Quark confinement and curved spaces

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In this work it will be shown how quark confinement appears when wave equations derived in curved spaces are considered. First, the equations and their solutions for Coulomb-like potentials will be presented, and then, how this theory leads to quark confinement. A comparison between different models of confinement will be also made.

I. INTRODUCTION

The introduction of quarks in the physical theory in 1964, by the Gell-Mann [1] and Zweig [2] hypothesis, has also introduced the annoying question of quark confinement. If by one hand this theory was able to organize and to explain the main proprieties of the hadrons with the scheme called the eightfold way [3], on the other hand, the impossibility of observing free quarks could be considered as a major problem. Since then, many authors proposed models in order to describe the structure of the hadrons in terms of confined quarks. A successful way to implement these ideas is to consider nonrelativistic constituent quark models, such as the the nonrelativistic oscillator model, proposed by Dalitz in 1967 [4] and by Faiman in 1968 [5], where the Baryons are supposed to be systems composed of three constituent quarks, confined by an oscillator potential, with the states determined by the Hamiltonian

\[ H_0 = \sum_i \frac{p_i^2}{2m_i} + \frac{K}{2} \sum_{i>j} (\vec{r}_i - \vec{r}_j)^2 . \] (1)

This model was improved by De Rújula [6] with the addition of a quark-quark spin-dependent potential, by Karl and Isgur, with the introduction of a quantum chromodynamics inspired potential [7] and also by Murthy [8] that considered a deformed oscillator. Heavy \( q\bar{q} \) systems are equally well described by nonrelativistic constituent quark models, such as the Cornell model [9], [10], that uses a linear plus Coulomb potential of the type

\[ V(r) = \frac{a}{r} + br , \] (2)

where \( a \) and \( b \) are constants determined phenomenologically. Some models, as for example [11], [12], are based on other mechanisms, and generate different potentials.

Despite the success of the nonrelativistic models in describing the hadrons proprieties, theoretically, it is more reasonable to think quarks as relativistic particles. In 1968, Bogolioubov [13] considered the Baryons as spherical cavities, and inside of them the three constituent quarks are Dirac particles, submitted to a self-consistent mean field, that was represented by a scalar potential

\[ V(r) = \begin{cases} 0 & \text{for } r \leq R \\ V_0 & \text{for } r > R \end{cases} , \] (3)

and quark confinement is achieved for \( V_0 \to \infty \). Further development of these ideas leads to the MIT bag model [14] where the vacuum pressure was included. Other models based on the Dirac equation [15]-[17], with \( r^n \) confining potentials, as in [17]

\[ V(r) = V_0 + \lambda r , \] (4)

may be found in the literature, and they also show good agreement with the experimental data.

In a recent paper [18], a relativistic wave equation based on the general covariance principle has been derived. This theory has been constructed taking into account the effect of different kind of interactions (electromagnetic and strong) in the metric of the space-time. In this work, the main objective is to investigate the quark confinement with this theory. As it will be seen in the next sections, very interesting results can be obtained this way, and in many aspects these results are qualitatively different from the ones obtained in the previously cited models.
This paper has the following structure: In Sec. II a brief review of the theory and the solution of the equation for a Coulomb-like potential are shown, in Sec. III, the quark confinement effect that comes from this theory is presented, and in Sec. IV, a comparison between the different confinement mechanisms discussed in this paper is made and the conclusions are drawn.

II. QUANTUM MECHANICS IN CURVED SPACE-TIME

The Einstein general theory of relativity is one great achievement in the understanding of Nature, and when applied to very large systems, such as planets or galaxies, gives very precise results. Taking this fact into account, fundamental questions may be asked, as for example why the general covariance principle does not apply to very small systems, such as atoms or elementary particles, and if the laws of physics depends on the size of the object. In quantum systems, the electromagnetic and strong interactions dominate and the gravitational interaction is negligible, as the masses of the considered particles are very small. So, the gravitational potential may be turned off, and then, the curvature of the space-time will be predominantly due to the other interactions (electromagnetic and strong).

With these aspects in mind, in [18] a theory was proposed, and an equation similar to Dirac equation was derived. In this section, a brief revision of this theory will be made, and some results, necessary to the development of this paper will be shown.

The simplest way to formulate this theory is to consider systems where spherical symmetry exists. In this case, the space-time is described by the Schwarzschild metric [19], [20],

\[ ds^2 = \xi \, d\tau^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) - \xi^{-1} dr^2 , \]

where the factor \( \xi(r) = (1 + V(r)/m_0c^2)^2 \) is determined by the interaction potential \( V(r) \), and is a function only of \( r \).

