Non-linear perturbations in multiple-field inflation

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We develop a non-linear framework for describing long-wavelength perturbations in multiple-field inflation. The basic variables describing inhomogeneities are defined in a non-perturbative manner, are invariant under changes of time slicing on large scales and include both matter and metric perturbations. They are combinations of spatial gradients generalising the gauge-invariant variables of linear theory. Dynamical equations are derived and supplemented with stochastic source terms which provide the long-wavelength initial conditions determined from short-wavelength modes. Solutions can be readily obtained via numerical simulations or analytic perturbative expansions. The latter are much simpler than the usual second-order perturbation theory. Applications are given in a companion paper.

I. INTRODUCTION

It is a well-established fact that the universe on large scales exhibits a high degree of uniformity. During most cosmological eras and for a large span of length scales, it can be well approximated by a Friedmann-Robertson-Walker (FRW) spacetime with inhomogeneities described as small linear perturbations around the highly symmetric background. This picture has proved particularly relevant for the early universe, as the smallness of the cosmic microwave background (CMB) temperature anisotropies indicates. An extrapolation of this observational fact suggests that the use of linear theory would also be justified during inflation when the perturbations leading to the CMB anisotropies are thought to have been created. So far, almost all studies of the generation and evolution of perturbations in inflation invoke the use of linear perturbation theory [1]. In principle it offers a tremendous simplification of the task of studying the true inhomogeneous spacetime.

However, even when attention is focused on the inflationary era, linear theory cannot be the whole picture. Since gravity is inherently non-linear and the potential of the inflationary model is likely to be interacting, some small non-linearity will be endemic to the perturbations. Given the accuracy of forthcoming CMB observations, it is worth investigating whether this non-linearity can be observationally relevant. The characteristic signature of non-linear effects will be deviations of the primordial fluctuations from Gaussian statistics. In order for this non-Gaussianity to be calculated one needs to go to second order in perturbation theory or develop a fully non-linear approach.

The issue of calculating non-linearity and the consequent non-Gaussianity in the primordial universe has been attracting increasing attention recently, although some early attempts to calculate it can also be found in the literature [2]. A tree-level calculation with a cubic action for the perturbations has been performed in [3] for the case of slow-roll single-field inflation (with a similar slow-roll calculation for more general single-field Lagrangeans given in [4]). At the level of the equations of motion, various authors have pursued perturbation theory to second order [5], with [6] providing a review of these techniques. Although interesting results can be obtained at second order, full exploration of the system of equations suffers from great computational complexity. On the one hand, the perturbation equations tend to be rather cumbersome to derive. On the other hand, gauge-invariant variables, which have proved very useful for computations and the interpretation of results in linear theory, are not as simple as their first-order counterparts when second-order perturbations are considered.

In this paper we take a different viewpoint on the study of non-linear perturbations during inflation, systematically formulating and extending ideas presented in [7], which incorporate the general multiple-field analysis of [8]. We use combinations of spatial gradients to construct variables describing the deviation from a spatially uniform spacetime and derive equations for these variables on super-horizon scales (section II). These variables are defined non-perturbatively and are invariant under changes of the time coordinate on such scales (appendix A). They were first used in [9], where we derived a non-linear generalisation of the familiar adiabatic conservation law of linear theory. More recently, the authors of [10] used similar combinations of gradients in the context of the covariant formalism [11] to also derive this conservation law. They showed the relation of these simple, yet fully non-linear, variables to others defined in second-order perturbation theory. An equivalent conservation law was derived in [12] without using such gradient variables. [Another, more recent approach in the study of non-linear perturbations can be found in [13], where the formalism of [14] is extended and used.]

In order to include the continuous influx of sub-horizon perturbations to the long-wavelength system, stochastic noise terms are added to the long-wavelength equations. For the definition of these noise terms a specific choice of
time turns out to be particularly convenient. Thus, we arrive at a set of fully non-linear stochastic equations which include both metric and matter perturbations (section III). For actual multiple-field calculations it is more convenient to use the explicit field basis of \( \mathcal{B} \). It is introduced and the relevant equations are rewritten in terms of this basis (section IV). In section V we show how the derived equations can be used to extract information about non-linearity in inflation. A perturbative analytic approach can be applied giving results to second order. At first order, this perturbative expansion is equivalent to the well-known linear gauge-invariant perturbation theory. At second order, however, it is much simpler than the corresponding second-order approaches pursued to date. Since the evolution equations are fully non-linear on long wavelengths, numerical simulations can be performed without the need for analytic approximations. In this paper we concentrate on presenting the methods, and we stress that no slow-roll approximation is made here. Detailed applications are provided in separate publications [13, 19, 20].

II. LONG-WAVELENGTH APPROACH

We start by considering the metric
\[
ds^2 = -N^2(t, \mathbf{x}) \, dt^2 + e^{2\alpha(t, \mathbf{x})} h_{ij}(t, \mathbf{x}) \, dx^i \, dx^j,\]
where we have fixed part of the gauge by setting the shift, i.e. the \( g_{0i} \) component, to zero. For convenience we will keep \( g_{00} = 0 \) throughout the whole paper. The spatial part of the metric has been decomposed into a spatial metric tensor \( h_{ij} \) with unit determinant and a determinant part which plays the role of a locally defined scale factor \( a(t, \mathbf{x}) \equiv \exp[\alpha(t, \mathbf{x})] \) [18]. The remaining gauge freedom is encoded in the lapse function \( N(t, \mathbf{x}) \); a choice of \( N \) corresponds to a choice of time slicing for the spacetime. The vector normal to these time slices is given by \( n_0 = -N, n_i = 0 \) and their embedding in the four-dimensional spacetime is characterised by the extrinsic curvature tensor \( K_{ij} \).

We will use the tensor
\[
H_{ij} = \frac{1}{3} \nabla (n_i n_j) = \frac{1}{6N} \partial_t (e^{2\alpha} h_{ij}),
\]
which is \(-(1/3)K_{ij}\). The quantity \( H_{ij} \) can be decomposed into a trace and a traceless part, respectively,
\[
H = \frac{1}{N} \partial_t \alpha, \quad \tilde{H}_{ij} = \frac{1}{6N} e^{2\alpha} \partial_t h_{ij}.
\]

We now focus on the first approximation we will be employing: the long wavelength approximation [19, 20]. (The second is the use of stochastic noise terms to describe the effect of quantum fluctuations and will be discussed in the next section.) We will be interested in length scales larger than the comoving Hubble radius \( 1/(aH) \), which are called super-horizon, because on those scales one expects non-linearities to build up. (On sub-horizon scales the sources of non-linearity, gravity and interacting potentials, do not play a role and linear theory is expected to hold to high accuracy.) Consider variations over a characteristic comoving length scale \( L \). For any quantity \( F(t, \mathbf{x}) \) constructed out of metric and matter variables typically we will have \( \partial_i F = \mathcal{O}(F/L) \) and \( \partial_i F/N = \mathcal{O}(H F) \). From this we see\(^1\) that for scales \( L \gg 1/(aH) \) we can expect \( |\frac{1}{L} \partial_i F| \ll |\frac{1}{N} \partial_i F| \). Hence the long-wavelength approximation means that we can ignore second-order spatial derivatives when compared to time derivatives.

