A Remark on Boundary Effects in Static Vacuum Initial Data sets

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Abstract

Let \((M, g)\) be an asymptotically flat static vacuum initial data set with non-empty compact boundary \(\Sigma\). We prove that \((M, g)\) is isometric to a spacelike slice \((\mathbb{R}^3 \setminus B_{\frac{1}{m}}(0), (1 + \frac{m}{2|x|}^4)\delta_{ij})\) of a Schwarzschild spacetime under the mere assumption that \(\Sigma\) has zero mean curvature, hence generalizing a classic result of Bunting and Masood-ul-Alam. In the case that \(\Sigma\) has constant positive mean curvature and satisfies a stability condition, we derive an upper bound of the ADM mass of \((M, g)\) in terms of the area and mean curvature of \(\Sigma\). Our discussion is motivated by Bartnik’s quasi-local mass definition.

1 Introduction

Throughout this paper we let \((M, g)\) be an asymptotically flat three dimensional Riemannian manifold that has one end and non-empty smooth compact boundary \(\Sigma\). Following [6], we say \((M, g)\) is static if there exists a function \(u\), called the static potential of \((M, g)\), satisfying \(u \to 1\) at the end of \((M, g)\) and

\[
\begin{aligned}
\{ \quad u\text{Ric}(g) &= D^2 u \\
\Delta u &= 0, 
\end{aligned}
\]

(1)

where \(D^2 u\) is the Hessian of \(u\) and \(\text{Ric}(g)\) is the Ricci curvature of \(g\). It is well known that if \((M, g)\) and \(u\) satisfy (1), the asymptotically flat spacetime metric \(\tilde{g} = -u^2 dt^2 + g\) solves the Vacuum Einstein Equation on \(M \setminus u^{-1}(0) \times \mathbb{R}\).
where $u^{-1}(0)$ is the zero-set of $u$. It is also known that $u^{-1}(0)$, if non-empty, is an embedded totally geodesic hypersurface in $M$. As a result, it implies that $(M, g)$ has zero scalar curvature.

A classic result of Bunting and Masood-ul-Alam states that, if $(M, g)$ is static and $u = 0$ on $\Sigma$, $(M, g)$ is isometric to a spacelike slice $(\mathbb{R}^3 \setminus B_{\frac{m}{2}}(0), (1 + \frac{m}{2|x|})^4 \delta_{ij})$ of a Schwarzschild spacetime with positive mass $m$. In particular, $\Sigma$ is a connected two sphere.

The condition $u = 0$ on $\Sigma$ is a natural assumption as it corresponds to that the static killing vector field $\partial_t$ of $\bar{g}$ vanishes at $\Sigma$. On the other hand, from a point of view of quasi-local mass question, the mean curvature of $\Sigma$, denoted by $H_\Sigma$, and the induced metric on $\Sigma$, denoted by $g|_\Sigma$, also represent important boundary condition. In [1], Bartnik proposed his quasi-local mass definition and $H_\Sigma$ and $g|_\Sigma$ constitute the geometric boundary constraint in his static metric extension conjecture [2] [10]. Thus, it is natural to ask whether Bunting and Masood-ul-Alam’s result holds under the mere assumption $H_\Sigma = 0$. We give a positive answer to this question.

**Theorem 1** Let $(M, g)$ be an asymptotically flat manifold with non-empty smooth compact boundary $\Sigma$. If $(M, g)$ is static and $\Sigma$ has zero mean curvature, then $(M, g)$ is isometric to a spacelike slice $(\mathbb{R}^3 \setminus B_{\frac{m}{2}}(0), (1 + \frac{m}{2|x|})^4 \delta_{ij})$ of a Schwarzschild spacetime with positive mass $m$. In particular, $\Sigma$ is a connected two sphere.

An immediate corollary of Theorem 1 is a non-existence result on horizons (stable minimal two spheres) in a static and asymptotically flat manifold whose boundary has non-negative mean curvature. Our sign convention on mean curvature gives $H_{\partial B_1(0)} = 2$, where $\partial B_1(0)$ is the boundary of the Euclidean exterior region $\mathbb{R}^3 \setminus B_1(0)$.

**Corollary 1** Let $(M, g)$ be an asymptotically flat manifold with non-empty smooth compact boundary $\Sigma$. If $(M, g)$ is static and $\Sigma$ has non-negative mean curvature, then there is no horizon (allowed to have multiple components) in $M$ enclosing $\Sigma$.

