Acoustic analogues of black hole singularities

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We search for acoustic analogues of a spherical symmetric black hole with a pointlike source. We show that the gravitational system has a dynamical counterpart in the constrained, steady motion of a fluid with a planar source. The equations governing the dynamics of the gravitational system can be exactly mapped in those governing the motion of the fluid. The different meaning that singularities and sources have in fluid dynamics and in general relativity is also discussed. Whereas in the latter a pointlike source is always associated with a (curvature) singularity in the former the presence of sources does not necessarily imply divergences of the fields.

I. INTRODUCTION

One of the most intriguing features of classical general relativity is the presence of curvature singularities. The classical gravitational dynamics described by general relativity generates singularities both in the collapse of massive bodies and in the cosmological evolution of the universe. The physical meaning of these spacetime singularities has been widely discussed since their discovery. By now it has become conventional wisdom the point of view that the resolution of the singularities of general relativity has to be found in a quantum theory of gravity governing its short distance behavior. For instance, string theory allows for the quantum resolution of some singularities, but we are far away from having a detailed description of how the Schwarzschild black hole (or the cosmological) singularity is resolved.

Building on a proposal of Unruh [1], in recent years there has been a flurry of activity about the use of condensed matter systems to mimic various kinematical aspects of general relativity [2, 3, 4, 5, 6]. Condensed matter systems provide us with analogue models of gravity that not only reproduce typical gravitational structures (event and cosmological horizons) but can be also used to indicate an answer to fundamental questions of gravitational physics (field theory in curved spacetime and Hawking radiation) [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The hope behind these efforts is that in a near future we could experimentally test the properties of the gravitational interaction using condensed matter systems analogues.

One is therefore tempted to use analogue systems of gravity, in particular fluids, to try to mimic and understand spacetime singularities. Till a few months ago this idea would have been immediately rejected because the analogy gravity/fluids was formulated only at a kinematical level. Spacetime singularities in general relativity have a dynamical origin. In order to describe them using analogue models one needs to formulate the correspondence at a dynamical level.

For spherically symmetric black holes such a dynamical correspondence has been recently found [19]. After gauge fixing, the four-dimensional dynamics governing a spherically symmetric black hole was shown to be equivalent to that governing the steady, constrained, cylindrically symmetric motion of a fluid. The results of Ref. [19] provide us with a framework to describe black hole singularities using acoustic analogues.

In this paper, extending the results of Ref. [19] we will find the dynamical acoustic analogue of a spherical symmetric black hole with a pointlike source. We will show that the gravitational system has a dynamical counterpart in the constrained, steady motion of a fluid with a source (Sect. II). Although the discussion will be purely classical it will shed light on the different meaning that singularities and sources have in fluid dynamics and in general relativity. Whereas in the latter a pointlike source is always associated with a (curvature) singularity, in the former the presence of sources does not necessarily imply divergences of the fields.

We will also discuss the implications of our results for black hole thermodynamics (Sect. III) and solve the constraint for the fluid dynamics in the case of a fluid with a generic power-law equation of state (Sect. IV).

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II. GRAVITATIONAL DYNAMICS, POINTLIKE SOURCES AND FLUID DYNAMICS

We start from generic four-dimensional Einstein gravity coupled to matter fields and a pointlike source of mass $m$. The action is (we adopt the notations of Ref. [20] and use natural units $c = \hbar = k_B = 1$)

$$A = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \, R - \int d^4x \sqrt{-g} \, \mathcal{L}_{MF} - m \int dt \sqrt{-g_{ij}} \frac{dx^i}{dt} \frac{dx^j}{dt}, \quad (1)$$

where $G$ is the Newton constant and $\mathcal{L}_{MF}$ is the Lagrangian for the matter fields. $\mathcal{L}_{MF}$ not only describes matter fields such as gauge and scalar fields, but may also contain a cosmological constant $\Lambda$. We shall consider for simplicity only the case in which the classical solutions for the matter fields are (or can be treated as) constant background configurations. In general the field equations stemming from the action (1) will admit spherically symmetric black hole solutions, whose mass $M$ will be equal to the mass $m$ of the pointlike source. Moreover, these solutions will exhibit curvature singularities at the location of the source. The simplest case is represented by the Schwarzschild solution, obtained setting $\mathcal{L}_{MF} = 0$ in Eq. (1). Introducing a cosmological constant term or an electromagnetic field in (1), we have the Schwarzschild-anti de Sitter or the Reissner-Nordstrom solution, respectively.