From the definition of the energy-momentum relations and the respective the quantum operators (mathematical details may be found in [18]) in the given metric, the general relativistic equation for spin-1/2 particles

\[ i\hbar \frac{\partial}{\xi} \Psi = \left( -i\hbar \vec{c} \cdot \vec{\nabla} + \beta m_0 c^2 \right) \Psi \]

has been deduced [18], where \( \Psi \) is a four-component spinor. One must note that despite of the fact that this theory is conceptually more complicated then the Dirac one, the final equation is very similar to the Dirac equation, what is surprisingly in accord with his simplicity ideal.

The spacial part of \( \Psi \) may be written as

\[ \psi = \left( \frac{F(r) \chi^\mu_k}{iG(r) \chi^\mu_{-k}} \right) , \]

where \( \chi^\mu_k \) are the usual two-component spinors, and \( k \) is related with the angular momentum by

\[ k = l \quad \text{for} \quad j = l - 1/2 , \]

\[ k = -l - 1 \quad \text{for} \quad j = l + 1/2 . \]

The radial part of eq. [9] may be rewritten as a pair of coupled equations for the and the \( F \) and \( G \) functions

\[ \sqrt{\xi} \frac{dF}{dr} + (1 + k) \frac{F}{r} = \left( \frac{E}{\sqrt{\xi}} + m_0 \right) G \]

\[ \sqrt{\xi} \frac{dG}{dr} + (1 - k) \frac{G}{r} = -\left( \frac{E}{\sqrt{\xi}} - m_0 \right) F . \]

Considering a coulomb-like potential \( V(r) = -\alpha Z/r \) the \( \xi \) function becomes

\[ \xi = \left( 1 - \frac{\alpha Z}{m_0c^2 r} \right)^2 , \]

and inserting it [10] in eq. [9] and making the substitution \( \rho = \beta r \), the equations may be put in the form

\[ \frac{\xi}{\rho} \frac{dF}{d\rho} + \sqrt{\xi}(1 + k) \frac{F}{\rho} = \left( \frac{E}{\beta} + \sqrt{\xi} \frac{m_0}{\beta} \right) G \]

\[ \frac{\xi}{\rho} \frac{dG}{d\rho} + \sqrt{\xi}(1 - k) \frac{G}{\rho} = -\left( \frac{E}{\beta} - \sqrt{\xi} \frac{m_0}{\beta} \right) F . \]
The equations may be solved by the Frobenius method, expressing the $F$ and $G$ functions as power series of the form

$$F = \rho^s \sum_{n=0}^{N} a_n \rho^n e^{-\rho},$$

$$G = \rho^s \sum_{n=0}^{N} b_n \rho^n e^{-\rho}. \quad (12)$$

Substituting this expressions in the equations we find that $s = 0$ and the relations between the coefficients are obtained

$$a_1 = \left[\frac{1 + k + \alpha \beta}{\alpha \beta}\right] a_0$$

$$b_1 = \left[\frac{1 - k + \alpha \beta}{\alpha \beta}\right] b_0, \quad (13)$$

$$2\alpha^2 \beta^2 a_2 - \alpha \beta (3 + k + \alpha \beta) a_1 + (1 + k + 2\alpha \beta) a_0 + a m_0 b_0 = 0$$

$$2\alpha^2 \beta^2 b_2 - \alpha \beta (3 - k + \alpha \beta) a_1 + (1 - k + 2\alpha \beta) b_0 + a m_0 a_0 = 0 \quad (14)$$

and

$$(n + 3)\alpha^2 \beta^2 a_{n+3} - \alpha \beta [2n + 5 + k + \alpha \beta] a_{n+2} + [n + 2 + k + 2\alpha \beta] a_{n+1} - a_n + a m_0 b_{n+1} - \left(\frac{E + m_0}{\beta}\right) b_n = 0$$

$$(n + 3)\alpha^2 \beta^2 b_{n+3} - \alpha \beta [2n + 5 - k + \alpha \beta] b_{n+2} + [n + 2 - k + 2\alpha \beta] b_{n+1} - b_n + a m_0 a_{n+1} + \left(\frac{E - m_0}{\beta}\right) a_n = 0 \quad (15)$$