From the long-wavelength evolution equation for \( \tilde{H}_{ij} \) we find [18, 19]
\[
\partial_t \tilde{H}_{ij} \simeq -3NH \tilde{H}_{ij} \quad \Rightarrow \quad \tilde{H}_{ij} = C_{ij}(\mathbf{x}) \frac{1}{a^3}.
\]

Here the approximate equality implies second-order spatial gradients are ignored. Hence, \( \tilde{H}_{ij} \) is a mode that decays exponentially fast in an inflationary universe. From now on we will set it to zero, thereby demanding that \( h_{ij} \) depends on \( \mathbf{x} \) only, as can be seen from equation (3). In other words, \( h_{ij} \) (which contains the tensor modes) does not participate in the long-wavelength dynamics (gravity waves freeze on such scales). For the sake of simplicity, henceforth we will ignore the tensor perturbations generated in inflation and consider the spacetime to be close to flat Robertson-Walker, thus setting \( h_{ij}(\mathbf{x}) = \delta_{ij} \).

\(^1\) As will be explained later, we will choose a gauge in which \( aH \) does not depend on \( \mathbf{x} \). In this case the statement \( L \gg 1/(aH) \) makes sense globally and not just locally.
For the matter sector, we consider a very general inflationary era driven by \( m \) scalar fields \( \phi^A \) with the energy-momentum-tensor

\[
T_{\mu\nu} = G_{AB} \partial_\mu \phi^A \partial_\nu \phi^B - g_{\mu\nu} \left( \frac{1}{2} G_{AB} \partial^\lambda \phi^A \partial_\lambda \phi^B + V \right),
\]

with \( A, B, \) etc. running from 1 to \( m \) and \( V \) the potential. The derivative of the fields with respect to proper time is denoted by

\[
\Pi^A \equiv \frac{\partial_t \phi^A}{N}.
\]

Note that in \( \text{[7]} \) we have taken a general field metric \( G_{AB} \) and all the equations we give below will be valid for such an arbitrary metric. We thus view the dynamics as taking place on a general \( m \)-dimensional field manifold parametrised by a set of \( m \) coordinate functions \( \phi^A \). Hence it makes sense to define covariant derivatives when considering spatial or temporal dependence. For a spacetime-dependent quantity \( L^A(t, \mathbf{x}) \) which transforms as a vector in field space, we define the covariant derivatives

\[
\mathcal{D}_i L^A = \partial_i L^A + \Gamma^A_{BC} \partial_i \phi^B L^C, \quad \mathcal{D}_i L^A = \partial_i L^A + \Gamma^A_{BC} \partial_i \phi^B L^C,
\]

with \( \Gamma^A_{BC} \) the symmetric connection formed from \( G_{AB} \). The quantities \( \partial_t \phi^B \) and \( N \Pi^B \) transform as vectors in field space but \( \phi^B \) does not. Then the long-wavelength equations of motion for this system are \( \text{[10]} \)

\[
\frac{dH}{dt} = -\frac{\kappa^2}{2} N \Pi_B \Pi^B, \quad \mathcal{D}_i \Pi^A = -3NH \Pi^A - NG^{AB}V_B, \quad H^2 = \frac{\kappa^2}{3} \left( \frac{1}{2} \Pi_B \Pi^B + V \right), \quad \partial_i H = -\frac{\kappa^2}{2} \Pi_B \partial_i \phi^B,
\]

where \( V_B \equiv \partial_B V \equiv \partial V / \partial \phi^B \) and \( \kappa^2 \equiv 8\pi G = 8\pi / m_{\text{pl}}^2 \). We will also use the notation \( \Pi \equiv \sqrt{\Pi_B \Pi^B} \).

The system of equations \( \text{[8]–[11]} \) provides a basis for what has been termed in the past the ‘separate universe picture’ for studying the evolution of perturbations on long wavelengths. According to this picture, each point of the long-wavelength inhomogeneous universe evolves like a separate FRW universe. In general, this picture must be supplemented by the constraint \( \text{[11]} \) which connects the separate universes together. Only when the scalar fields are all slowly rolling can the gradient constraint \( \text{[11]} \) be ignored, since in this case it can be obtained from the Friedmann equation \( \text{[10]} \) with the \( \Pi^2 \) term dropped and the field equation \( \text{[9]} \) with the left-hand side set to zero. However, we will not make any slow-roll assumptions for the formalism we develop and hence we keep both \( \text{[10]} \) and \( \text{[11]} \). All formulae in this paper are valid regardless of slow roll.

Equations \( \text{[8]–[11]} \) can be reformulated in a convenient way to make easy contact with the well-known theory of gauge-invariant linear perturbations, but which is fully non-linear. In the separate universe picture deviations from uniformity can be described as differences in the properties of ‘neighbouring universes’. It thus makes sense to consider spatial gradients in order to describe perturbations. The use of spatial gradients was first advocated in \( \text{[11]} \) in the context of the covariant formalism. Here we consider two spacetime scalars \( A \) and \( B \), and define the following combination of their spatial gradients:

\[
C_i \equiv \partial_i A - \frac{\partial_i A}{\partial_i B} \partial_i B,
\]

as in \( \text{[7]} \). We show in the appendix that under long-wavelength changes of time slicing \( (t, \mathbf{x}) \rightarrow (\tilde{t}, \tilde{\mathbf{x}}) \),

\[
C_i = \delta^i_1 C_j,
\]

which means that such combinations of spatial gradients are invariant under these specific coordinate transformations. One such quantity that will be very useful is \( \text{[2]} \)

\[
Q^A_i = e^\alpha \left( \partial_i \phi^A - \frac{\Pi^A}{H} \partial_i \alpha \right).
\]
Although it looks linear in $Q^A_{\text{lin}}$ placed on the magnitude of $\tilde{\epsilon}$, second order. However, we stress again that no slow-roll approximation is made in this paper, so no restriction is in the long-wavelength approximation.

From $[8]$–$[11]$ an equation for $Q^A_i$ can be derived $\tilde{\Pi}$:

$$D^2_i Q^A_i - \left( \frac{\dot{N}}{N} - NH \right) D_i Q^A_i + (NH)^2 \Omega^A_i B Q^B_i = 0,$$

with the "mass matrix"

$$\Omega^A_B \equiv \frac{V^A_B}{H^2} - \frac{2\tilde{\epsilon}}{\kappa^2} R^A_D C B \frac{\Pi^D \Pi^C}{\Pi} - \left( 2 - \tilde{\epsilon} \right) \delta^A_B - 2\tilde{\epsilon} \left[ (3 + \tilde{\epsilon}) \frac{\Pi^A \Pi_B}{\Pi} + \frac{\Pi^A}{\Pi} \eta_B + \tilde{\eta}^A \eta_B \right],$$

where $V^A_B \equiv G^{AC} D_B V_C \equiv G^{AC} (\partial_B V_C - \Gamma^D_{BC} V_D)$, and $R^A_{DCB}$ is the curvature tensor of the field manifold. We define

$$\tilde{\epsilon} \equiv \frac{\kappa^2 \Pi^2}{2 H^2}, \quad \tilde{\eta}^A \equiv \frac{3H \Pi^A + G^{AB} V_B}{H \Pi}, \quad \tilde{\xi}^A \equiv -\frac{V^A_B \Pi^B}{H \Pi} + 3\tilde{\epsilon} \frac{\Pi^A}{\Pi} - 3\tilde{\eta}^A.$$

In the absence of stochastic source terms (see next section) the relations $[18]$ are equivalent to the standard definitions of the multiple-field slow-roll parameters $\tilde{\Pi}$ (where one should read $(1/N)\mathcal{D}_t$ instead of $\mathcal{D}$) by using $\tilde{\Pi}$ and $\tilde{\eta}$ and its time derivative. If a slow-roll approximation were to be made, $\tilde{\epsilon}$ and $\tilde{\eta}^A$ would be first order, while $\tilde{\xi}^A$ would be second order. However, we stress again that no slow-roll approximation is made in this paper, so no restriction is placed on the magnitude of $\tilde{\epsilon}$, $\tilde{\eta}^A$, and $\tilde{\xi}^A$. Equation $[19]$ describes the full non-linear dynamics on long wavelengths. Although it looks linear in $Q^A_i$, its coefficients are spatially dependent functions which depend implicitly on $Q^A_i$ (see $\tilde{\Pi}$ for details). It is valid for any choice of time slicing.