We end this paper by applying the idea of the proof of Theorem 1 to the case where $\Sigma$ has constant positive mean curvature. As a result, we derive an upper bound of the ADM mass of $(M, g)$ in terms of the area and mean curvature of $\Sigma$(see Section 3).
2 Proof of the Theorem

We assume that $g$ has sufficient boundary regularity, say $g$ is $C^3$ on the closure of $M$, then $u$ is $C^2$ on the closure of $M$ \[6\]. We first prove Theorem \[1\] under an additional assumption that $u$ is entirely positive in $M$.

Proposition 1 Assume that $(M, g)$ is asymptotically flat and static. Suppose that $(M, g)$ admits a positive static potential $u$. If $\Sigma$ has zero mean curvature, then $(M, g)$ is isometric to a spacelike slice $(\mathbb{R}^3 \setminus B_m(0), (1 + \frac{m}{2|x|})^4 \delta_{ij})$ of a Schwarzschild spacetime with positive mass $m$.

Proof: The asymptotic flatness of $g$ and the equation $\triangle u = 0$ guarantee that $u$ has an asymptotic expansion at $\infty$

$$u = 1 - \frac{m}{|x|} + O(|x|^{-2}), \quad (2)$$

where $m$ is some constant. It is a non-trivial fact of \[1\] that $m$ is indeed the ADM mass of $(M, g)$ \[3\]. We now consider the Green function $G$ at $\infty$ defined by

$$\begin{cases}
\triangle G &= 0 \quad \text{in } M \\
G &\to 1 \quad \text{at } \infty \\
G &= 0 \quad \text{at } \Sigma.
\end{cases} \quad (3)$$

$G$ has its own asymptotic expansion

$$G = 1 - \frac{A}{|x|} + O(|x|^{-2}). \quad (4)$$

Since $u \geq 0$ on $\Sigma$, it follows from the maximum principle that $u \geq G$ on $M$, hence $m \leq A$. On the other hand, it is proved by Bray (Theorem 9 on page 206 in \[3\]) that $m \geq A$ for any asymptotically flat manifold $M$ whose scalar curvature is non-negative and boundary mean curvature is zero, furthermore the equality holds if and only if $(M, g)$ is isometric to a manifold $(\mathbb{R}^3 \setminus B_m(0), (1 + \frac{m}{2|x|})^4 \delta_{ij})$. Therefore, we have $m = A$ and Proposition 1 follows from Bray’s theorem.

Next, we show that Theorem \[1\] holds if $\Sigma$ consists of outermost minimal surfaces of $(M, g)$. A minimal surface is called outermost \[3\] if it is not contained entirely inside another minimal surface. Here “contained entirely inside” is defined with respect to the end of $(M, g)$.

Proposition 2 Assume that $(M, g)$ is asymptotically flat and static. If $\Sigma$ consists of outermost minimal surfaces of $(M, g)$, then $(M, g)$ is isometric...
to a spacelike slice \( (\mathbb{R}^3 \setminus B_{\frac{m}{2}}(0), (1 + \frac{m}{2|x|})^4 \delta_{ij}) \) of a Schwarzschild spacetime with positive mass \( m \).

To prove Proposition 2, we need a lemma concerning the equation of \( u \) when restricted to a hypersurface. For a given hypersurface \( \Sigma \subset M \), we let \( \triangle_{\Sigma} \) denote the Laplacian operator of the induced metric on \( \Sigma \), \( \nu \) denote the unit normal vector at \( \Sigma \) pointing to \( \infty \), \( A_{\Sigma} \) denote the second fundamental form of \( \Sigma \) and \( K_{\Sigma} \) be the Gaussian curvature of \((\Sigma, g|_{\Sigma})\).

**Lemma 1.** Assume that \((M, g)\) is static and \( u \) is a static potential. Suppose that \( \Sigma \subset M \) is a smooth compact hypersurface. Then the restriction of \( u \) to \( \Sigma \) satisfies

\[
\triangle_{\Sigma} u + H_{\Sigma} \frac{\partial u}{\partial \nu} + \left( \frac{1}{2} H_{\Sigma}^2 - \frac{1}{2} |A_{\Sigma}|^2 - K_{\Sigma}\right) u = 0.
\]  

(5)

**Proof:** Let \( U \) be a Gaussian tubular neighborhood of \( \Sigma \) in \( M \) such that \( U \) is diffeomorphic to \( \Sigma \times (-\epsilon, \epsilon) \) and \( g|_{U} \) has the form \( g = g_t + dt^2 \), where \( t \) is the coordinate of \((-\epsilon, \epsilon)\) and \( g_t \) is the induced metric on the slice \( \Sigma \times \{t\} \).