We want to find a classical, non-relativistic fluid whose dynamics is equivalent to that governing the static, spherically symmetric solutions of the theory (1). When the pointlike source is not present, this problem can be drastically simplified by noticing that once spherical symmetry is imposed the gravitational theory (1) becomes essentially two-dimensional (2D). Recently this result has been extended also to the case where sources are present [21]. The 2D model is obtained by retaining only the radial modes in the action (1) and it has the form of a 2D dilaton gravity model (see for instance Ref. [22]). A scalar field $\phi$ (the dilaton) parametrizes the volume of the transverse 2-dimensional sphere,

$$ds^2_{(2)} = ds^2_{(4)} + \frac{2}{\lambda^2} \phi d\Omega_2^2, \quad (2)$$

where $\lambda$ is a parameter with dimensions of length$^{-1}$, related to the 4D Newton constant, $G = \lambda^{-2}$. Inserting the ansatz (2) into the action (1) and performing the Weyl rescaling of the 2D metric

$$g \rightarrow \frac{1}{\sqrt{2\phi}} g, \quad \quad (3)$$

needed to get rid of the kinetic terms for the dilaton, one obtains a 2D model characterized by a dilaton potential $V(\phi)$ and a coupling function $W(\phi)$ [21],

$$A = A_G + A_M = \frac{1}{2} \int d^2x \sqrt{-g} \left( \phi R + \lambda^2 V(\phi) \right) - m \int dt W(\phi) \sqrt{-g_{\mu\nu}} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}. \quad (4)$$

The dilaton potential $V(\phi)$ depends on the form of the matter Lagrangian $\mathcal{L}_{MF}$, whereas the coupling function $W(\phi)$ reads

$$W(\phi) = (2\phi)^{-1/4}. \quad (5)$$

For static solutions in the Schwarzschild gauge,

$$ds^2 = -U(r) dr^2 + U^{-1}(r) d\tau^2, \quad (6)$$

with a source particle at rest in the origin, the field equations coming from the 2D action (4) become [21],

$$\frac{d^2 U}{dr^2} + \frac{U}{r} \frac{dU}{dr} = \lambda^2 \frac{dV}{d\phi} - \frac{2m}{\lambda^2} \frac{dW}{d\phi} \delta(r),$$

$$\frac{dU}{dr} \frac{d\phi}{dr} = \lambda^2 V,$$

$$2U \frac{d^2 \phi}{dr^2} + \frac{dU}{dr} \frac{d\phi}{dr} = \lambda^2 V - 2mW \delta(r). \quad (7)$$

In order to find the acoustic analog of the gravitational dynamics (7) we will consider separately the gravitational dynamics in vacuo, $m = 0$, and in the presence of the source $m \neq 0$. 

A. Gravitational dynamics in vacuo and fluid dynamics

In this case the problem of finding a fluid dynamics equivalent to the gravitational dynamics \( \mathbf{17} \) has been already solved in Ref. \( \mathbf{19} \). However, in that paper the dilaton has not been considered as dynamical, but rather as constrained by the equations of motion to be proportional to the spacelike coordinate \( r \) of the 2D spacetime. Here we will improve the derivation of Ref. \( \mathbf{19} \) by considering the full gravitational dynamics with both degrees of freedom \( U \) and \( \phi \).

When considering the field equations \( \mathbf{17} \) in the vacuum or, equivalently, the spacetime region \( r \neq 0 \) away from the source, we can define a scalar mass function \( M(r) \) \( \mathbf{23} \),

\[
M(r) = \frac{F_0}{2} \left( \lambda^2 \int V(\phi)d\phi - (\nabla\phi)^2 \right),
\]

which is constant by virtue of the field equations \( \mathbf{17} \), with \( F_0 \) arbitrary normalization constant. On shell we therefore have \( M(r) = M = \text{const} \), where \( M \) is the mass associated with the classical solution of the field equations under consideration.