Equation $[19]$ easily connects with linear perturbation theory, since its linearised version is the spatial gradient of the long-wavelength equation for $Q^A_i$, $\tilde{\Pi}$ in the long-wavelength approximation. For our purposes another variable will be more convenient than $Q^A_i$. We define

$$\zeta^A_i \equiv -\frac{\kappa}{e^\alpha \sqrt{2\tilde{\epsilon}}} Q^A_i,$$

which when linearised is just the spatial gradient of the well-known comoving curvature perturbation $\zeta$. Just as $Q^A_i$, the variable $\zeta^A_i$ is invariant under choices of time slicing within the long-wavelength approximation. Expressing the long-wavelength evolution equation $[19]$ in terms of $\zeta^A_i$ we get

$$D^2_i \zeta^A_i - \left( \frac{\dot{N}}{N} - 2NH \left( \frac{3}{2} + \tilde{\epsilon} + \tilde{\eta}^\parallel \right) \right) D_i \zeta^A_i + (NH)^2 \Xi^A_i B \zeta^B_i = 0,$$

with

$$\Xi^A_B \equiv \frac{V^A_B}{H^2} - \frac{2\tilde{\epsilon}}{\kappa^2} R^A_D C B \frac{\Pi^D \Pi^C}{\Pi} + \left( 3\tilde{\epsilon} + 3\tilde{\eta}^\parallel + 2\tilde{\epsilon}^2 + 4\tilde{\epsilon} \tilde{\eta}^\parallel + (\tilde{\eta}^\parallel)^2 + \tilde{\xi}^\parallel \right) \delta^A_B.$$

2 We note that we can write $(\Pi^A_i / \Pi) \zeta^A_i = \partial_i \alpha - (\dot{\partial}_i \alpha / \Pi) \Pi$, where $\rho$ is the energy density of the scalar fields (here we have used $[11]$, the definition of $\tilde{\epsilon}$ in $[15]$, $[16]$, $[19]$ in the form $H^2 = \kappa^2 / 3$, $[8]$ and $[9]$). This shows that in the long-wavelength approximation the variable used in $[15]$ is identical to the component of our $\zeta^A_i$ parallel to the field velocity, i.e. its single-field version.
With these constraint relations the spatial derivative of any quantity of interest can be calculated in terms of system 7:

\[ \eta^A = \frac{\Pi A}{\Pi} \tilde{\eta}_A, \quad \tilde{\eta}^A = \frac{1}{\Pi} \frac{\Pi A \Pi B}{\Pi^2} \tilde{\eta}_B, \quad \tilde{\xi}^A = -\frac{\Pi A V_{AB} \Pi B}{\Pi} + 3\tilde{\epsilon} - 3\tilde{\eta}^A. \]  

(22)

Since the coefficients appearing in (20) are spatially dependent, a set of constraint equations is needed to close the system 7:

\[ \frac{d}{dt} \partial_i \alpha = -NH\tilde{\epsilon} \partial_i \alpha + H \partial_i N + NH\tilde{\epsilon} \frac{\Pi A}{\Pi} \tilde{\zeta}^A, \]  

\[ \partial_i \ln H = \tilde{\epsilon} \left( \frac{\Pi A}{\Pi} \tilde{\zeta}^A - \partial_i \alpha \right), \]  

\[ \partial_i \phi^A = \frac{\sqrt{2\kappa}}{\kappa} \left( \frac{\Pi A}{\Pi} \partial_i \alpha - \tilde{\zeta}^A \right), \]  

and

\[ D_i \Pi A = -\frac{\sqrt{2\kappa}}{\kappa} \left[ \frac{1}{N} D_i \tilde{\zeta}^A + H \left( \tilde{\epsilon} + \tilde{\eta}^A \right) \delta A_B - \tilde{\epsilon} \frac{\Pi A \Pi B}{\Pi} \right] \tilde{\zeta}_B - H \tilde{\eta}^A \partial_i \alpha \]. \]

(26)

With these constraint relations the spatial derivative of any quantity of interest can be calculated in terms of \( \zeta^A \) and its time derivative, for a given choice of time slicing (i.e. the lapse function \( N \)). Equation (20) along with the constraints (23)–(26) are the main results of this section.

For single-field inflation an important result can be obtained immediately, since in that case \( \Xi^A B = 0 \). Hence, \( \zeta_i \) is seen to be conserved in single-field inflation, as was first shown in 7, 9. The authors of 10 recently reached a similar conclusion. With a choice of time slicing which sets \( \partial_i \phi = 0 \), \( \zeta_i = \partial_i \alpha \) corresponds to the gradient of the integrated expansion \( \alpha(t, x) = \int N H \, dt \) of different points. So, in single-field inflation the difference in the number of e-folds of the separate universes is conserved. The conservation law has been formulated in these terms in 12.

Before closing this section we would like to make a few remarks on the ‘\( \delta N \) formalism’, which has been used in other, more recent work (see e.g. 13) as an alternative for studying non-linear perturbations. In this formalism the perturbed expansion \( \alpha(t, x) \) on comoving time slices is related to the perturbations of the scalar fields \( \delta \phi^A(x) \) on an initial flat time slice where the perturbations are supposed to have been generated after horizon crossing. In our notation \( \delta \alpha(t, x) = \partial^{-2} \partial_i \left( (\Pi A / \Pi) \zeta^A \right) \) since \( (\Pi A / \Pi) \zeta^A = \partial_i \alpha \) on comoving time slices. All calculations in the \( \delta N \) formalism are based on the knowledge of \( \alpha(t, \phi^A(x)) \), i.e. the local expansion expressed in terms of the initial fields at each spatial point. All the initial scalar field values which appear in this solution are then made spatially dependent. Practical calculations require an accurate multi-dimensional solution for the expansion \( \alpha(t, \phi^A) \) of the homogeneous universe, from which all its field derivatives can be derived. Furthermore, this entails the approximation that the momentum dependence is completely ignored, since in general \( \alpha(t, \phi^A, \Pi A) \) is a function of both the fields and the momenta. The conditions for such an approach to be valid can be seen from 8, 11. Since the evolution equations on long wavelengths are the same as those of a homogeneous cosmology valid locally, any analytic solution of a homogeneous universe with the initial conditions made spatially varying will provide a solution describing the evolution of the true inhomogeneous spacetime. The initial conditions must satisfy both of the constraints (10) and (11). In 13 equation (11) is ignored, which is the same approximation as ignoring the momenta (see the discussion below (11)). These conditions restrict the type of multiple-field models that can be investigated. In contrast, the formalism presented here is applicable in a practical way to any expansion history, slow-roll or otherwise, and it is tractable numerically when analytic solutions cannot be obtained in full.