with $a_0, a_1, a_2, b_0, b_1, b_2 \neq 0$. From these relations, one obtains $\beta = \sqrt{m^2 - E^2}$ that determines the factor $e^{-\sqrt{m^2 - E^2}}$ in the wave functions (12) what determines the same behavior that was obtained with the Dirac equation. The relation

$$[N + 2\alpha \beta] \beta - a m^2 = 0 \quad (16)$$

is also obtained from eq. (13)-(14), and gives the relation for the energy levels

$$E_N = \pm m_e c^2 \left[\frac{1}{2} - \frac{N^2}{8a^2} \pm \frac{N}{4a} \sqrt{\frac{N^2}{4a^2} + 2}\right], \quad (17)$$

where the physical values are the positive ones. This spectrum may be compared with the one obtained with the Dirac equation \[22, 23\] (also deduced by Sommerfeld \[25\]),

$$E_N = \frac{m_e c^2}{\sqrt{1 + \alpha^2/a^2}}, \quad (18)$$

where $a = N - j + 1/2 + \sqrt{(j + 1/2)^2 - a^2}$.

If one takes numerical results. For example, for the electron-proton interactions in the deuterium atom, the results may be obtained considering the fine-structure constant \[26\] $\alpha = 1/137.03599976$, $Z = 1$ and the electron mass $m_0 = 0.510998902$ MeV/c$^2$ \[20\]. The experimental ground state energy for the deuterium atom is $E_0 = -13.60214$ eV \[24\], calculating it with the Dirac spectrum \[18\], one has -13.60587 eV, and with eq. \[17\], -13.60298 eV. More numerical results may be found in \[18\]. Observing these results, one can see that the accord of both theories with the deuterium experimental data is very good, but the results from eq. \[17\] are closer to the experimental data then the results from eq. \[18\]. The results form the Dirac theory \[18\], one can see that the deviations from the data are of the order of 0.027%. Considering the spectrum of eq. \[17\], the deviations are of the order of 0.005%, almost five times smaller, what is a significant improvement. The same pattern occurs when the other energy levels are compared \[18\].
III. QUARK CONFINEMENT

At this point it is instructive to investigate the solutions of the wave equations (11) for the Coulomb potential near the classical horizon of events, at \( r = r_0 \) and apply these ideas to strongly interacting systems. One must remark that the solution presented in Sec. II is not valid at the surface \( r = r_0 \), and for this reason, this case must be studied separately. The solution of the equation in the neighborhood of \( r_0 \), may be given by an expression similar to (12), but replacing \( \rho \) for \( \rho - \alpha \beta /m_0 \),

\[
F = \rho^s \sum_{n=0}^{N} a_n \left( \rho - \frac{\alpha \beta}{m_0} \right)^n e^{-\rho},
\]

\[
G = \rho^s \sum_{n=0}^{N} b_n \left( \rho - \frac{\alpha \beta}{m_0} \right)^n e^{-\rho}.
\]

With this procedure, one finds that near the horizon of events just one energy value is possible, \( E = 0 \). The other conditions for the existence of a solution are \( k = 0 \) (\( l = 0 \)) and \( s = -1 - \alpha \), what means an infinite discontinuity of the wave function at \( r = r_0 \). If this solution is discarded, the trivial solution \( \psi(r_0) = 0 \) must be considered, what can be interpreted as a boundary condition at \( r = r_0 \). Consequently, this solution tells us that the space is divided in two parts (fact that is true in both cases) inside and outside the horizon, which does not communicate.

So, if one considers hadrons composed of quarks, that generate a self-consistent field, described by a potential (here, as an example we considered a Coulomb potential), inside the horizon of events, the quarks may be described by solutions of the type (12) with the energy levels given by the expression (17). At \( r \sim r_0 \), the solution is (19), that imposes the confinement of quarks inside this region. Classically thinking, the quarks are confined by a trapping surface [27] that is generated by the potential.

Considering the \( Y \) meson for example, that is a \( b \bar{b} \) state, the theory may be applied just considering \( b \) constituent quarks with mass \( m = 5.5 \) GeV, and a Coulomb potential with \( \alpha = 1.05 \), what is a reasonable value for quark interactions. From the expression (17) one has \( E_0 = 9.47 \) GeV (\( m_\Upsilon = 9.46 \) GeV), and \( E_{\text{max}} = 11.0 \) GeV, that is the mass of \( \Upsilon(11020) \). The quarks will be confined inside the region \( r < r_0 = 0.05 \) fm, that is a reasonable size for a core of a meson. Some estimates of these quantities for other hadrons may be found in Table I [18].