Using the mass function \( M(r) \) we can find an equivalent form for the system \( \mathbf{17} \). Only two of the three equations in \( \mathbf{17} \) are independent and the the system can be written, for \( \phi \neq \text{const} \), as

\[
\frac{dU}{dr} \frac{d\phi}{dr} = \lambda^2 V, \quad \frac{dM(r)}{dr} = 0. \tag{9}
\]

By subtracting the third from the second equation we can also rewrite Eqs. \( \mathbf{17} \) in a second equivalent form,

\[
\frac{dU}{dr} \frac{d\phi}{dr} = \lambda^2 V, \quad \frac{d\phi^2}{dr^2} = 0. \tag{10}
\]

The existence of two equivalent forms for the equations of the gravitational dynamics can be traced back to the existence of two constant of motion: \( M \) and \( \phi_0 = (1/\lambda)d\phi/dr \). In the solution of Eqs. \( \mathbf{17} \) these constants of motion appear as integration constants,

\[
ds^2 = -\phi_0^{-2} \left( J(\phi) - \frac{2M\phi_0}{\lambda} \right) d\tau^2 + \phi_0^2 \left( J(\phi) - \frac{2M\phi_0}{\lambda} \right)^{-1} dr^2, \quad \phi = \phi_0 \lambda r, \tag{11}
\]

where \( J = \int Vd\phi \). Moreover, they are related one to the other in a geometrical way: \( d\phi/dr \) determines the Killing vector of the metric \( \mathbf{11} \), \( \chi^\mu = \epsilon^{\mu\nu} \partial_\nu \phi \), whereas \( M \) is its associated conserved charge. In general the solution \( \mathbf{11} \) describes a black hole with an event horizon at \( r = r_h \), with \( J(r_h) = 2M\phi_0/\lambda \).

The form \( \mathbf{11} \) of the gravitational field equations appears more suitable for making contact with fluid dynamics. For this reason in the following we will consider the equations for the gravitational dynamics in the form given by Eqs. \( \mathbf{17} \). With this choice the constant \( \phi_0 \) becomes irrelevant. Therefore we will set \( \phi_0 = 1 \), and correspondingly \( F_0 = 1/\lambda \) in Eq. \( \mathbf{8} \).

The gravitational metric \( \mathbf{11} \) can be related to the acoustic metric associated with the steady, cylindrically symmetric flow of a barotropic, inviscid and locally irrotational 3D fluid \( \mathbf{10} \),

\[
ds^2 = \frac{\rho_0}{c} \left[ - (c^2 - v_0^2) dt^2 - 2v_0 dx dt + dx^2 \right], \tag{12}
\]

where \( x \) is the coordinate along the flux tube, \( \bar{\rho}_0(x) \) is the density of the fluid, \( v_0(x) \) its velocity and \( c(x) = \sqrt{dP/d\bar{\rho}} \) is the velocity of sound (\( P \) is the pressure of the fluid). Owing to the cylindrical symmetry of the flux tube the fluid parameters \( \bar{\rho}, v_0, c \), the section of the flux tube and the potential for the external forces acting on the fluid are independent of the transverse coordinates \( y, z \) of the flux tube. For notational convenience, in \( \mathbf{12} \) we use the dimensionless variable \( \rho_0 = \bar{\rho}_0/\lambda^4 \). The mapping between the gravitational \( \mathbf{11} \) and acoustic \( \mathbf{12} \) metric is realized by the coordinate transformation

\[
r = \int \rho_0 dx, \quad \tau = t + \int dx \frac{v_0}{c^2 - v_0^2}, \quad J = \frac{2M}{\lambda} = \frac{\rho_0}{c} \left( c^2 - v_0^2 \right). \tag{13}
\]
Considering a flux tube of section $\bar{A}(x)$, the dynamical equations governing the motion of the fluid are the Euler and the continuity equations

$$\bar{\rho}_0 v_0 \frac{dv_0}{dx} + \frac{dP}{dx} + \bar{\rho}_0 \frac{d\psi}{dx} = 0,$$

$$\frac{dF}{dx} \equiv \frac{d}{dx}(\bar{\rho}_0 v_0 \bar{A}) = 0,$$  (14)

where $F$ is the flux of fluid mass and $\psi$ is the potential for external forces acting on the fluid.