### III. STOCHASTIC EQUATIONS

In the previous section we derived equations for the long-wavelength evolution of the variables \( \zeta^A \) defined in (19). These are (20) along with the constraints (23)–(26). In order to solve this long-wavelength system initial conditions must be provided. In inflation these originate in the short-wavelength quantum regime, before the fluctuations in

\(^3\) Note that in these other works \( N(t, x) \) is the local expansion, not the lapse function. In our notation it would be the ‘\( \delta \alpha \) formalism’.
the metric and matter cross the horizon, so we have to incorporate these sub-horizon effects in our long-wavelength equations. We achieve this in the following way. Once super-horizon, the quantum fluctuations can be considered as classical stochastic quantities. This is the stochastic picture for the generation of inflationary perturbations [21]. Linear theory is used to describe the perturbations until shortly after horizon crossing. These are then used in stochastic source terms as the initial conditions for the full non-linear long-wavelength system. A smoothing window function is used to separate short and long wavelengths as in [7]. However, improving on [7], we start from a first-order system, in order to correctly reproduce the velocity modes as well. We also note that in many references on stochastic inflation, metric fluctuations are not consistently included. They are included in our formalism.

For the definition of the source terms it is convenient to fix the remaining gauge freedom \((N)\) by choosing \(\tau = \ln(aH)\) as the time variable.\(^4\) One advantage of this gauge choice is that horizon crossing of modes \((k = aH)\) happens simultaneously throughout the whole universe. Other advantages will become clear below. For this choice we have

\[
N H = \frac{1}{1 - \epsilon},
\]

and our basic equations \((20), (24) - (26)\) read (note that \((23)\) is no longer an independent equation)

\[
D_t^2 \zeta_i^A + \frac{3 - 2\epsilon + 2\eta^\parallel - 3\epsilon^2 - 4\epsilon\eta^\parallel}{(1 - \epsilon)^2} D_t \zeta_i^A + \frac{1}{(1 - \epsilon)^2} \Xi^A B \zeta_i^B = 0,
\]

with

\[
\begin{align*}
\partial_t a &= -\partial_t (\ln H) = -\frac{\dot{\epsilon}}{1 - \epsilon} \Pi A, \\
\partial_t \phi^A &= -\sqrt{2\epsilon} \left( \delta^A B + \frac{\epsilon}{1 - \epsilon} \Pi A \Pi B \right) \zeta_i^B,
\end{align*}
\]

and

\[
D_t \Pi^A = -\frac{\sqrt{2\epsilon}}{\kappa} H \left[ (1 - \epsilon) D_t \zeta_i^A + \left( (\epsilon + \eta^\parallel) \delta^A B - \epsilon \frac{\Pi A \Pi B}{\Pi} + \frac{\dot{\epsilon}}{1 - \epsilon} \eta^A \Pi B \right) \zeta_i^B \right].
\]

Defining the velocity \(\theta_i^A\) we express the dynamical equation \((22)\) as

\[
\begin{align*}
D_t \zeta_i^A - \theta_i^A &= 0 \\
D_t \theta_i^A + \frac{3 - 2\epsilon + 2\eta^\parallel - 3\epsilon^2 - 4\epsilon\eta^\parallel}{(1 - \epsilon)^2} \theta_i^A + \frac{1}{(1 - \epsilon)^2} \Xi^A B \zeta_i^B &= 0
\end{align*}
\]

We then add to the right-hand side of these equations stochastic source terms to emulate the continuous influx of short-wavelength modes when they cross the horizon, in this way setting up the proper initial conditions for the long-wavelength system. The derivation of these source terms goes as follows (for more details see [7]). We start from the exact (i.e. no long-wavelength approximation) equations for the linear theory, written as two first-order differential equations like \((32)\). We define smoothed long-wavelength variables using a window function \(W\). In Fourier space this means \(\zeta_{\text{lin},lw}^A(k) = W(k) \zeta_{\text{lin}}^A(k)\), and an identical expression for \(\theta^A\). Rewriting the exact linear equations in terms of the smoothed variables \(\zeta_{\text{lin},lw}^A\), one is left with terms that depend on the non-smoothed variables \(\zeta_{\text{lin}}^A\). These terms together form the source, and are put on the right-hand side of the equations, while the other terms are on the left-hand side. At the linearised level these equations are still exact. Next, in order to go to the non-linear case one uses the fact that the left-hand side is exactly the linearised version of the long-wavelength system \((32)\). It is then postulated that the right-hand side can be taken as the source term for the non-linear long-wavelength equations, if one replaces all background quantities with their fully inhomogeneous versions. The final result is:

\[
\begin{align*}
D_t \zeta_i^A - \theta_i^A &= S_i^A \\
D_t \theta_i^A + \frac{3 - 2\epsilon + 2\eta^\parallel - 3\epsilon^2 - 4\epsilon\eta^\parallel}{(1 - \epsilon)^2} \theta_i^A + \frac{1}{(1 - \epsilon)^2} \Xi^A B \zeta_i^B &= J_i^A
\end{align*}
\]

\(^4\) This is a useful time variable as long as it is monotonic, a condition that is invalid only around reheating. Since the source terms have disappeared by then, a different choice can be made near the end of inflation.
with the source terms $S^A_i$ and $J^A_i$ given by

$$S^A_i = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{W}(k) \zeta^A_{\text{lin}}(k, x) i k_i e^{i k \cdot x} + \text{c.c.}, \quad J^A_i = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{W}(k) \theta^A_{\text{lin}}(k, x) i k_i e^{i k \cdot x} + \text{c.c.},$$

(34)

where c.c. denotes the complex conjugate and $\zeta^A_{\text{lin}}$ and $\theta^A_{\text{lin}}$ are the full, non-smoothed solutions from linear perturbation theory, that is, incorporating short-wavelength information. The fact that they depend on $x$ as well as on $k$ represents the fact that all background quantities in these solutions should be made inhomogeneous.

The following relations hold:

$$\zeta^A_{\text{lin}} = -\frac{\kappa}{a \sqrt{2} \epsilon} q^A_{\text{lin}}, \quad \theta^A_{\text{lin}} = D\tau \zeta^A_{\text{lin}}, \quad q^A_{\text{lin}}(k) = Q^A_{\text{lin} B}(k) \alpha^B(k),$$

(35)

where $q^A_{\text{lin}}$ is the solution from linear theory for the Sasaki-Mukhanov variable $\zeta$. In other words, $Q^A_{\text{lin} B}(k)$ is the solution of

$$\mathcal{D}^2 Q^A_{\text{lin} B} + \frac{1 - 2 \epsilon - \hat{c}^2 - 2 \hat{c} \hat{\eta}_\parallel}{(1 - \hat{\eta})^2} \mathcal{D}_\tau Q^A_{\text{lin} B} + \frac{\Omega^A C}{(1 - \hat{\eta})^2} Q^A_{\text{lin} B} + \frac{k^2}{(a H)^2 (1 - \hat{\eta})^2} Q^A_{\text{lin} B} = 0,$$

