1 T(x)^2 S^{(0)}(x)^2 + \tilde{\chi}_2 \tilde{H}(x)^2 S^{(0)}(x)^2 \]

\[ + \tilde{\xi} \tilde{H}(x) \tilde{T}(x) \tilde{H}(x) S^{(0)}(x) . \]  

If the bulk singlet is localized close to the TeV brane, it then acquires a weak scale VEV, which leads to a large linear term for the triplet potential through the \( \xi \) term, similar to the case with a bulk triplet discussed previously. Thus the bulk singlet must be localized close to the Planck
brane, \textit{i.e.} \( \alpha_S > 1 \). In this case, the effective couplings are given by,

\[
\tilde{\chi}_i \simeq \chi_i \left( \frac{k}{M_{pl}} \right) (\alpha_S - 1) e^{-2(\alpha_S-1)\pi k R} \tag{41}
\]

\[
\tilde{\xi} \simeq \xi \sqrt{\frac{k}{M_{pl}}} \sqrt{\alpha_S - 1} e^{-(\alpha_S-1)\pi k R} . \tag{42}
\]

By minimizing the effective potential given in Eq.\,(40), we obtain the following conditions,

\[
2 \tilde{\lambda}_H v^2 + [\tilde{\mu}_H + \tilde{\eta}u^2 + \tilde{\chi}_2 w^2 + \tilde{\xi} w v] = 0 \tag{43}
\]

\[
2u \left[ 2 \tilde{\lambda}_T u^2 + \tilde{\mu}_T + \tilde{\eta}_v^2 + \tilde{\chi}_1 w^2 \right] + \tilde{\xi}_v^2 w = 0 \tag{44}
\]

\[
2w \left[ 2 \tilde{\lambda}_S w^2 + \tilde{\chi}_1 u^2 + \tilde{\chi}_2 v^2 \right] + \tilde{\xi}_v^2 u = 0 \tag{45}
\]

where we have defined \(< \tilde{H}(x) >= v, < \tilde{\xi}_0(x) >= u \) and \(< \tilde{S}^{(0)}(x) >= w \), with \( \tilde{\xi}_0 \) being the neutral component of the triplet, \( \tilde{T}(x) \). If \( \chi_1 > 0 \) and \( \chi_2 < 0 \), assuming \( v \gg w \gg u \) we obtain the following solutions,

\[
v = \sqrt{\frac{-\tilde{\mu}_H}{2 \lambda_H}} = \sqrt{\frac{-\mu_H}{2 \lambda_H} v} \tag{46}
\]

\[
w = \sqrt{\frac{-\tilde{\chi}_2 v^2}{2 \lambda_S}} = \sqrt{\frac{-\chi_2}{2 \lambda_S} \frac{k}{M_{pl}} \sqrt{\alpha_S - 1} v e^{-(\alpha_S-1)\pi k R}} \tag{47}
\]

\[
u = -\frac{\tilde{\xi} v^2 w}{2 \mu_T + 2 \tilde{\eta} u^2} = -\frac{1}{2(\mu_T + \eta)} \xi \sqrt{\frac{-\chi_2}{2 \lambda_S} (\alpha_S - 1)} \left( \frac{k}{M_{pl}} \right) v e^{-2(\alpha_S-1)\pi k R} . \tag{48}
\]

As we can see the assumption \( v \gg w \gg u \) is justified. A crucial point to note is that, in order for \( w \) given in Eq. \,(47) to be the true vacuum, \( \chi_2 \) must be negative. Otherwise the VEV will be determined by the linear term of \( S \), leading a large VEV for the singlet. Similarly, in order for \( u \) given in Eq. \,(48) to be the true vacuum, \( \chi_1 \) must be positive. Otherwise the VEV will be determined by the quartic and quadratic terms of \( T \) in the potential, leading to a large VEV for the triplet.

The Yukawa coupling for the charged fermions to the Higgs doublet reads

\[
\frac{Y_{ij}}{M_{pl}} \int d^4x \int dy \sqrt{-g} \Psi_{R,i}(x,y) \Psi_{L,j}(x,y) H(x) \delta(y - \pi R) , \tag{49}
\]

where \( Y_{ij} \) are dimensionless \( \mathcal{O}(1) \) coefficients. The \( SU(2) \) Higgs doublet is assumed to be confined to the TeV brane as we have explained in Sec.\,II. The effective Yukawa coupling in four dimensions
is obtained after integrating out the fifth coordinate, $y$:

$$
\tilde{Y}_{ij} = \frac{Y_{ij}}{M_{pl}} \left( \frac{1}{\pi R} \right)^{1/2} \int_{-\pi R}^{\pi R} dy \sqrt{-g} \left( e^{(2c_{R,i})} \right)^{\sigma} e^{(2c_{L,j})} \sigma e^{\sigma} \delta(y - \pi R) \\
= \frac{Y_{ij}}{2 M_{pl}} \frac{k}{\sqrt{(e^{(1-2c_{R,i})} - 1)(e^{(1-2c_{L,j})} - 1)}} \int_{-\pi R}^{\pi R} dy \sqrt{-g} e^{(2c_{R,i})} e^{(2c_{L,j})} \sigma e^{\sigma} \delta(y - \pi R). 
$$