<table>
<thead>
<tr>
<th></th>
<th>( M ) (GeV)</th>
<th>( \alpha )</th>
<th>( r_0 ) (fm)</th>
<th>( M ) (GeV)</th>
<th>( M_{\text{exp}} ) (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nucleon (( qqq ))</td>
<td>0.38</td>
<td>1.60</td>
<td>0.83</td>
<td>0.938</td>
<td>0.938 (proton)</td>
</tr>
<tr>
<td>( J/\psi(\epsilon \bar{\epsilon}) )</td>
<td>1.79</td>
<td>1.06</td>
<td>0.11</td>
<td>3.10</td>
<td>3.10</td>
</tr>
<tr>
<td>( \Upsilon(b\bar{b}) )</td>
<td>5.50</td>
<td>1.05</td>
<td>0.05</td>
<td>9.47</td>
<td>9.46</td>
</tr>
</tbody>
</table>

One must remark that in order to describe the spectra of the particles of Table I, other terms must be added in the potential. This fact (that is widely used [6]-[12]) may be understood considering that for short-range interactions many effects may occur, generating corrections to the potential. Another factor that must be considered to improve the description is that spherical symmetry is not the best one for \( q\bar{q} \) mesons, and this fact must be corrected in future works.

In fact, the horizon of events is not an exclusive feature of the Coulomb potential, it may appear for any attractive potential, when the condition

\[
\xi(r_0) = 0,
\]

is satisfied, what occurs for \( V(r_0) = -mc^2 \). In these cases, the only energy value is \( E = 0 \), and the solution will present the discontinuity showed above,

\[
\psi \propto \frac{f(r)}{(r-r_0) \delta},
\]

with \( \delta > 0 \), results that lead to quark confinement for general attractive potentials.
IV. DISCUSSION OF THE RESULTS

In this work it was shown how quark confinement appears when relativistic wave equations in curved spaces are used. Now the obtained results will be compared with the results of the existing models.

As it was said in the introduction, many authors succeeded in explaining quark confinement with phenomenological potentials. In the nonrelativistic oscillator model \[5\]-\[7\], the Hamiltonian \(\phi\) leads to confining oscillator-type wave functions that contain a factor \(e^{-\beta r^2}\) and in the Cornell model \[3\], where the potential of the type \(\phi\) is used, a similar behavior occurs. In the Bogolioubov and in the MIT bag models that consider a potential of the type \(\phi\), the wave functions contain a factor \(e^{-\beta \sqrt{m^2-E^2(r-r_0)}}\), and quark confinement appears in the limit \(V_0 \to \infty\), where the wave function is constant for \(r = R\) and \(0\) for \(r > R\).

As it was seen in last section, quark confinement appears in a different way when the equations derived in curved spaces are used. Differently from the other models, for an internal particle it is not possible to reach the surface \(r = r_0\), as \(\psi(r_0) = 0\). One must observe that even in the MIT bag model, with an infinite potential, a strong condition is not reached. The result of this condition is that the space-time is divided in two disconnected regions, inside and outside the surface. This fact is an intrinsic property of the space-time, due to attractive potentials, as for example the Coulomb potential, and there is no need of introducing confining potentials to obtain this effect. In fact, potentials of the type \(11-14\), represent in an approximate way, in plane space-time formulations, systems that are described in a natural way by curved spaces.

Another interesting aspect of the theory is that classically it is expected the collapse at the origin, but here we are dealing with quantum systems, and the uncertainty principle forbids this collapse. The solution of the equation shows that the wave function is \(0\) at the origin, confirming this statement.

The Dirac theory \[28,29\] introduced the special relativity in quantum mechanics, so it is very reasonable to think that the next step is to formulate the quantum mechanics in terms of the general relativity principles. The deuterium spectrum obtained in this way shows that the corrections of the energy levels, due to this general formulation of quantum mechanics (or general quantum mechanics) with the inclusion of the electric curvature of the space-time, provide a quite impressive agreement with the experimental data. Another strong evidence in the validation the theory is the quark confinement mechanism proposed in this paper, where the quarks are confined by a trapping surface, similar to the one defined by Penrose \[27\]. Conceptually, these are very important results, as they show a successful way to join the quantum mechanics and the general relativity.

Acknowledgments

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[24] The experimental values for the energy levels of the hydrogen and deuterium may be found in [http://physics.nist.gov](http://physics.nist.gov).