Let us now compare the gravitational equations with the fluid-dynamical equations obtained defining the variables

$$X = \frac{\bar{\rho}_0}{c} (c^2 - v_0^2), \quad Y = \rho_0 c, \quad F = \ln \left(\frac{c}{\rho_0}\right).$$  (15)

The Euler and continuity equations take, respectively, the form

$$\frac{dX}{dr} = 2\frac{dY}{dr} - X\frac{dF}{dr} + 2e^{-F}\frac{d\psi}{dr},$$

$$\frac{d}{dr}\sqrt{A^2Y(Y - X)} = 0,$$  (16)

where we have introduced the dimensionless flux tube section $A = \lambda^2 \bar{A}$.

The fluid dynamic equations (16) are put in the gravitational form (9) by the identification

$$X = U, \quad \sqrt{A^2Y(Y - X)} = M(r)\lambda,$$  (17)

(actually, on the right hand side of the second equation one could choose any function of $M(r)$), and by introducing the constraint

$$2\frac{dY}{dr} - X\frac{dF}{dr} + 2e^{-F}\frac{d\psi}{dr} = \lambda^2 V(\phi)\frac{dr}{d\phi}.$$  (18)

In Eq. (17) the $\lambda$ factor has been introduced for dimensional reasons. Notice that the identification (17) and the constraint (18) hold both for a flux tube of constant and non-constant section.

In terms of the solutions of the gravitational field equations (17) the identifications (18) read

$$X = \left(J(\lambda r) - \frac{2M}{\lambda}\right), \quad \sqrt{A^2Y(Y - X)} = \frac{F}{\lambda^2} = \frac{M}{\lambda},$$  (19)

where now $M$ is $M(r)$ evaluated on-shell. The identification given by the second equation in (19) is very natural and has a simple physical meaning. It identifies the conserved quantity of fluid dynamics, the flux of matter fluid $F$, with the conserved quantity of the gravitational dynamics, the black hole mass $M$.

### B. Gravitational dynamics with pointlike sources and fluid dynamics

Let us now generalize the previous results to the case $m \neq 0$. Also in presence of a pointlike source the equations (17) can be rewritten in terms of $M(r)$, in a form similar to (18), as

$$\frac{dU}{dr} \frac{d\phi}{dr} = \lambda^2 V, \quad \frac{dM}{dr} = \frac{m}{\lambda} W \frac{d\phi}{dr} \delta(r).$$  (20)

Introducing the variables (19), we find that also in this case the identification (17), allows us to find the acoustic analog of the gravitational system (20). The Euler equation in (17) and the constraint (18) are unchanged by the coupling to matter, and are still equivalent to the first equation in (20). The second equation in (20) implies a modification of the continuity equation in (16), which acquires a term proportional to the parameter $m$,

$$\frac{d}{dr}(A\rho_0 v) = \frac{m}{\lambda^2} W \frac{d\phi}{dr} \delta(r).$$  (21)
This can be recognized as the continuity equation in the presence of a flux source of strength
\[
\Phi = \frac{m}{\lambda^2} \left[ W(\phi) \frac{d\phi}{dr} \right]_{r=0} \delta(r).
\] (22)

The acoustic analog of the gravitational singularity is therefore simply a source term for the flux of the fluid. Owing to the cylindrical symmetry of our flux tube the source is planar. Using the first equation in (13) one can trade in Eqs. (21),(22) the radial coordinate \( r \) for the the flux tube coordinate \( x \). In terms of the coordinate \( x \) the source is located at the position \( x = x_0 \) corresponding to \( r = 0 \) and it extends along the perpendicular \( y, z \) directions to cover the whole section of the flux tube.