We arrange the direction of \( \partial_t \) so that \( \partial_t \) points to \( \infty \). At \( \Sigma \), we have

\[
\triangle u = \triangle_{\Sigma} u + H_{\Sigma} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2},
\]  

(6)

where \( \frac{\partial^2 u}{\partial t^2} \) agrees with \( D^2 u(\partial_t, \partial_t) \) because \( \partial_t \) is the velocity vector of a geodesic. Hence it follows from (1) that

\[
\triangle_{\Sigma} u + H_{\Sigma} \frac{\partial u}{\partial \nu} + \text{Ric}(\nu, \nu) u = 0.
\]  

(7)

Applying the Gauss equation and using the fact \( g \) has zero scalar curvature, we have

\[
\text{Ric}(\nu, \nu) = \frac{1}{2} H_{\Sigma}^2 - \frac{1}{2} |A_{\Sigma}|^2 - K_{\Sigma}.
\]  

(8)

Lemma 1 follows from (7) and (8).

**Proof of Proposition 2:** The assumption that \( \Sigma \) consists of outermost minimal surfaces implies that \( \Sigma \) is area outer minimizing \[3 \; 8], i.e. there is no other closed surface enclosing \( \Sigma \) which has less area than \( \Sigma \). In particular, \( \Sigma \) is stable with respect to the second variation of area inside \((M, g)\), hence we have

\[
\int_{\Sigma} |\nabla_{\Sigma} f|^2 - (\text{Ric}(\nu, \nu) + |A_{\Sigma}|^2) f^2 \geq 0
\]  

(9)
for any $f \geq 0$ on $\Sigma$. Now by a general fact that, for any fixed $h$,

$$\inf_{f>0} \left\{ \frac{\int_{\Sigma} |\nabla_{\Sigma} f|^2 - hf^2}{\int_{\Sigma} |f|^2} \right\} = \inf_{f \in W^{1,2}(\Sigma)} \left\{ \frac{\int_{\Sigma} |\nabla_{\Sigma} f|^2 - hf^2}{\int_{\Sigma} |f|^2} \right\},$$

(10)

we see (9) holds without requiring $f \geq 0$. In particular, we can choose $f = u$ to have

$$\int_{\Sigma} |\nabla_{\Sigma} u|^2 - (Ric(\nu, \nu) + |A_{\Sigma}|^2)u^2 \geq 0.$$  

(11)

On the other hand, it follows from Lemma 1 and the assumption $H_{\Sigma} = 0$ that

$$\Delta_{\Sigma} u + Ric(\nu, \nu) u = 0.$$  

(12)

Multiplying it by $u$ and integrating by parts, we have

$$\int_{\Sigma} \left[ |\nabla_{\Sigma} u|^2 - (Ric(\nu, \nu) + |A_{\Sigma}|^2)u^2 \right] + |A_{\Sigma}|^2 u^2 = 0.$$  

(13)

Hence, (11) and (13) imply that

$$\int_{\Sigma} |\nabla_{\Sigma} u|^2 - (Ric(\nu, \nu) + |A_{\Sigma}|^2)u^2 = 0.$$  

(14)

It follows from (9) and (14) that, on each connected component of $\Sigma$, either $u$ is identically zero or $u$ is the first eigenfunction with eigenvalue 0 of the operator $\Delta_{\Sigma} - (Ric(\nu, \nu) + |A_{\Sigma}|^2)$, in which case $u$ must not change sign.