(36)

where all coefficients take their homogeneous background values and with initial conditions deep within the horizon $Q^A_{\text{lin} B}(k) = U^A_B / \sqrt{2k}$ and $\mathcal{D}_\tau Q^A_{\text{lin} B}(k) = 1 / [a H (1 - \hat{\eta})] \sqrt{k / 2 U^A_B}$, where $U^A_B$ is a physically irrelevant unitary matrix. Then the sources can be written as

$$S^A_i = -\frac{\kappa}{a \sqrt{2} \epsilon} \int \frac{d^3k}{(2\pi)^{3/2}} \hat{W}(k) Q^A_{\text{lin} B}(k) \alpha^B(k) i k_i e^{i k \cdot x} + \text{c.c.},$$

$$J^A_i = -\frac{\kappa}{a \sqrt{2} \epsilon} \int \frac{d^3k}{(2\pi)^{3/2}} \hat{W}(k) \left[ D\tau Q^A_{\text{lin} B}(k) - \frac{1 + \hat{c} + \hat{\eta}_\parallel}{1 - \hat{\eta}} Q^A_{\text{lin} B}(k) \right] \alpha^B(k) i k_i e^{i k \cdot x} + \text{c.c.}$$

(37)

Following [8] we have introduced a set of complex Gaussian stochastic quantities $\alpha^A(k)$ to replace the quantum creation and annihilation operators, satisfying

$$\langle \alpha^A(k) \alpha^*_B(k') \rangle = \delta^A_B \delta(\mathbf{k} - \mathbf{k}'), \quad \langle \alpha^A(k) \alpha_B(k') \rangle = 0,$$

(38)

where $\langle \ldots \rangle$ denotes an ensemble average. This is why the source terms are called stochastic. Because $\zeta^A$ and $\theta^A$ now represent smoothed, long-wavelength variables, it is clear that they have to be zero at early times when all the modes are sub-horizon:

$$\lim_{\tau \to -\infty} \zeta^A = 0, \quad \lim_{\tau \to -\infty} \theta^A = 0.$$

(39)

The appropriate short-wavelength initial conditions are then introduced into the system later via the stochastic source terms.

The linear perturbation equation (36) can either be solved exactly numerically to obtain $Q^A_{\text{lin} B}$, or analytically within the slow-roll approximation. The latter was done in [8]. From that analytic slow-roll solution one finds that

$$\mathcal{D}_\tau Q^A_{\text{lin} B} = \bar{D}^A C Q^C_{\text{lin} B},$$

with $\bar{D}^A C$ a matrix containing slow-roll parameters (given explicitly in [8]), and $Q^A_{\text{lin} B} = c / (2k^{3/2} R) \delta^A_B +$ first-order slow-roll terms (omitting physically irrelevant overall unitary factors). Even though in this paper we make no use of the slow-roll approximation whatsoever, for applications of the formalism it is useful to observe that this means that, in our gauge, non-linear corrections to $Q^A_{\text{lin}}$ are higher order in slow roll and can be neglected in a leading-order treatment. (This reflects the fact that $q$, not $\zeta$, is the proper well-behaved quantity to use on short wavelengths.) Hence in such a treatment the only quantities that have to be made inhomogeneous in the source terms [15] are the matrix $\bar{D}^A C$ and the $1 / (a \sqrt{2} \epsilon)$ prefactors. This point is explicitly worked out in [15, 16].

We can take the Fourier transform of the window function $\hat{W}(k)$ to be a Gaussian,

$$\hat{W}(k) \equiv e^{-k^2 R^2 / 2}, \quad R \equiv \frac{c}{a H} = c e^{-\tau},$$

(40)

with $R$ the smoothing length above which the system is considered to be ‘long wavelength’. We take it to be a small multiple of the comoving Hubble radius, since the latter is the natural scale that separates short and long wavelengths in inflation; $c \approx 3 - 5$ will do for our purposes. Since the smoothing length is decreasing quasi-exponentially during inflation, more and more modes will populate the long-wavelength system. For our choice of window function, $\hat{W}(k) = (k R)^2 \hat{W}(k)$, where a dot denotes a time derivative with respect to $\tau = \ln(a H)$. If the smoothing length $R$ were
constant, $\mathcal{S}_i^A$ and $\mathcal{J}_i^A$ would be zero. Note that with our choice of time the smoothing length remains uniform across space even in the presence of perturbations. Had we not made this choice any mode would enter the long-wavelength system at different times for different points in space, which would complicate matters. The system of equations (33) along with the constraints (29)–(31) (where $\mathcal{D}_\tau \xi_i^A$ is replaced by $\theta_i^A$) forms a consistent and self-contained system of non-linear stochastic equations and is the main result of this paper. When linearised it is exact and reproduces the well-known gauge-invariant linear perturbation theory [1].

At first sight it might seem that any results obtained from solving (33) will depend on the \textit{ad hoc} choice for the window function $W$. It turns out that the exact form of $W$ is for the most part irrelevant. At linear order there is no dependence of the final results on the exact form of $W$, by construction; any properly normalised function with $W(k) \rightarrow 1$ for scales sufficiently larger than the horizon will produce the same final answer. Beyond linear order there is some dependence on the choice of $W$, but this dependence is limited to terms that involve only non-linear effects around horizon crossing, which turn out to be small and hence uninteresting observationally. The non-Gaussianity in single-field inflation is an example of this, see [15], but even here only the limit with all three momenta of the same order is affected by the choice of window function. On the contrary, any effects which involve super-horizon evolution are independent of the functional form of $W$ for scales sufficiently larger than the horizon. We discuss these issues in greater detail in [16, 17], where we also show that such super-horizon evolution effects can lead to potentially observably large non-Gaussianity.

IV. FIELD BASIS

So far we have used a field component notation, with indices $A, B$, etc. labeling the fields. However, in actual analytic multiple-field calculations it is more convenient to work with a certain explicit basis on the field manifold, which allows us to clearly distinguish effectively single-field effects from truly multiple-field ones, to work again with normal time derivatives instead of covariant ones, and which is also a necessary ingredient for quantisation. This basis was first defined in [22], and is also described in [8].

The first basis vector $e^A_1$ is the direction of the field velocity. Next, $e^A_2$ is defined as the direction of that part of the field acceleration that is perpendicular to $e^A_1$. Hence,

$$e^A_1 \equiv \frac{H^A}{\Pi}, \quad e^A_2 \equiv \frac{\mathcal{D}_t H^A - e^A_1 e^B_1 \mathcal{D}_t H^B}{|\mathcal{D}_t H^A - e^A_1 e^B_1 \mathcal{D}_t H^B|}.$$  \hspace{1cm} (41)

One continues this orthogonalisation process with higher derivatives until a complete basis is found. Explicitly,\(^5\)

$$e^A_m = \frac{(P_{m-1}^\perp)^A_B (\frac{1}{\delta} \mathcal{D}_t)^{m-1} H^B}{|(P_{m-1}^\perp)^A_B (\frac{1}{\delta} \mathcal{D}_t)^{m-1} H^B|}, \quad (P_m)^A_B = e^A_m e^B_m, \quad (P_m^\perp)^A_B = \delta^A_B - \sum_{q=1}^m (P_q)^A_B.$$  \hspace{1cm} (42)