In the gravitational description the position of the source is associated with a singularity of the dynamics (curvature singularity). This is not the case with the acoustic description where the delta function singularity at \( r = 0 \) appears as a pure source term for the fluid flux. The Euler equation and the constraint (13) are unaffected by the presence of the source. We see here an important difference between the gravitational and the acoustic case. In general relativity the presence of a pointlike source implies a (curvature) singularity for the gravitational field. This is not the case for the fluid where the presence of the source is completely disentangled from the appearance of a singularity in the dynamics.

The solutions of the gravitational field equations [17] have been derived in Ref. [21]. The delta function singularity at \( r = 0 \) is generated by taking the solution as function of \( |r| \),
\[
U = \frac{J(\phi)}{\sigma^2} - \gamma, \quad \phi = \sigma \lambda |r| + \beta,
\] (23)
where as usual \( J = \int d\phi V \) and \( \sigma, \beta, \gamma \) are integration constants satisfying
\[
J(\beta) - \sigma^2 \gamma = -\frac{m \sigma}{2\lambda} W(\beta), \quad V(\beta) = -\frac{m \sigma}{\lambda} \frac{dW(\beta)}{d\phi}.
\] (24)
A third equation constraining the values of the integration constants has to be added to ensure the equality between the mass \( M \) of the solution and the mass \( m \) of the source [21]:
\[
\gamma = \frac{2m}{\lambda \sigma^2}.
\] (25)
Notice that, as expected, solution (23) differs from the vacuum solution [11] only at \( r = 0 \). For \( r > 0 \) the two solutions are related by a translation of the coordinate \( r \) and a scale transformation of \( r \) and \( \tau \). One can now easily work out the coordinate transformations relating the gravitational black hole [28] to the acoustic black hole [12]. They are given by Eq. (15) with the third equation replaced by
\[
\sigma^{-2} J - \gamma = \frac{\rho_0}{c} \left( c^2 - v_a^2 \right).
\] (26)

### III. BLACK HOLE THERMODYNAMICS

The correspondence between gravitational and fluid dynamics allows us to straightforwardly define the thermodynamical parameters, \( T, M_a, S_a \) associated with the acoustic black hole, which satisfy the first principle \( dM_a = T_a dS_a \) [19]. The relevant formulas expressing the thermodynamical parameters in terms of the variables \( X, Y, F \) of Eq. (15) have been already given in Ref. [10]. However, the acoustic black hole thermodynamics developed in that paper is not completely satisfactory. The thermodynamical parameters associated with the acoustic black hole are a simple “translation” of those associated with the gravitational black hole. A physical interpretation of them, in particular of the mass \( M_a = M \), in terms of fluid parameters is difficult.

The results derived in the previous sections of this paper, in particular Eq. (19), allow us to give a more transparent “fluid-dynamical” interpretation of the thermodynamical parameters associated with the acoustic black hole. From Eq. (19) it follows that the black hole mass \( M \) is the flux of matter fluid measured in units of \( \lambda \), \( M_a = F/\lambda \). Because \( F \) (and \( M \)) are constant of motion we can evaluate them on the horizon \( r = r_h \). We have,
\[
M_a = M(r_h) = \lambda A(r_h) Y(r_h).
\] (27)

The entropy of the acoustic black hole is [13], \( S_a = 2\pi \lambda r_h \). For a flux tube of constant section the entropy is proportional to the total mass of the fluid \( M(r_h) \) inside the horizon. In fact from Eq. (13) it follows
\[
S_a = 2\pi \lambda \int_{r=0}^{r_h} \rho_0 \, dx = \frac{2\pi}{\lambda A} M(r_h).
\] (28)
The thermodynamical parameters satisfy the first principle. Making use of Eq. (27) we therefore have for the black hole temperature,

$$T_a = \frac{1}{2\pi} \frac{d M_a}{d S_a} = \frac{1}{2\pi} \frac{d F(r_h)}{d r_h}.$$  

The temperature of the acoustic black hole measures the rate of change of the flux of fluid mass when the position of the horizon is changed. For a flux tube of constant section, for which $r_h$ is proportional to the mass of the fluid inside the black hole, the temperature measures the rate of change of the flux when the mass of the fluid inside the hole is changed.

