Structure Functions and Pair Correlations of the Quark-Gluon Plasma

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Abstract

Recent experiments at RHIC and theoretical considerations indicate that the quark-gluon plasma, present in the fireball of relativistic heavy-ion collisions, might be in a liquid phase. The liquid state can be identified by characteristic correlation and structure functions. Here definitions of the structure functions and pair correlations of the quark-gluon plasma are presented as well as perturbative results. These definitions might be useful for verifying the quark-gluon-plasma liquid in QCD lattice calculations.

I. INTRODUCTION

Relativistic heavy-ion collision experiments at RHIC have found evidences for a new state of matter, the quark-gluon plasma (QGP) [1,2]. It is expected to exist in the hot phase of the fireball in ultrarelativistic nucleus-nucleus collisions as well as in the early Universe for the first few microseconds after the Big Bang at temperatures above $T_c = 170$ MeV [3]. In the limit of an infinitely high temperature the QGP should be an ideal gas because the effective temperature dependent coupling constant becomes small due to asymptotic freedom. Therefore in the (extremely) high temperature regime the interaction between quarks and gluons is weak and the QGP can be described by perturbative methods [4]. However, at temperatures which can be realized in accelerator experiments, i.e., at maximum a few times of the transition temperature, the QGP is a strongly interacting many-body system.

In classical, non-relativistic electromagnetic plasmas one distinguishes between weakly coupled and strongly coupled plasmas. For this purpose the Coulomb coupling parameter, $\Gamma = Q^2/(dT)$, where $Q$ is the charge of the plasma particles, $d$ the inter-particle distance, and $T$ the plasma temperature, is considered [5]. In the case of an one-component plasma with a pure Coulomb interaction the Coulomb coupling parameter corresponds to the ratio of the interaction energy to the thermal energy per particle. Most plasmas in nature and in the laboratory are weakly coupled, i.e., $\Gamma \ll 1$. If $\Gamma$ is not much smaller than one, the plasma is called strongly coupled. These plasmas can be in a gas, liquid, or even solid phase [5]. For $\Gamma > O(1)$ the plasma behaves like a liquid and for $\Gamma > 172$ the plasma particles
are predicted to arrange in ordered structures, the plasma crystal [5]. The latter state was discovered in so-called complex or dusty plasmas [6].

In real plasmas the Coulomb interaction is modified to a Yukawa type interaction due to screening. Then the Coulomb coupling parameter, defined above, is no longer the ratio of the interaction to thermal energy. An additional parameter, the ratio of the inter-particle distance to the Debye screening length, \( \kappa = d/\lambda_D \), becomes important. If \( \kappa \gg 1 \) the plasma is weakly coupled. Now the transitions to the liquid and solid phases are considered within a phase diagram in the \( \Gamma-\kappa \)-plane (see e.g. Ref. [7]). For larger values of \( \kappa \) a higher value of \( \Gamma \) is required to achieve a phase transition to a more ordered phase (gas-liquid or liquid-solid). However, as long as \( \kappa \) is of the order of one or smaller, the phase transitions are not much shifted to higher values of \( \Gamma \) [7]. In addition, it should be noted that in strongly coupled plasmas the screening potential contains a long-range power law contribution [5].

In the case of a QGP the coupling parameter is estimated to be \( \Gamma = 2C g^2/(4\pi d T) = 1.5 - 5 \) [8,9]. Here \( C \) is the Casimir invariant (\( C = 4/3 \) for quarks and \( C = 3 \) for gluons), \( d \simeq 0.5 \) fm the inter-particle distance, and \( T \) the temperature assumed to be about 200 Mev corresponding to a strong coupling constant \( g \simeq 2 \). The factor 2 in the numerator comes from taking into account the magnetic interaction in addition to the static electric (Coulomb) interaction, which are of the same magnitude in ultrarelativistic plasmas.