(50)

The interaction between the leptons and the triplet gives the Majorana neutrino masses,

$$
\lambda_{ij} \int d^4x \int dy \sqrt{-g} \nu^T_L(x, y) \nu_L(x, y) \xi_0(x) \delta(y - \pi R).
$$

(51)

Because the triplet is also confined to the TeV brane, the effective coupling of the leptons to the triplet is of the same form as that to the SM Higgs doublet,

$$
\tilde{\lambda}_{ij} = \frac{\lambda_{ij}}{2 M_{pl}} \frac{k}{\sqrt{(e^{(1-2c_{R,i})} - 1)(e^{(1-2c_{L,j})} - 1)}} \int_{-\pi R}^{\pi R} dy \sqrt{-g} e^{(2c_{R,i})} e^{(2c_{L,j})} \sigma e^{\sigma} \delta(y - \pi R).
$$

(52)

In a large parameter space, $\tilde{\lambda}_{ij}$ and $\tilde{Y}_{ij}$ are of the same order. Thus small neutrino Majorana masses (see Eq. (23)) are obtained because $\langle \xi_0(x) \rangle$ is highly suppressed relative to $\langle \tilde{H}(x) \rangle$ due to the warped geometry.

**IV. NUMERICAL RESULTS**

In order to analyze the phenomenological implications of our model, we define

$$
\epsilon(x) \equiv \frac{\sqrt{x}}{\sqrt{e^{x\pi k R} - 1}}, \quad \eta(x) \equiv e^{-x\pi k R},
$$

(53)

and write the coupling constants, $\tilde{\lambda}_{ij}$ and $\tilde{Y}_{ij}$, as

$$
\tilde{\lambda}_{ij} = \frac{\lambda_{ij}}{2 M_{pl}} \frac{k}{\sqrt{(e^{(1-2c_{R,i})} - 1)(e^{(1-2c_{L,j})} - 1)}} \int_{-\pi R}^{\pi R} dy \sqrt{-g} e^{(2c_{R,i})} e^{(2c_{L,j})} \sigma e^{\sigma} \delta(y - \pi R)
$$

(54)

$$
\tilde{Y}_{ij} = \frac{Y_{ij}}{2 M_{pl}} \frac{k}{\sqrt{(e^{(1-2c_{R,i})} - 1)(e^{(1-2c_{L,j})} - 1)}} \int_{-\pi R}^{\pi R} dy \sqrt{-g} e^{(2c_{R,i})} e^{(2c_{L,j})} \sigma e^{\sigma} \delta(y - \pi R).
$$

(55)

Without imposing any additional symmetry, the 5D Yukawa couplings to the triplet for all the three families are universal, that is, $\lambda_{ij} = 1$ for all $(ij)$. In this case, the neutrino mass matrix arising from this mechanism has the nearest neighbor structure, and to the leading order has the following form,

$$
M_\nu \sim \begin{pmatrix}
 t^2 & t & t \\
 t & 1 & 1 \\
 t & 1 & 1
\end{pmatrix},
$$

(56)
where we have set $c_{L, \mu} = c_{L, \tau}$, which is implied by the maximal mixing angle in the atmospheric neutrino sector, $\sin^2 2\theta_{\mu\tau} = 1.00$ [14]. A neutrino mass matrix of this form, nonetheless, always gives a solar mixing angle $\tan^2 \theta_{\odot} > \pi/4$, while the presence of the matter effect requires the solar mixing angle to be in the dark side, $\tan^2 \theta_{\odot} < \pi/4$ [15]. We thus impose a $Z_4$ symmetry, under which, the fermions transform in the following way,

$$\begin{align*}
(\Phi_{L,\mu}, \Phi_{R,e}, \Phi_{R,\tau}) &\xrightarrow{Z_4} (i) (\Phi_{L,\mu}, \Phi_{R,e}, \Phi_{R,\tau}) \quad (57) \\
(\Phi_{L,e}, \Phi_{L,\tau}, \Phi_{R,\mu}) &\xrightarrow{Z_4} (-i) (\Phi_{L,e}, \Phi_{L,\tau}, \Phi_{R,\mu}) \quad (58)
\end{align*}$$