Suppose $u < 0$ on some component of $\Sigma$, we consider $M_+^{\infty} = \{ x \in M \mid u(x) > 0 \}$. It follows from the fact $u^{-1}(0)$ is an embedded totally geodesic hypersurface on which $\frac{\partial u}{\partial \nu}$ is a non-zero constant that $M_+^{\infty}$ has smooth compact boundary $\partial M_+^{\infty}$ on which $u \geq 0$. Let $M_+^{\infty}$ be the connected component of $M_+^{\infty}$ that contains the asymptotic flat end of $M$, it follows from Proposition 1 that $(M_+^{\infty}, g)$ is isometric to a Schwarzschild spacetime slice $((\mathbb{R}^3 \setminus B_{\mathbb{R}}(0)), (1 + \frac{m}{2|x|})^4 \delta_{ij})$. In particular, $\partial M_+^{\infty}$ is the unique connected outermost horizon of $(M, g)$ on which $u = 0$. Therefore, $\Sigma$ must agree with $\partial M_+^{\infty}$ by the assumption that $\Sigma$ consists of outermost minimal surfaces of $(M, g)$. Hence, $u = 0$ on $\Sigma$ which contradicts to the assumption that $u < 0$ on some component of $\Sigma$. We conclude that $u$ must be non-negative on $\Sigma$ and Proposition 2 follows from Proposition 1.

We now can finish the proof of Theorem.
Proof of Theorem\[1\] The assumption that $\Sigma$ is a minimal surface implies that the outermost minimal surface of $(M, g)$, denoted by $\Sigma'$, always exists in $M$ (or $\Sigma'$ may have multiple components). Let $M^\infty$ be the region of $M$ outside $\Sigma'$, Proposition\[2\] implies that $(M^\infty, g)$ is isometric to a Schwarzschild spacetime slice $(\mathbb{R}^3 \setminus B_m^0(0), (1 + \frac{m}{2|x|})^4 \delta_{ij})$. We now resort to a recent result of Chruściel\[5\] on the analyticity of static vacuum metrics at non-degenerate horizons. By the section 4 in\[5\], $(M, g)$ admits a global analytic atlas (even across $u^{-1}(0)$) with respect to which $u$ and $g$ are both analytic. Since the unique analytic continuation of the manifold $(\mathbb{R}^3 \setminus B_m^0(0), (1 + \frac{m}{2|x|})^4 \delta_{ij})$ is the whole Schwarzschild spacetime slice $M^S = (\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{2|x|})^4 \delta_{ij})$, we see that $(M, g)$ is isometric to a region $D$ of $M^S$, where $\partial D$ is a minimal surface.

However, $M^S$ contains no minimal surface other than its neck at $\{|x| = m^2\}$, we conclude that $\Sigma$ agrees with $\Sigma'$. Theorem \[1\] is proved.

A similar argument also gives a proof of Corollary \[1\].

Proof of Corollary\[1\]. Suppose that $(M, g)$ admits a horizon $\Sigma'$ enclosing $\Sigma$ ($\Sigma'$ may have multiple components). Let $M^\infty$ be the region of $M$ outside $\Sigma'$, Theorem \[1\] implies that $(M^\infty, g)$ is isometric to a Schwarzschild spacetime slice $(\mathbb{R}^3 \setminus B_m^0(0), (1 + \frac{m}{2|x|})^4 \delta_{ij})$. By the same reasoning as in the proof of Theorem \[1\] we know that $(M, g)$ is isometric to a region $D$ in a whole Schwarzschild spacetime slice $M^S = (\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{2|x|})^4 \delta_{ij})$, where $\partial D$ is a closed surface in the lower half end $(B_m^0(0) \setminus \{0\}, (1 + \frac{m}{2|x|})^4 \delta_{ij})$ of $M^S$. Since $H_{\Sigma} \geq 0$, the mean curvature of $\partial D$ computed with respect to the unit normal vector pointing towards the horizon neck $\{|x| = m^2\}$ is non-negative. However, $(B_m^0(0) \setminus \{0\}, (1 + \frac{m}{2|x|})^4 \delta_{ij})$ is foliated by a family of constant negative mean curvature surfaces $\{|x| = r\}$, the maximum principle implies that $\partial D$ does not exist. Hence, there is no horizon in $(M, g)$ enclosing $\Sigma$ and Corollary \[1\] is proved.

3 An upper bound of the ADM mass

In \[1\] Bartnik proposed his quasi-local mass definition for a compact region $(\Omega, g_\Omega)$ isometrically contained in an asymptotically flat manifold with non-negative scalar curvature. He further conjectured that there exists a static and asymptotically flat manifold $(M_0, g_0)$ with boundary, depending only on the boundary data $H_{\partial \Omega}$ and $(\partial \Omega, g_{\Omega|\partial \Omega})$, such that the ADM mass of $(M_0, g_0)$ achieves the Bartnik’s quasi-local mass of $(\Omega, g_\Omega)$. Therefore, as a
step to understand Bartnik’s quasi-local mass of a boundary surface, it is interesting to estimate the ADM mass of a given static and asymptotically flat manifold \((M, g)\) in terms of its boundary data \(H_\Sigma\) and \((\Sigma, g|_\Sigma)\).