As a matter of fact, the true coupling parameter of the QGP might be even larger because the potential might differ from a simple Coulomb or Yukawa potential corresponding to a one-gluon exchange. Indeed the cross section enhancement by about a factor of 80 due to non-perturbative effects discussed in Ref. [10] indicates a stronger interaction potential. The fact that the cross section enhancement by a factor 2 - 9 due to modifications of the scattering for \( \Gamma = 1.5 - 5 \) [9] does not explain the observed cross section enhancement suggests that a realistic coupling parameter could be up to an order of magnitude larger.

Following the investigations on non-relativistic plasmas, we proposed that the QGP in relativistic heavy-ion collisions is a liquid rather than a gas [9] which is supported by experimental results [2] as well as other theoretical considerations [11–14]. Among those are an observed strong elliptic flow, a strong jet quenching and a fast thermalization corresponding to large parton cross sections, and a small viscosity indicating the QGP as an ideal fluid.

The presence of a liquid QGP expands the phase diagram of strongly interacting matter by a new phase transition from a QGP liquid to a gas at high temperatures. Also a new critical point in the phase diagram should show up, above which only a supercritical fluid exists. Assuming that this phase transition occurs at \( \Gamma \simeq 1 \) - the distance parameter in the QGP, \( \kappa = 1 - 3 \), is sufficiently small - and that the temperature dependent coupling constant decreases logarithmically with the temperature, the transition temperature is estimated to be of the order of a few \( T_c \). Such a temperature could be reached at LHC, where the liquid-gas transition might occur during the expansion of the fireball [15].

The phase diagram of strongly interacting matter can also be studied within lattice QCD. However, the phase transition from the liquid to the gas phase at high temperatures is probably difficult to observe in lattice calculations [16]. However, correlation functions, which can be investigated by lattice simulations, can also provide information on the state of the QGP.
II. CORRELATION FUNCTIONS IN LIQUIDS

Quantitative investigations of liquids are possible by considering correlation functions [17]. In particular, the pair correlation or radial distribution functions in coordinate space as well as the dynamic and static structure functions in momentum space show characteristic properties within the different phases. For example, in the case of a liquid the pair correlation and its Fourier transform, the static structure function, exhibit a pronounced peak and one or two small and broad additional peaks. The first peak corresponds in coordinate space to the inter-particle distance, which is fixed in an incompressible liquid corresponding to a short-range order. In the case of a solid crystalline phase, where also a long-range order exists, a number of sharp peaks are observed. In the gas phase, where no order is present, the pair correlation function shows no clear structures.

Here we want to consider the static structure function, which shows a similar peak structure as the pair correlation function [17]. Furthermore we will also discuss the dynamic structure function giving additional information about the system. First we will define these structure functions. These definitions might be in particular useful for confirming and investigating the liquid phase of the QGP in QCD lattice simulations. To demonstrate the structure functions and for comparison with future lattice calculations we will calculate them perturbatively. Of course, in this case we do not expect any liquid behavior, e.g., a single clear peak, as the perturbative regime corresponds to the gas phase of the QGP with $\Gamma \ll 1$.

The static density-density autocorrelation function is defined as [17,18]

$$ G(r) = \frac{1}{N} \int d^3r' \langle \rho(r + r', t)\rho(r', t) \rangle, $$

(1)

where $N$ is the total particle number and

$$ \rho(r, t) = \sum_{i=1}^{N} \delta[r - r_i(t)] \] $$

(2)

the local density of point particles at the positions $r_i$ at time $t$.