The scalar fields are neutral under this $Z_4$ symmetry. This $Z_4$ symmetry is softly broken in the bulk through the Yukawa couplings to the triplet scalar field,

$$\lambda_{ij} = \begin{pmatrix} x & 1 & x \\ 1 & x & 1 \\ x & 1 & x \end{pmatrix}. \quad (59)$$

where the parameter $x < 1$ characterizes the amount of breaking, and the neutrino mass matrix is then given by $(M_\nu)_{ij} = \bar{\lambda}_{ij} m_\nu$, where the overall mass scale $m_\nu$ is,

$$m_\nu = \frac{1}{2(\mu_T + \eta)} \xi \sqrt{-\frac{\chi_2}{2\lambda_S}} (\alpha_S - 1) \left( \frac{k}{M_{pl}} \right) v e^{-2(\alpha_S-1)\pi kR}.$$ 

(60)

The bounds from the electroweak precision measurements on the fermionic mass parameters are rather weak for $c \geq 1/2$; however, the constraints grow strong rapidly for $c < 1/2$ [16]. As a result, we consider only the region $c \geq 1/2$. Clearly, many possible sets of bulk mass parameters can be chosen to accommodate the observed fermion masses and mixing angles. Let us assume that both the $\mu$ and $\tau$ lepton doublets are de-localized, that is, $c_{L,\mu} = c_{L,\tau} = 1/2$.

We consider $k \simeq M_{pl}$ and $kR = 11$ which translates into a warp factor $e^{-\pi kR} = 9.8 \times 10^{-16}$. The dimensionless parameters in the scalar potential are chosen to be $(\lambda_S, \mu_T, \eta, \xi, \chi_2) = (1/2, 1/2, 1/2, -1, -1)$. The rest of the parameters in the scalar sector are irrelevant, as long as they are of $O(1)$. With the choice $c_{L,\mu} = c_{L,\tau} = 1/2$, using $\Delta m^2_{\text{atm}} \equiv m_3^2 - m_2^2 \simeq m_3^2 = 2.3 \times 10^{-3} \text{eV}^2$ as an input, the bulk scalar mass parameter is determined to be $\alpha_S = 1.35$, which corresponds to a bulk mass term $a_S k^2 = -2.18 k^2$. The VEV of the neutral component of the triplet is highly suppressed, $\langle \xi^{(0)}(x) \rangle \simeq (1 - \alpha_S) e^{-2(\alpha_S-1)\pi kR} v \simeq 1.11 \times 10^{-11} v$, where the VEV of the SM doublet, $v$, is $v = 174 \text{ GeV}$. At the first glance one might think that this bulk mass term is larger than the 5D Planck scale. This turns out not to be the case. In order to derive the metric given in Eq. (1), one has to assume that the 5-dimensional curvature scalar, $R_5 = -20 k^2$, satisfies $|R_5| < M_{pl}^2 [1].$
This condition translates into an upper bound on $k$. It can be seen easily that, by slightly lowering the value of $k$, the condition $|a_S k^2| < |R_5| < M^2 \mu$ is satisfied, and thus our result can be trusted.

Using the LMA solution, $\Delta m^2_{\odot} \equiv m^2_2 - m^2_1 = 8.0 \times 10^{-5} \text{eV}^2$ as an input \cite{17, 18, 19, 20}, we find,

$$c_{L,e} = 0.55 \text{ .}$$

With $x = 0.797$, the neutrino mass matrix is then given by

$$M_\nu = \begin{pmatrix}
0.00130 & 0.00486 & 0.00387 \\
0.00486 & 0.0115 & -0.0145 \\
0.00387 & -0.0145 & 0.0115
\end{pmatrix} \cdot m_\nu \text{ , where } m_\nu = \left\langle \xi^0(x) \right\rangle = 1.11 \times 10^{-11} \text{ eV} \text{ .}$$

These parameters give

$$m_1 = 0.0110 \text{ eV}, \quad m_2 = 0.0142 \text{ eV}, \quad m_3 = 0.0501 \text{ eV} \text{ ,}$$

and the full neutrino mixing matrix reads

$$V_{\alpha i} \simeq \begin{pmatrix}
0.813 & -0.581 & 0.0298 \\
0.392 & 0.584 & 0.711 \\
0.431 & 0.567 & -0.702
\end{pmatrix}$$

where $\alpha = e, \mu, \tau$ (flavor eigenstates) and $i = 1, 2, 3$ (mass eigenstates). The predictions with these parameters are

$$\tan^2 \theta_\odot = 0.512 \text{ (agrees with LMA solution \cite{19, 20})},$$

$$|U_{e\nu_3}| \sim 0.0298 \text{ (agrees with CHOOZ 1\sigma bound } |U_{e\nu_3}| < 0.12 \text{ \cite{19, 20, 21})} \text{ .}$$