IV. SOLUTIONS OF THE CONSTRAINED FLUID DYNAMICS

In the previous sections we have shown that the gravitational dynamics governing the spherically symmetric solution of the action has an acoustic analogue given by a fluid dynamics constrained by Eq. (18). If the geometry of the flux tube and the external forces are fixed, this constraint determines the equation of state of the fluid. The physically relevant situation, which we consider here, is the opposite. One works with a fluid with a given equation of state and Eq. (18) becomes a constraint on the geometry of the flux tube and/or on the external forces acting on the fluid.

In Ref. [19] the constraint (18) has been solved for a perfect fluid. In the following we will solve the constrained fluid dynamics for a fluid with a generic power-law equation of state

$$P = \lambda \frac{\rho^n}{a^n},$$  

where $a$ and $n \neq 0$ are real dimensionless constants. This equation of state describes almost all physically interesting fluids: perfect fluid ($n = 1$), Bose-Einstein condensate ($n = 2$), Chaplygin gas ($n = -1$), fluid mechanics in $d$ spatial dimensions invariant under the nonrelativistic conformal group $SO(1, 2)$ ($n = 1 + 2/d$) [24].

In terms of the variables (15) the equation of state reads

$$Y^{3-n} e^{(n+1)F} = a^4.$$  

The particular cases $n = 1$ (perfect fluid), $n = -1$ (Chaplygin gas) $n = 3$ (conformal invariant fluid mechanics in $d$ spatial dimensions), corresponding respectively to $Ye^F = const$, $Y = const$, $F = const$ will be considered separately.

Using Eq. (31) in (the inverse of) Eqs. (15) we find the fluid velocity, the fluid density and the speed of sound

$$v_0 = a \frac{\alpha^2}{Y} \sqrt{Y - X},$$  

$$\rho_0 = a \frac{\alpha^2}{Y} Y^{2/n}, \quad c = a \frac{\alpha^2}{Y} Y^{n+1}.$$  

We will discuss separately the two cases of a flux tube of constant and non-constant section.

A. Flux tube with constant section

In this case $A = const$ and we can use the continuity equation to solve for $X = X(Y)$:

$$X = Y - \frac{\alpha^2}{Y},$$  

where $\alpha = M/\lambda$. Using Eq. (34) into Eq. (32) we can express also $v_0$ as a function of $Y$:

$$v_0 = \alpha \left( \frac{a}{Y} \right)^{\frac{2}{n+1}}.$$  

The constraint (18) can be written as

$$\frac{d X}{d Y} - 2 + X \frac{d F}{d Y} - 2e^F \frac{d \psi}{d Y} = 0,$$
which using Eq. (31) can be easily solved for $\psi(Y)$,

$$\psi = -a^{\frac{4}{n+1}} \left( \frac{\alpha^2}{2} Y^{-\frac{n+1}{n+1}} + \frac{1}{n-1} Y^{2\frac{n-1}{n+1}} \right). \quad (37)$$

The acoustic horizon forms at $Y = \alpha$, corresponding to $X = 0$. The subsonic region ($X > 0$), describing the external region of the gravitational black hole, is given by $Y > \alpha$. The source, corresponding to the black hole singularity, is located at $Y = Y_\gamma$ with $0 < Y_\gamma < \alpha$. The fluid parameters $v_0, \rho_0, c$ and the external potential $\psi$ remain finite both on the horizon and on the location of the source. In the case under consideration the constraint necessary to have a correspondence between gravitational and fluid dynamics takes the form of a constraint on the external forces acting on the fluid, given by Eq. (37). Moreover, this constraint implies a “null force condition” on the acoustic horizon. In fact the external potential $\psi$ given by Eq. (37) becomes extremal for $Y = \alpha$ (a local maximum for $n > -1$ and a local minimum for $n < -1$).