The density-density autocorrelation function is related to the pair correlation or radial distribution function,

$$ g(r) = \frac{1}{N} \sum_{i\neq j}^{N} \delta(r + r_i - r_j), $$

(3)

by

$$ G(r) = g(r) + \delta(r). $$

(4)
The static structure function, defined by

\[ S(p) = \frac{1}{N} \langle \rho(p) \rho(-p) \rangle \]  

with the Fourier transformed particle density,

\[ \rho(p) = \int d^3r \rho(r) e^{-ip \cdot r}, \]  

is the Fourier transform of the density-density autocorrelation function

\[ S(p) = \int d^3r G(r) e^{-ip \cdot r}. \]  

For uncorrelated particles the structure function is constant for \( p \neq 0 \) [18]. The typical behavior of the static structure function in an interacting gas and in a liquid is shown in Fig.1 according to Ref. [5]. There a one-component Yukawa system was assumed. Note that in a one-component Yukawa system there is no gas-liquid phase transition but only a supercritical fluid exists. After all the structure function shows a gas like behavior for small coupling parameters, whereas a peak structure (fluid like behavior) develops only at rather high values of \( \Gamma \) [5]. Only in the presence of repulsive and attractive forces at the same time a gas-liquid transition shows up, ending at a critical point. In the QGP, where a gas-liquid phase transition could exist [19], the appearance of peaks in the static structure function could correspond to a real liquid phase.

The time-dependent generalization of the density-density autocorrelation function,

\[ G(r, t) = \frac{1}{N} \int d^3r' \langle \rho(r + r', t) \rho(r', 0) \rangle, \]  

FIG. 1. Sketch of the static structure functions vs. momentum in the gas and liquid phase in arbitrary units [5].
is called van Hove function and its Fourier transform,

$$S(p, \omega) = \int dt \int d^3 r \ G(r, t) \ e^{i(\omega t - p \cdot r)},$$

(9)

dynamic structure function. It is convenient to decompose the van Hove function into a self and a distinct part

$$G(r, t) = G_s(r, t) + G_d(r, t),$$

(10)

with

$$G_s(r, t) = \frac{1}{N} \langle \sum_{i=1}^{N} \delta[r + r_i(0) - r_i(t)] \rangle,$$

$$G_d(r, t) = \frac{1}{N} \langle \sum_{i \neq j} \delta[r + r_i(0) - r_j(t)] \rangle.$$

(11)

The distinct part looks similar as the pair correlation function, i.e., showing a pronounced peak plus one or two broad side peaks in the liquid case, if $t \ll \tau$ where $\tau$ is the characteristic relaxation time of the system. For large $t$, however, it becomes a smooth and flat function of $r$.

III. STRUCTURE FUNCTIONS IN A QUARK-GLUON-PLASMA

Now we want to define these correlation functions in the case of the QGP. For simplicity we consider here only the quark component. The local quark density is given by

$$\rho(r, t) = \bar{q}(r, t) \gamma_0 q(r, t),$$

(12)

where $q$ and $\bar{q}$ are the quark wave functions. In a homogeneous and isotropic plasma, in which $\langle \rho(r)\rho(r') \rangle$ depends only on $r - r'$, the static quark density-density autocorrelation function is given by

$$G(r) = \frac{1}{n} \langle \rho(r, t)\rho(0, t) \rangle.$$

(13)

Here $n = N/V = \langle \rho(r) \rangle$ is the average particle density in a homogeneous system. At finite temperature its Fourier transform $\chi$ follows from [20]

$$G(r) = \frac{1}{n} \lim_{\Delta \tau \to 0} T \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} e^{-i(\omega_n \Delta \tau - p \cdot r)} \chi(\omega_n, p),$$

(14)

where $\omega_n = 2n\pi T$ are the discrete Matsubara frequencies.

The corresponding spectral function is given by [20]

$$\sigma(\omega, p) = \frac{1}{\pi} \text{Im} \chi(\omega, p),$$

(15)

where $\omega$ is a real number now. The static structure function follows from the spectral function according to [20]
\[ S(p) = \frac{1}{n} \int_{0}^{\infty} d\omega \sigma(\omega, p) \coth \frac{\omega}{2T}, \] (16)

which is a consequence of the fluctuation-dissipation theorem [18].