Here the projection operators $P_m$ project on the $e_m$, while the $P_m^\perp$ project on the subspace that is perpendicular to $e_1, \ldots, e_m$, and we define $(P_0^\perp)^A_B = \delta^A_B$. The basis is orthonormal: $e^A_m e^A_{nA} = \delta_{mn}$. Now one can take components of vectors in this basis and we define, for example for $\xi^A_i$,

$$\xi^m_i \equiv e_m A \xi_i^A.$$

Note that, unlike for the index $A$, there is no difference between upper and lower indices for the $m$. The slow-roll parameters $\tilde{\eta}^\parallel$ and $\tilde{\eta}^\perp$ given in (22) are now simply $\tilde{\eta}^\parallel \equiv e^A_1 \tilde{\eta}_A$ and $\tilde{\eta}^\perp \equiv e^A_2 \tilde{\eta}_A$ (by construction there are no other components of $\tilde{\eta}^A$) and equivalently $\tilde{\xi}^\parallel \equiv e^A_1 \tilde{\xi}_A$. The definitions (22) can be rewritten as (including $\tilde{\epsilon}$ for completeness’ sake)

$$\tilde{\epsilon} = \frac{\kappa^2 H^2}{2H^2}, \quad \tilde{\eta}^\parallel = -3 - e^A_1 V_A H^A, \quad (\tilde{\eta}^\perp)^2 = \frac{V^AV_A - (e^A_1 V_A)^2}{H^2 H^2}, \quad \tilde{\xi}^\parallel = 3 \tilde{\epsilon} - 3 \tilde{\eta}^\parallel - e^A_1 V_A e^B_1 e^B_1.$$  \hspace{1cm} (44)

\(^5\) Again, as in the definition of the slow-roll parameters, using (11) and its time derivatives one can rewrite all time derivatives of $H^A$ in terms of $H^A$ itself and field derivatives of the potential, and define the basis vectors in that way. However, once the stochastic source terms are included, equation (13) changes subtly and the two ways of defining the basis vectors (except for $e^A_1$) are no longer equivalent. At that point it is actually the second way that is our real definition, but since those equations are rather awkward and have to be written down for each value of $m$ separately, we give here the compact definition in terms of time derivatives of $H^A$. 
Finally we define for later use
\[
Z_{mn} = \frac{1}{NH} \, e_m^A D_i e_n^A. \tag{45}
\]

It is antisymmetric and only non-zero just above and below the diagonal. If a slow-roll approximation were to be made, it would be first order in slow roll. Its explicit form in terms of slow-roll parameters can be found in [8]; up to \(m, n = 3\) the only non-zero terms are \(Z_{21} = -Z_{21} = -\dot{\eta}^\perp\) and \(Z_{23} = -Z_{32} = -\xi_3/\dot{\eta}^\perp\).

The equation of motion (28) for \(\zeta_i^A\) can now be rewritten as an equation of motion for \(\zeta_i^m\):
\[
\dot{\zeta}_i^m + \left(\frac{3 - 2\dot{\xi} + 2\dot{\eta}^\parallel - 3\dot{\xi}^2 - 4\ddot{\eta}^\parallel}{(1 - \ddot{\xi})^2} + \frac{2Z_{mn}}{1 - \ddot{\xi}}\right) \zeta_i^m + \frac{\Xi_{mn}}{(1 - \ddot{\xi})^2} \zeta_i^n = 0,
\]
where \(\dot{\zeta}_i^m = \frac{\partial}{\partial \tau}(\zeta_i^m) = \frac{\partial}{\partial \tau}(e_m^A e_i^A B)\), with \(\tau\) the time variable defined in (27), and with
\[
\Xi_{mn} = \frac{V_{mn}}{H^2} - \frac{2\dot{\xi}}{\kappa^2} R_{mn1B} + (1 - \dot{\xi}) \dot{Z}_{mn} + Z_{mp} Z_{pn} + \left(3 - 2\dot{\xi} + 2\ddot{\eta}^\parallel - \ddot{\xi}^2 - 2\dddot{\eta}^\parallel\right) \frac{Z_{mn}}{1 - \ddot{\xi}}
\]
\[
+ \left(3\dot{\xi} + 3\dot{\eta}^\parallel + 2\ddot{\eta}^\parallel + 4\dddot{\eta}^\parallel (\dddot{\eta}^\perp)^2 + \dddot{\eta}^\parallel\right) \frac{d_m n_1 + \dddot{\eta}^\perp (\delta_m n_1 + \delta_m n_2 + \delta_m n_1)\right),
\]
where \(R_{mn1B} = e_m^A R_{ABCD} e_1^C e_1^D\) and \(V_{mn} = e_m^A \dot{e}_A^B e_n^D\). The matrix \(\Xi_{mn}\) is not just \(e_m^A \Xi_{AB} e_n^B\), but contains additional \(Z_{mn}\) terms. Note that \(\Xi_{11} = 0\) identically, which means that in the single-field case \(\Xi\) is zero, and hence \(\zeta_i\) conserved on long wavelengths, as we mentioned before. In a procedure completely identical to the one described in the previous section, we can then split the second-order differential equation into two first-order equations and add stochastic source terms to find the analogue of (39):
\[
\begin{align*}
\dot{\zeta}_i^m - \theta_i^m &= S_i^m, \\
\dot{\theta}_i^m + \left(3 - 2\dot{\xi} + 2\ddot{\eta}^\parallel - 3\dot{\xi}^2 - 4\dddot{\eta}^\parallel\right) \frac{d_m n_1 + \dddot{\eta}^\perp (\delta_m n_1 + \delta_m n_2 + \delta_m n_1)\right) \zeta_i^m + \frac{\Xi_{mn}}{(1 - \ddot{\xi})^2} \zeta_i^n = J_i^m,
\end{align*}
\]
where the source terms \(S_i^m\) and \(J_i^m\) are defined analogously to (39) with \(\zeta_i^m\) and \(\theta_i^m\) instead of \(\zeta_i^A\) and \(\theta_i^A\). It is important to notice that for a velocity, like \(\theta_i^m\), the relation between \(\theta_i^m\) and \(\theta_i^A\) is not simply a contraction with the basis vector, since the time derivative of the basis vector has to be taken into account as well. They are related by
\[
\theta_i^m = e_m^A \theta_i^A - NH Z_{mn} \zeta_i^n.
\]
Of course (39) has to be supplemented with the constraints (24) - (31). From these constraints and the definitions (44) we can derive the following expressions for the spatial derivatives of \(a, H\) and the slow-roll parameters:
\[
\begin{align*}
\partial_i \ln a &= -\partial_i \ln H = -\frac{\ddot{\xi}}{1 - \ddot{\xi}} \zeta_i^1, \\
\partial_i \dot{\xi} &= -2\dot{\xi} \left(1 - \dot{\xi}\right) \theta_i^1 + \frac{\ddot{\xi} + \dddot{\eta}^\parallel}{1 - \ddot{\xi}} \zeta_i^1 - \dddot{\eta}^\perp \zeta_i^2, \\
\partial_i \dddot{\eta}^\parallel &= (1 - \ddot{\eta}^\parallel) \left(3 + \dddot{\eta}^\parallel\right) \theta_i^1 - \dddot{\eta}^\perp \theta_i^2 - \frac{1}{1 - \ddot{\xi}} \left(3 + \dddot{\eta}^\parallel\right) \dddot{\eta}^\perp \zeta_i^2
\end{align*}
\]
\[
+ \dddot{\xi} \zeta_i^3 + \sum_{n \geq 2} \frac{V_{i n}}{H^2} \zeta_i^n, \tag{49}
\]
\[
\partial_i \dddot{\eta}^\perp = (1 - \ddot{\eta}^\parallel) \left(3 + \dddot{\eta}^\parallel\right) \theta_i^1 + (3 + \dddot{\eta}^\parallel) \theta_i^2 + \frac{1}{1 - \ddot{\xi}} \left(3 + \dddot{\eta}^\parallel\right) \dddot{\eta}^\perp + \frac{V_{i 2}}{H^2} \zeta_i^1 + \left(3 + \dddot{\eta}^\parallel\right) \left(\dddot{\eta}^\parallel - \dddot{\eta}^\perp\right) \zeta_i^2
\]
\[
- (3 + \dddot{\eta}^\parallel) \dddot{\eta}^\perp \zeta_i^3 + \sum_{n \geq 2} \frac{V_{i n}}{H^2} \zeta_i^n.
\]