Now we consider the charged lepton sector. The Yukawa coupling is dictated by the $Z_4$ charge assignment given in Eq\cite{57} and \cite{58} it is,

$$Y_e = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}$$

Using the charged lepton masses \cite{13}

$$m_e = 0.504 \text{ MeV}, \quad m_\mu = 108 \text{ MeV}, \quad m_\tau = 1.77 \text{ GeV} \text{ ,}$$

we found the following solutions for $c_R$:

$$c_{R,e} = 0.776, \quad c_{R,\mu} = 0.622, \quad c_{R,\tau} = 0.521 \text{ ,}$$

13
FIG. 1: The profile for the zero mode of various bulk fields with corresponding values of $c$-parameters considered in the model. Taking into account the measure due to the warp geometry, we have $z = 1/2$ for fermions, and $z = -1$ for bosons. In this case the triplet and the SM Higgs doublet are both confined to the TeV brane.

and the corresponding charged lepton mass matrix is given by,

$$
M_e = \begin{pmatrix}
-2.66 \times 10^{-4} & 0 & 7.92 \times 10^{-4} \\
0 & 0.108 & 0 \\
0.563 & 0 & 1.68
\end{pmatrix} \text{GeV}.
$$

The $1-3$ mixing in the charged lepton mass matrix is of the order of $O(10^{-4})$, which is negligibly small and do not affect the predictions given in Eq. 65 and 66 to the accuracy we are considering. This set of $c$-parameters give rise to the profiles of the zero mode of various bulk fields as shown in Fig. 1.

We comment that the mass of the $SU(2)$-triplet scalar field is of the order of a TeV, which is substantially lower than that in the usual triplet mechanism implemented in 4D [11]. In the usual 4D case, the triplet mass is required to be of the order of $10^{13}$GeV. The difference between these two cases arises because the tri-linear coupling constant between the SM Higgs doublet and the $SU(2)$ triplet in the usual 4D case is of the order of the weak scale, while it is suppressed by the warped factor in our case. The triplet with a TeV scale mass could be produced at the upgraded Tevatron or at the Large Hadron Collider. By measuring the decay branching ratio of

$$
\xi^{++} \rightarrow l_i^+ l_j^+
$$

it is possible to map out the whole triplet coupling matrix, $\tilde{\lambda}_{ij}$, and thus the mixing matrix in the
leptonic sector, making this model verifiable. The presence of this $SU(2)$-triplet Higgs might also have interesting implication for leptogenesis through the decay of the doubly charged component of the triplet scalar field \cite{22}. The TeV-brane induced mass of the singlet zero mode, on the other hand, is suppressed by $e^{-(\alpha_S-1)\pi kR}$ relative to the weak scale, due to the small overlap between its wave function and the SM Higgs. The induced mass of the singlet zero mode is closely linked to the absolute mass scale of the neutrinos. For $\alpha_S = 1.35$, it is $\sim ve^{-(\alpha_S-1)\pi kR} \simeq 0.5$ MeV. As the singlet does not have any SM interactions (it only couples weakly to the SM Higgs and the triplet through the couplings $\chi_i$ and $\xi$), there are no experimental constraints on the mass of the singlet.

V. CONCLUSIONS

In this paper we propose a new mechanism which naturally gives rise to small neutrino Majorana masses in 5D with warped geometry. This is realized at tree level by coupling the lepton doublets to the $SU(2)$ triplet scalar field. The smallness of neutrino masses is due to the small overlap between the profile of a bulk singlet zero mode, which is localized close to the Planck brane, and the TeV-brane confined SM Higgs and the triplet. We emphasize that even though this mechanism is not predictive in the sense that it does not reduce the number of parameters in the Yukawa sector, it provides a way to generate large mass hierarchy from parameters that are all naturally of $\mathcal{O}(1)$. The generic form for the neutrino mass matrix due to the overlap between the fermions is not compatible with the LMA solution. This is overcome by imposing a $Z_4$ symmetry, which is softly broken by the couplings of the triplet Higgs to the lepton doublets. This model successfully reproduces the observed masses and mixing angles in charged lepton sector as well as in the neutrino sector, in addition to having a prediction of small $|U_{e3}| \sim \mathcal{O}(0.01)$, which is in the range accessible to the future long baseline neutrino experiments \cite{23}. As the mass of the triplet scalar field is of the TeV scale, it could be produced at the upcoming collider experiments. Once it is produced, by measuring the decay branching ratio of $\xi^{++} \rightarrow l_i^+ l_j^+$, it is possible to map out the whole triplet coupling matrix, thus the mixing matrix in the neutrino sector.
Acknowledgments

This work was supported, in part, by the U.S. Department of Energy under grant number DE-AC02-98CH10886. I thank Ryu Kitano and K.T. Mahanthappa for helpful conversations.