The function \( \chi(\omega, p) \) can be expressed by the one-loop polarization diagram containing dressed quark propagators [20]

\[ \chi(\omega, p) = -2N_c T \sum_{n=-\infty}^{\infty} \int \frac{d^3 k}{(2\pi)^3} \text{Tr}[\gamma_0 S_F(k_0, k) \gamma_0 S_F(\omega - k_0, p - k)] \] (17)

with the number of colors \( N_c \) and \( k_0 = 2n_i \pi T \). This expression is related to the longitudinal part of the QCD polarization tensor containing only the quark loop, \( \Pi_L = \Pi_{00} \), by

\[ \chi(\omega, p) = -\frac{4N_c}{g^2} \Pi_L(\omega, p), \] (18)

where \( g \) is the QCD coupling constant.

The longitudinal part of the polarization tensor also determines the quark number susceptibility \( \chi_q(T) \), which is proportional to the \( p = 0 \) limit of the static structure function (16), \( \chi_q(T) = (n/2T) S(0) \) [21]. In the case of an isotropic and homogeneous plasma the polarization tensor depends only on \( \omega \) and \( p = |p| \).

According to (9) and (14) for \( t \neq 0 \), the dynamic structure function is given by

\[ S(\omega, p) = \frac{1}{n} \chi(\omega, p). \] (19)

Using (14) to (19) the density-density autocorrelation and structure functions can be derived in the strongly coupled phase of the QGP by using lattice QCD and investigated for signatures of the liquid phase as discussed above. In lattice calculations the correlation functions can be derived directly whereas the structure functions and spectral densities require a maximum entropy analysis [22]. For comparison, we calculate in the following the static and dynamic structure functions within perturbation theory, i.e., in a weakly coupled QGP, where no pronounced structures in these functions are expected.

**IV. HARD THERMAL LOOP APPROACH TO THE STATIC STRUCTURE FUNCTION**

Let us first consider the high-temperature or hard thermal loop (HTL) limit [23–25]. In this case \( \chi \) is given by the high-temperature limit of the 1-loop polarization tensor with bare quark propagators,

\[ \chi^{\text{HTL}}(\omega, p) = \frac{2N_c N_f}{3} T^2 \left[ 1 - \frac{\omega}{2p} \ln \frac{\omega + p}{\omega - p} \right], \] (20)

where \( N_c \) is the number of colors, i.e., \( N_c = 3 \) and \( N_f \) the number of light flavors in the QGP. For the interesting regime of small times, \( t \ll \tau \), corresponding to \( \omega \gg p \) in (20), \( \chi \) behaves as
\[ \chi_{\text{HTL}}(\omega \gg p) = -\frac{2N_cN_f}{9} T^2 \frac{p^2}{\omega^2} \]  

and the dynamic structure function (19) is a monotonically decreasing function of \( \omega \). In the case of a liquid this function also decreases with \( \omega \) but shows a peak structure in addition [17].

The spectral density, following from the imaginary part of the polarization tensor according to (15), within the HTL limit is given by

\[ \sigma(\omega, p) = \frac{N_cN_f}{3} T^2 \frac{\omega}{p} \theta(p^2 - \omega^2). \]  

The imaginary part of (20) corresponds to Landau damping. The static structure function according to (16) reads

\[ S(p) = \frac{2N_fT^3}{n}. \]  

Here we have used consistently the high-temperature approximation, \( \text{coth}(\omega/2T) = 2T/\omega \), and \( N_c = 3 \). The high-temperature limit corresponds to the classical limit, in which the polarization tensor, related to the dielectric function (see below), can also be derived by the Vlasov equation [4]. Using the HTL polarization function together with \( \text{coth}(\omega/2T) \) in (16) leads to the unphysical result of a linearly increasing static structure function for large \( p \). The constant structure function (23), on the other hand, indicates an uncorrelated QGP corresponding to an ideal gas – the corresponding pair correlation function (4) is equal to zero – in the limit of infinitely high temperature. This result also agrees with the free quark number susceptibility \( \chi_q(T) = N_fT^2 \) [21]. Let us note here that using the 1-loop polarization tensor beyond the HTL limit [26–28] in (16) also leads to an unphysical result, namely a monotonically decreasing \( S(p) \). This shows once more that the naive use of perturbation theory for gauge theories at finite temperature is inconsistent [25].