When no stochastic source terms are present, or when all slow-roll parameters take their homogeneous background values, the relation \(V_{i n}/H^2 = (3\ddot{\xi} - 3\dddot{\eta}^\parallel - \dddot{\xi}) \delta_{n1} - (3\dddot{\eta}^\perp + \dddot{\xi}) \delta_{n2} - \dddot{\xi} \delta_{n3}\) can be used. This relation is derived by taking the time derivative of (43) and using the definitions of the slow-roll parameters. Let us reiterate that all these expressions are not slow-roll approximated, even though they contain slow-roll parameters. The equations in this section will be the basis of more detailed multiple-field calculations in [16].
V. APPLYING THE FORMALISM

The closed system of non-linear equations for \( \zeta_i^A \) is, in principle, amenable to direct numerical solution on a multi-
dimensional spatial grid. However, the inhomogeneous coefficients in \( \delta \zeta_i^A \) have an implicit dependence on \( \zeta_i^A \), meaning that \( a(t, \mathbf{x}), \phi^A(t, \mathbf{x}) \) and \( \Pi^A(t, \mathbf{x}) \) need to be reconstructed from the constraints \( 29-31 \) at each timestep. In the
long-wavelength approximation and in the absence of noise, the \( \zeta_i^A \) equation plus the constraints are equivalent
to the separate universe equations \( \delta \zeta_i^A(k, t) \) from which they were derived, that is, the original evolution equations for
\( a(t, \mathbf{x}), \phi^A(t, \mathbf{x}) \) and \( \Pi^A(t, \mathbf{x}) \). Numerically, it turns out to be simplest to solve the \( \zeta_i^A \) equations by supplementing
them with the separate universe equations, so that the inhomogeneous coefficients depending on \( a, \phi^A \) and \( \Pi^A \) are
given explicitly, rather than being reconstructed from \( \zeta_i^A \). (The redundancy entailed in this can be viewed positively,
since it gives a check on the accuracy of the finite difference scheme through the relations \( 14 \) and \( 19 \).)

In order to add the stochastic noise on the right-hand side of \( \delta \zeta_i^A(k, t) \), we evaluate the mode functions of the \( k \)-dependent
linear perturbation equations to high accuracy, beginning with flat-space initial conditions well inside the horizon (no
slow-roll approximation is made). As in other stochastic approaches, a realisation of the linear perturbation solution
\( q^A_{\text{lin}}(k, t) \) is then constructed by multiplying random numbers satisfying \( 38 \) for each \( k \) on the numerical grid by the
corresponding mode function for \( k = |k| \). At a finely-spaced set of times, the linear perturbations \( q^A_{\text{lin}}(k, t) \) are convolved
with the window function \( W(k) \) to determine the necessary long-wavelength correction \( \delta \zeta_i^A(x, t) \delta t \) in real space, where \( \delta t \) is the timestep. Since any change \( \delta \zeta_i^A \) necessarily entails changes to \( a, \phi^A, \Pi^A \), the corrections \( \delta a, \delta \phi^A, \delta \Pi^A \) also
need to be determined from the constraints \( 29-31 \). The procedure then iterates through successive non-linear
approximation and the construction of the stochastic source terms for the non-linear equations. The
single- and multiple-field inflation contexts will be presented elsewhere.

Up to this point, the only approximations that have been made are the basic assumptions of the method: the
long-wavelength approximation and in the absence of noise, the

The general form of the system of non-linear stochastic equations with the constraints is
\[
D_\tau v_i + A(u) v_i = G_i(u), \quad \partial_j u = C(u) v_i .
\]

We use the vectors \( v_i \equiv (\zeta_i^A, \phi_i^A)^T \) and \( u \equiv (a, \phi^A, \Pi^A)^T \) and \( G_i \equiv (S_i^A, T_i^A)^T \) (where \( T \) denotes the transpose) to
simplify the notation. The quantities \( A \) and \( C \) are matrices, functions of the components of \( u \), which can be read off from
\( 38 \) and \( 29-31 \). Now everything can be expanded as
\[
v_i = v_i^{(1)} + v_i^{(2)} + \ldots , \quad u = u^{(0)} + u^{(1)} + u^{(2)} + \ldots
\]

Note that \( v_i \) does not have a zeroth-order spatially homogeneous background part. At first order we have
\[
D_\tau v_i^{(1)} + A^{(0)} v_i^{(1)} = G_i^{(1)} , \quad \partial_j u^{(1)} = C^{(0)} v_i^{(1)} .
\]

The first equation is linear in \( v_i^{(1)} \) and can be solved since the background solution and the linear source term \( G_i^{(1)} \)
are assumed to be known. The procedure is equivalent to standard gauge-invariant linear perturbation theory. With
this solution the constraint can then be solved for \( u^{(1)} \):
\[
u^{(1)} = C^{(0)} \partial^{-2} \phi_i v_i^{(1)} .
\]

It is now straightforward to obtain the second-order perturbation equations:
\[
D_\tau v_i^{(2)} + A^{(0)} v_i^{(2)} = G_i^{(2)} - A^{(1)} v_i^{(1)} , \quad \partial_j u^{(2)} = C^{(0)} v_i^{(2)} + C^{(1)} v_i^{(1)} .
\]

Here \( A^{(1)} \) is given by
\[
A^{(1)} = \left( \frac{\partial A}{\partial u} \right)^{(0)} u^{(1)} ,
\]
and similarly for $C^{(1)}$ and $\xi^{(2)}$. We see that the equation of motion for $\xi_i^{(2)}$ is again linear, with the right-hand side known from the solution at first order, and it can be solved. Again, given the solution for $\xi_i^{(2)}$ we can then solve the constraint for $u^{(2)}$:

$$u^{(2)} = C^{(0)} \partial^{-2} \partial^i \xi_i^{(2)} + \partial^{-2} \partial^i \left( C^{(1)} \xi_i^{(1)} \right).$$

Knowledge of $\xi_i^{(2)}$ and $u^{(2)}$ gives $\xi_i^{(3)}$ which in turn determines $u^{(3)}$, etc. Continuing in this fashion one can in principle obtain all quantities of interest to any desired order. Even though the discussion in this section might seem rather abstract, we show in [15, 16, 17] that this approach works well in explicit calculations. The expressions on the right-hand side of (56) are relatively simple to derive, which makes the calculations tractable.