We consider the simplest case where \(H_\Sigma = H_0\), a positive constant. Motivated by the role played by stable minimal surfaces in the proof of Theorem 1, we assume \(\Sigma\) satisfies an additional stability assumption. For a constant mean curvature surface \(\Sigma'\), we say \(\Sigma'\) is mean-stable if

\[
\int_{\Sigma'} |\nabla_{\Sigma'} \phi|^2 - (Ric(\nu, \nu) + |A_{\Sigma'}|^2)\phi^2 \geq 0
\]

for any \(\phi\) satisfying \(\int_{\Sigma'} \phi = 0\). A result of Huisken and Yau [9] states that, for any asymptotically flat manifold \((M^3, g)\) whose mass is strictly positive, there is a unique foliation of mean-stable constant mean curvature spheres in the asymptotic region.

**Proposition 3** Let \((M, g)\) be an asymptotically flat and static manifold with connected boundary \(\Sigma\). Assume that \(H_\Sigma = H_0\) and \(K_\Sigma \geq \frac{1}{4} H_0^2\). Then, if \(\Sigma\) is mean-stable,

\[
m \leq 4 \sqrt{\frac{16\pi}{H_0^2 |\Sigma|}} m_H(\Sigma),
\]

where \(m\) is the ADM mass of \((M, g)\), \(H_0\) is a positive constant, \(|\Sigma|\) is the area of \(\Sigma\) and \(m_H(\Sigma)\) is the Hawking quasi-local mass of \(\Sigma\). The equality holds if and only if \((M, g)\) is Euclidean.

**Proof:** We first show that the assumption \(K_\Sigma \geq \frac{1}{4} H_0^2\) and \(H_\Sigma > 0\) guarantees that \(0 < u < 1\) in \(M\) unless \(g\) is flat. Applying Lemma 1 to \(\Sigma = \partial M\), we have

\[
-H_\Sigma \frac{\partial u}{\partial \nu} = \Delta_\Sigma u + \left(\frac{1}{2} H_\Sigma^2 - \frac{1}{2} |A_{\Sigma}|^2 - K_\Sigma\right)u.
\]

Let \(p \in \Sigma\) such that \(u(p) = \min_\Sigma u\). Suppose \(u(p) \leq 0\), then \(u(p) = \min_M u\) because \(\Delta u = 0\) and \(u \to 1\) at \(\infty\). Hence, \(\frac{\partial u}{\partial \nu}(p) > 0\) by the Hopf strong maximum principle. Therefore, \(H_\Sigma(p) \frac{\partial u}{\partial \nu}(p) > 0\) by the assumption \(H_\Sigma > 0\).

On the other hand,

\[
\frac{1}{2} H_\Sigma^2(p) - \frac{1}{2} |A_{\Sigma}|^2(p) - K_\Sigma(p) \leq \frac{1}{4} H_\Sigma^2(p) - K_\Sigma(p) \leq 0
\]

by the assumption \(\frac{1}{4} H_\Sigma^2 \leq K_\Sigma\), and \(\Delta_\Sigma u(p) \geq 0\) by the maximum principle. Therefore, we get a contradiction to (15). Hence, \(\min_\Sigma u > 0\). A similar argument shows that \(\max_\Sigma u < 1\) unless \(u \equiv 1\), in which case \((M, g)\) is flat.
It then follows from the maximum principle that $0 < u < 1$ in $M$ unless $u \equiv 1$.

Next, we define $v = \log u$, it follows from (5) that
\begin{equation}
\Delta_\Sigma v + |\nabla_\Sigma v|^2 + H_\Sigma \frac{\partial v}{\partial \nu} = \frac{1}{2}(2K_\Sigma - H_0^2 + |A_\Sigma|^2).
\end{equation}