Once a specific action for the four-dimensional gravity model (1), and hence a specific $V(\phi)$ in the 2D model (3), is given, one can use Eq. (34) to find $Y$. Use of Eqs. (32), (33) and (37) allows one to get the fluid parameters as a function of $r$.

**B. Flux tube with non-constant section**

In this case $A$ is not constant and we can use the continuity equation to express it as function of $X$ and $Y$,

$$A^2 = \frac{\alpha^2}{Y(Y - X)}. \quad (38)$$

The presence of external forces is now an unnecessary complication, and therefore we set them to zero. With $\psi = \text{const}$, and using Eq. (31) the constraint (13) becomes

$$2 \frac{dY}{dX} - \frac{n - 3}{n + 1} \frac{X dY}{Y dX} = 1, \quad (39)$$

which is readily solved to give

$$X = \frac{n + 1}{n - 1} \left( Y - \omega^2 Y^{-\frac{n-3}{n+1}} \right), \quad (40)$$

where $\omega$ is an integration constant. Inserting the previous equation into Eqs. (32), (33) and (38), we get the solution of the constrained fluid dynamics. The acoustic horizon forms at $Y = Y_h = \omega^{(n+1)(n-1)}$ and the source is located at $Y = Y_\gamma$, with $0 < Y_\gamma < Y_h$. The supersonic region, corresponding to the black hole interior, is given by $Y_\gamma < Y < Y_h$. Conversely, the subsonic region (black hole exterior) is given by $Y > Y_h$. The fluid parameters $v_0, \rho_0$ and the speed of sound $c$ remain finite both at the horizon and at the location of the source.

We can also show that for $n > -1$ the acoustic horizon always forms at a minimum of the section $A(x)$, i.e. the flux tube must have the form of a so-called Laval nozzle. In fact it follows from Eq. (35) and Eq. (39) that

$$\frac{dA}{dX} = \frac{4X}{Y - X} \frac{1}{2(n + 1)Y - (n - 3)X}. \quad (41)$$

Taking into account that by definition $Y > 0$ and that Eq. (35) requires $Y > X$, one finds that for $n > -1$, $dA/dX = 0$ at the horizon ($X = 0$), whereas it is positive (negative) for $X > 0$ ($X < 0$). This is a highly non-trivial result. The constraint, which is necessary to have a correspondence between gravitational and fluid dynamics becomes a geometrical constraint on the form of the flux tube. This geometrical constraint forces the fluid to develop an acoustic horizon.

**C. Particular cases: $n = 1, -1, 3$**

The case $n = 1$ corresponding to a perfect fluid has been already discussed in Ref. [10]. For $n = -1$ we have the Chaplygin gas. The pressure is negative and the equation of state becomes $P = -\lambda a^2 / \rho_0$, which in terms of the new variables (15) reads $Y = a$. In the case of a flux tube of constant section, constant $Y$ implies, owing to the continuity equation, also constant $X$. This forbids the realization of an acoustic Chaplygin analogue of a gravitational
black hole, independently of the form of the potential $V$. This difficulty can be circumvented by considering a flux tube of non-constant section. In this case, considering for simplicity $\psi = 0$, the constraint $\frac{\omega}{b} = \frac{\alpha}{X}$ can be solved to give $\exp(-F) = bX$, with integration constant $b > 0$. The fluid parameters take therefore the form

$$\rho_0 = \sqrt{abX}, \quad v_0 = \sqrt{\frac{1}{b} \left( \frac{a}{X} - 1 \right)}, \quad c = \sqrt{\frac{a}{bX}}, \quad A^2 = \frac{\alpha^2}{a^2 - aX}. \quad (42)$$