VI. SUMMARY

In this paper we described a scheme for studying the non-linear evolution of inhomogeneities during multiple-field inflation, extending ideas presented in [7]. Here we summarise the main ingredients of the formalism. (i) We advocate the use of variables like $\zeta^A$, defined in [13], using spatial gradients to describe deviations from a homogeneous spacetime. These variables are invariant under long-wavelength changes of time slicing and include both matter and metric perturbations. (ii) We derived a set of exact, non-linear, long-wavelength evolution equations for $\zeta_i^{(2)}$ [20]. To close the system, a set of constraints [28]–[30] determines all local quantities appearing in the coefficients of $\zeta_i^{(2)}$. An obvious consequence of these equations is that for single-field inflation the variable $\zeta_i^{(2)}$ is exactly conserved. (iii) In order to incorporate the influence of short-wavelength quantum modes, stochastic sources are added on the right-hand side of the evolution equations, leading to [33, 34]. When linearised, the equations are exactly equivalent to linear, gauge-invariant perturbation theory. We also introduced the field basis of [8] and rewrote our equations in terms of it, which is very convenient for actual multiple-field calculations. The full stochastic system is derived without any slow-roll approximations and is well-suited for numerical simulations. If slow roll is assumed, analytic results can be obtained via perturbative expansions. The merit of perturbing this set of non-linear equations for the variables $\zeta^A$ is a significant simplification compared to the standard approach of perturbing the Einstein equations from the outset. This paper concentrated on describing the method, while quantitative calculations using this formalism are given in related publications [15, 16, 17].

Acknowledgements

This research is supported by PPARC grant PP/C501676/1.

APPENDIX A: CHANGING THE TIME SLICING ON LONG WAVELENGTHS

In this appendix we demonstrate that [12] is invariant under changes of time slicing. Again, we restrict attention to metrics like [1] which do not mix temporal and spatial intervals. The transformations we consider are defined by going to a new time slicing $t \rightarrow \tilde{t}(t, \textbf{x})$ and then fixing the spatial coordinates by demanding that $g_{\tilde{t}i} = 0$, that is, making the coordinate lines $\tilde{\textbf{x}}^j = \text{const.}$ normal to the $\tilde{t} = \text{const.}$ hypersurfaces. We show, following [10], that, up to second order in spatial gradients, the spatial coordinates can be taken to transform trivially.

Suppose we are interested in a patch of spacetime which at early enough times is homogeneous, in accordance with the initial conditions, that is, the comoving horizon is initially taken to be larger than the spatial extent of the patch. Perturbations are introduced later as wavelengths shorter than the spatial region of interest exit the horizon. While the patch is homogeneous, there is a preferred time slicing which respects the spatial symmetries, and spatial coordinates are chosen on these initial time slices. After perturbations are introduced there is no preferred choice of time slicing. Hence one should consider the consequences of changes $t \rightarrow \tilde{t}(t, \textbf{x})$, for example between uniform field and uniform expansion time slices. To define the new spatial coordinate lines we demand that the curves $\tilde{\textbf{x}}^j = \text{const.}$ be the integral curves of $w^\mu$ with

$$w^\mu = \frac{dx^\mu}{ds} = \partial^\mu \tilde{t}.$$  \hspace{1cm} (A1)

Roman indices run from 1 to 3 and Greek from 0 to 3. The vector $w^\mu$ is normal to the $\tilde{t} = \text{const.}$ hypersurfaces. For a displacement along its integral curves $\delta x^\mu = \partial^\mu \tilde{t} \delta s$, where $s$ is an arbitrary parameter. For such a displacement $\delta x^\mu$
we have \( \delta \tilde{t} = \partial_{\mu} \tilde{t} \delta x^\mu = \partial_{\mu} \tilde{t} \partial^{\mu} \tilde{t} \delta s \). Therefore, the time-time and space-time parts of the transformation are defined to be
\[
\frac{\partial x^\mu}{\partial \tilde{t}} = \frac{\partial^{\mu} \tilde{t}}{\partial_{\mu} \tilde{t} \partial^{\nu} \tilde{t}} \quad (A2)
\]
by eliminating \( ds \) in (A1). We can now derive the time-space and space-space parts. Defining \( \Lambda^\mu_j = \partial x^\mu / \partial \tilde{x}^j \), consider another displacement \( \delta x^\mu \) on a \( \tilde{t} = \text{const.} \) hypersurface:
\[
\delta x^\mu \partial_{\mu} \tilde{t} = 0. \quad (A3)
\]
Writing \( \delta x^\mu = \Lambda^\mu_j \delta \tilde{x}^j \) (\( \delta \tilde{t} \) is zero for this displacement), we find from (A3) for an arbitrary \( \delta \tilde{x}^j \) that
\[
\Lambda^\mu_j \partial_{\mu} \tilde{t} = 0 \quad \Rightarrow \quad \Lambda^0_j = -\frac{\Lambda^i_j \partial_i \tilde{t}}{\partial t \tilde{t}}. \quad (A4)
\]
So, we need to specify only the \( \Lambda^i_j \) part of the transformation. Integrating equation (A2) along a \( \tilde{x}^j = \text{const.} \) curve gives
\[
x^j = f^j(\tilde{x}^i) + \int d\tilde{t} \frac{\partial x^j}{\partial_{\mu} \tilde{t} \partial^{\nu} \tilde{t}}, \quad \text{(A5)}
\]
with \( f^j \) some arbitrary function depending on the initial labelling of the new spatial coordinates. Since at early times spacetime is homogeneous, useful time slicings are those coinciding with the preferred one initially, \( \tilde{t} = t = t_{\text{pref}} \). Hence it is natural to consider spatial coordinates that reduce to the preferred set on these initial time slices. We thus take \( f^j(\tilde{x}^i) = \delta^j_i \tilde{x}^i \). Of course, we could perform an arbitrary spatial transformation, but on long wavelengths it would be independent of time, as shown in [19]. Given this result, equations (20) and (23)--(26) are covariant under such a transformation. Taking a spatial derivative of (A5) with respect to \( \tilde{x}^j \), using the chain rule and (A4), one arrives at
\[
\left[ \delta^i_j + \frac{\partial \tilde{t}}{\partial t \tilde{t}} \frac{\partial x^j}{\partial_{\mu} \tilde{t} \partial^{\nu} \tilde{t}} - \partial_i \int d\tilde{t} \frac{\partial x^j}{\partial_{\mu} \tilde{t} \partial^{\nu} \tilde{t}} \right] N^i_k = \delta^i_k, \quad \text{(A6)}
\]
so that
\[
\Lambda^i_j = \delta^i_j + \mathcal{O} \left( (\partial t)^2 \right), \quad \Lambda^0_j = -\delta^i_j \frac{\partial_i \tilde{t}}{\partial t \tilde{t}} + \mathcal{O} \left( (\partial t)^2 \right). \quad (A7)
\]
The full transformation has now been specified. With these formulae one can verify that the metric in the new coordinates reads
\[
e^{2\alpha(t,x)} h_{ij}(\tilde{x}) = \delta_{ij} \frac{\partial^k \partial^l e^{2\alpha(t,x)} h_{kl}(x)}{\partial_{\mu} \tilde{t} \partial^{\nu} \tilde{t}} e^{2\alpha(t,x)} h_{kl}(x), \quad g_{\tilde{t} \tilde{t}} = 0, \quad \tilde{N} = N/\partial_i \tilde{t}. \quad \text{(A8)}
\]
We note that time derivatives transform as
\[
\frac{1}{\tilde{N}} \partial_{\tilde{t}} = \frac{1}{N} \partial_t + \mathcal{O} \left( (\partial t)^2 \right). \quad \text{(A9)}
\]
Under these long-wavelength time-slicing transformations the determinant of the spatial metric, \( \exp[2\alpha] \), transforms as a spacetime scalar. Hence, it is easy to see that a variable like (12), in particular (14), is invariant under these changes.