To go beyond the uncorrelated QGP, we resum the HTL polarization tensor (18) within a Dyson-Schwinger equation,

\[ \Pi^*_L = \Pi^{\text{HTL}}_L \sum_{n=0}^{\infty} (D^0_L \Pi^{\text{HTL}}_L)^n, \]  

where \( D^0_L = 1/p^2 \) is the bare longitudinal gluon propagator in Coulomb gauge [4]. Using the relation between the longitudinal polarization tensor and the longitudinal dielectric function [4], holding in particular for the HTL limit,

\[ \epsilon^{\text{HTL}}_L(\omega, p) = 1 - \frac{\Pi^{\text{HTL}}_L(\omega, p)}{p^2}, \]  

the resummed polarization tensor is given by

\[ \Pi^*_L(\omega, p) = p^2 \left( \frac{1}{\epsilon^{\text{HTL}}_L(\omega, p)} - 1 \right). \]
The static structure function, following from (15), (16), (18), and (26) together with the high-temperature approximation, \( \coth(\omega/2T) = 2T/\omega \), using a Kramers-Kronig relation is given by [18]

\[
S(p) = \frac{4N_c T}{g^2} p^2 \left[ 1 - \text{Re} \frac{1}{\epsilon_L^{\text{HTL}}(\omega = 0, p)} \right] = \frac{2N_f T^3}{n} \frac{p^2}{p^2 + m_D^2},
\]  

(27)

where \( m_D^2 = -\Pi_L^{\text{HTL}}(\omega = 0, p) = N_f g^2 T^2 / 6 \) is the quark contribution to the classical Debye screening mass. The static structure function (27) starts at zero for \( p = 0 \) and saturates at the uncorrelated structure function (23) for large \( p \). Such a structure function corresponds to a Yukawa system in the gas phase (see Fig.1). Indeed, the pair correlation function following from the Fourier transform of \( S(p) - 1 \),

\[
g(r) = \frac{N_f T^3}{2\pi n} \frac{m_D^2}{r} e^{-m_D r},
\]

(28)

reproduces the Yukawa potential. The radial distribution function, giving the probability of finding other quarks, in Ref. [5] is given by \( 1 + g(r) \). Using (28) the radial distribution function becomes unphysical, i.e., negative, for small \( r \) indicating the failure of the Vlasov approach at small inter-particle distances [18].

In a next step one could use HTL-resummed quark propagators and quark-gluon vertices in the polarization tensor as done for the quark number susceptibility [21]. However, to look for a strongly-coupled liquid phase within the structure functions and pair correlation functions, one has to adopt non-perturbative methods. At the classical level molecular dynamics, generalized to the relativistic case, could be useful [5]. In general, of course, QCD lattice simulations would be the ultimate choice.

V. SUMMARY

In summary, we have presented definitions of the pair correlation function, the density-density autocorrelation function, its time-dependent generalization (van Hove function) and their Fourier transforms, the static and dynamic structure functions in the case of a QGP. The latter are closely related to the longitudinal part of the polarization tensor. In liquids these correlation functions and their Fourier transforms show a specific peak structure. Hence they could be used to investigate the existence of a liquid phase in a strongly coupled QGP, if calculated within non-perturbative methods, in particular lattice QCD. Adopting the perturbative high-temperature limit of the polarization tensor and its resummation corresponding to the weak coupling limit, we have shown, as expected, that there is no indication of a liquid phase in the dynamic as well as static structure functions. These results can be used as a reference for non-perturbative calculations.

Acknowledgment: I would like to thank W. Cassing, A. Peshier, F. Karsch, and W. Zajc for helpful discussions.
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