This solution is defined only for $X \leq a$ and becomes singular on the black hole horizon $X = 0$. For $n = 3$ the equation of state implies $F = const$ and the solutions take a very simple form. In the case of a flux tube of constant section Eqs. (33), (34), (35), (37) give

$$v_0 = \alpha \left( \frac{a}{Y} \right)^{1/2}, \quad \rho_0 = \left( \frac{a}{Y} \right)^{-1/2}, \quad c = (\alpha Y)^{1/2}, \quad Y = \frac{1}{2} \left( X - \sqrt{X^2 + 4\alpha^2} \right), \quad \psi = -\frac{a}{2} \left( \frac{\alpha^2}{\sqrt{Y}} + Y \right). \quad (43)$$

In the case of a non-constant flux tube section we get instead

$$v_0 = \sqrt{a \left( d - \frac{X}{2} \right)}, \quad \rho_0 = \sqrt{\frac{1}{a} \left( d + \frac{X}{2} \right)}, \quad c = a\rho_0, \quad A = \frac{\alpha}{v_0c}. \quad (45)$$

where $d > 0$ is an integration constant related to the integration constant $\omega$ appearing in Eq. (40). It is evident from the previous formulae that $-2d \leq X \leq 2d$. The horizon is located at $X = 0$, which is also a minimum of $A$. The section of the flux tube diverges at $X = \pm 2d$. Therefore the acoustic black hole cannot describe the entire black hole spacetime.

V. CONCLUSIONS

The resolution of the singularities of general relativity relies heavily on the understanding of the short distance, quantum behavior of gravity. Nevertheless, investigations of singularities at the classical level may also help to improve our understanding of the subject. In this paper we have presented acoustic analogues of the black hole singularities generated by a pointlike source. The acoustic analogues of the singularities are constructed finding a fluid whose dynamics is completely equivalent to that governing the spherically symmetric solutions of (gauge fixed) general relativity with pointlike sources.

This result enhances the analogy between gravity and condensed matter systems. This analogy is not restricted to kinematical features of gravity such as acceleration horizons. We can use a fluid to mimic the full black hole dynamics, spacetime singularity included. Analogue models of gravity have also the nice feature of being, at least in principle, experimentally testable in laboratory. In the near future this could open the way to experimental tests of the full classical black hole structure realized using condensed matter systems.

Another important result of our investigation concerns the notion of singularity and source in classical field theory and, in particular, in general relativity. In a classical field theory, such as Maxwell electromagnetism, the (delta function) singularity associated with a pointlike source is conceptually independent from the field singularity, the point where the fields diverges. It is the classical dynamics (e.g. the Laplace equation) that forces the identification of the two types of singularities. The delta function singularity becomes the source of the Green function for the field. In general relativity this relationship between pointlike sources and field singularities becomes even deeper. The equivalence principle treats on equal footing sources and test particles, and the field singularities become geometrical (curvature singularities).

In this paper we have shown that for a classical field theory describing the motion of a fluid we may have a slightly different situation. The presence of a source for the mass of the fluid is not necessarily related to a divergence of the fields describing the fluid. It is related with a much milder form of singularity, a cusp point in the field configuration. This result is consequence of two features of the the fluid dynamics we have considered. First the effective theory which describes the fluid is two-dimensional. In two dimensions the delta-like singularity of the source can be generated by a second derivative of $|r|$ in the Green equation. Second, differently from other field theories, in fluid dynamics the source couples, through the continuity equation, to the flux of matter fluid only.

By mapping gravitational and fluid dynamics, we have also shown that the singularity of spherically symmetric black hole spacetime generated by a pointlike source can be transformed in the milder singularity of fluid dynamics described above. The responsible for this miracle is the 2D dilaton-dependent Weyl rescaling of the metric. The
price that we have to pay for this is that the Weyl transformation itself is singular at the position of the source (see eq. 23). The Weyl transformation changes the 4D black hole singularity into a 2D cusp singularity.

In this paper we have considered only spherically symmetric solutions of general relativity. It is an interesting open question if our results hold also in more general situations. It seems very difficult to mimic a gravitational dynamics with propagating degrees of freedom or a system with two or more interacting masses, using a fluid. Nevertheless a description similar to that presented in this paper may still hold for classes of solutions with high degree of symmetry, such as for instance cosmological solutions.