On the other hand,
\begin{equation}
\int_{S_\infty} \frac{\partial v}{\partial \nu} = \int_{S_\infty} \frac{\partial u}{\partial \nu} = 4\pi m
\end{equation}
and
\begin{equation}
\Delta v + |\nabla v|^2 = \frac{1}{2} \Delta u = 0.
\end{equation}
Integration by parts, we have
\begin{equation}
4\pi m + \int_M |\nabla v|^2 = \int_{\Sigma} \frac{\partial v}{\partial \nu}.
\end{equation}
Integrating (20) on $\Sigma$ and applying $H_\Sigma = H_0$ and (23), we have
\begin{equation}
\int_\Sigma |\nabla_\Sigma v|^2 + H_\Sigma \int_M |\nabla v|^2 + 4\pi m H_\Sigma = \frac{1}{2} \int_\Sigma (|A_\Sigma|^2 - H_0^2_\Sigma) + 4\pi,
\end{equation}
where we used $\int_\Sigma K_\Sigma = 4\pi$ by the Gauss-Bonnet theorem and the fact that $K_\Sigma > 0$. We now apply the mean-stable condition to get a $L^2$ estimate of $|A_\Sigma|$. We follow an idea in [7] and choose $\psi$ to be a conformal map of degree 1 which maps $(\Sigma, g_{|\Sigma})$ onto the standard sphere $S^2 \subset \mathbb{R}^3$. Using the conformal group of $S^2$, we can arrange that each component $\psi_i$ of $\psi$, $i = 1, 2, 3$, satisfies $\int_\Sigma \psi_i = 0$. On the other hand, the Dirichlet integral is conformal invariant in dimension 2, so
\begin{equation}
\int_\Sigma |\nabla_\Sigma \psi_i|^2 = \int_{S^2} |\nabla_{S^2} x_i|^2 = \frac{8\pi}{3}.
\end{equation}
Applying the mean-stability condition (16) to $\psi_i$ and summing over $i$, we get
\begin{equation}
8\pi \geq \int_\Sigma (\text{Ric}(\nu, \nu) + |A_\Sigma|^2).
\end{equation}
It follows from the Gauss equation and the fact $g$ has zero scalar curvature that
\begin{equation}
\text{Ric}(\nu, \nu) + |A_\Sigma|^2 = \frac{1}{2} H_0^2 + \frac{1}{2} |A_\Sigma|^2 - K_\Sigma.
\end{equation}
Hence, \(25\) implies

\[
12\pi - \frac{1}{2} \int_{\Sigma} H_0^2 \geq \frac{1}{2} \int_{\Sigma} |A_{\Sigma}|^2
\]

by the Gauss-Bonnet theorem. It follows from \(24\) that

\[
\int_{\Sigma} |\nabla_{\Sigma} v|^2 + H_0 \int_M |\nabla v|^2 + 4\pi m H_0 \leq 16\pi - \int_{\Sigma} H_0^2. \quad (28)
\]

Hence,

\[
\frac{1}{4\pi H_0} \left[ \int_{\Sigma} |\nabla_{\Sigma} v|^2 + H_0 \int_M |\nabla v|^2 \right] + m \leq 4 \sqrt{\frac{16\pi}{H_0^2 |\Sigma|}} m_H(\Sigma), \quad (29)
\]

where

\[
m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \right) \quad (30)
\]

is the Hawking quasi-local mass of \(\Sigma\). Proposition \(3\) follows from \(29\).

**Corollary 2** Let \((M, g)\) be an asymptotically flat and static manifold with boundary \(\Sigma\). Assume that \((\Sigma, g|_{\Sigma})\) is isometric to the standard unit sphere \(S^2 \subset \mathbb{R}^3\) and \(\Sigma\) has constant mean curvature \(2\). If \(\Sigma\) is mean-stable, then \((M^3, g)\) is isometric to \(\mathbb{R}^3 \setminus B_1(0)\).

**Proof:** The boundary assumption implies that \(m_H(\Sigma) = 0\). Hence, Proposition \(3\) implies that \(m \leq 0\). On the other hand, we can glue the Euclidean ball \(B_1(0)\) and \((M, g)\) along the boundary and the generalized positive mass theorem in [11] implies that \(m \geq 0\). Therefore, \(m = 0\) and \((M, g)\) is isometric to the Euclidean exterior region \(\mathbb{R}^3 \setminus B_1(0)\) by the theorem in [11].

**Remark** One can also prove Corollary \(2\) by showing that \(27\) implies \(A_{\Sigma} = g|_{\Sigma}\) and proving \(u \equiv 1\) in a way similar to the derivation of \(0 < u < 1\) in Proposition \(3\). We choose the above proof to demonstrate the expectation that a static metric might be the minimal mass metric, hence minimizes the ADM mass.

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References


