Masslessness of ghosts in equivariantly gauge-fixed Yang–Mills theories

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Abstract

We show that the one-loop ghost self-energy in an equivariantly gauge-fixed Yang–Mills theory vanishes at zero momentum. A ghost mass is forbidden by equivariant BRST symmetry, and our calculation confirms this explicitly. The four-ghost self interaction which appears in the equivariantly gauge-fixed Yang–Mills theory is needed in order to obtain this result.

PACS numbers: 11.15.Ha, 11.15.Bt
I. INTRODUCTION

When one considers Yang–Mills (YM) theories in the continuum, one needs to gauge fix the theory, and this is usually done in the framework of BRST symmetry. On the other hand, it is well known that non-perturbatively gauge fixing is not needed if the theory is defined on the lattice in terms of link variables which are elements of a compact gauge group \( G \). The volume of a compact group is finite, and if the theory is formulated on a finite lattice, it follows that the path integral defining the (euclidean) YM theory is finite without the need to fix a gauge. However, the possibility of gauge fixing a lattice YM theory non-perturbatively following the standard BRST procedure would lead to interesting applications. As an example, we mention the construction non-abelian chiral gauge theories on the lattice, where one recent approach uses non-perturbative gauge fixing as a key ingredient to solving the problem \([1]\).

The problem of defining a gauge-fixed lattice YM theory with BRST symmetry was considered before \([2, 3]\). In particular, it was shown that the lattice partition function of a YM theory with standard BRST symmetry vanishes identically on a finite lattice, when formulated in terms of the link variables \([3]\). This result extends trivially to the (un-normalized) expectation value of any gauge-invariant operator. The result is a “no-go” theorem saying that YM theories cannot be gauge fixed on the lattice while maintaining BRST symmetry. Since the lattice is the only presently known tool for defining YM theories non-perturbatively, this relegates the standard BRST approach to a purely perturbative construct, also on the lattice.

More recently, it has been shown that this problem can be partially solved. It was conjectured that non-abelian theories with a gauge group \( G \) can be gauge fixed non-perturbatively down to a subgroup \( H \subset G \) provided that the Euler characteristic of the coset manifold \( G/H \) does not vanish \([4]\). Ref. \([4]\) worked this out in detail for \( G = SU(2), H = U(1) \), and it was generalized in Ref. \([5]\) for \( G = SU(N) \), with \( H \) the maximal abelian subgroup \( U(1)^{N-1} \). (See Ref. \([6]\) for an alternative approach dealing only with \( G = U(1) \), which is not the subject of this note.)

In this approach, ghosts are only introduced for coset generators, and the usual BRST transformation rules are changed to maintain this constraint. As a result, the standard BRST algebra is changed into an “equivariant” BRST (eBRST) algebra, i.e. a BRST algebra in which the square of the BRST transformation is a gauge transformation in \( H \), instead of zero. The gauge-fixing action is changed into an action invariant under eBRST transformations. The gauge-fixed theory is still \( H \)-gauge invariant, and the modified BRST charge is still nilpotent on \( H \)-gauge-invariant operators, which is sufficient to guarantee unitary. In fact, correlation functions of gauge invariant operators in the original un-fixed theory are identically equal to those in the equivariantly gauge-fixed theory \([5]\).

This alternative construction of a \( G/H \) gauge-fixed lattice YM theory leads to new interactions in the gauge-fixing lagrangian. Since \( H \)-gauge invariance is maintained, one cannot choose the Lorenz gauge, because all derivatives need to be \( H \)-covariant. A renormalizable gauge is thus always non-linear, and leads to new gauge-field self-interactions. It turns out that the gauge-fixed theory also contains four-ghost self-interactions, which play a role in proving that the partition function does not vanish in the equivariant case \([4, 5]\). No ghost mass term appears in the gauge-fixed action. And indeed, one expects the ghosts to remain (perturbatively) massless, if they are to play their usual role in guaranteeing perturbative unitarity of scattering amplitudes of the massless gauge fields.
While the gauge-fixed theory is thus constructed to have all the desired properties of a
gauge theory, the way this works out in detail is not trivial, because of the presence of the new
interaction terms introduced by the equivariant gauge fixing. In Ref. [5] explicit examples
of one-loop unitarity of scattering amplitudes were worked out. These examples however
do not involve the ghost self-interaction. In this note, we show by explicit calculation that
these ghost self-interactions are needed to keep the ghost mass equal to zero to one loop.
In the standard BRST case, this is rather trivial, due to the fact that, in Lorenz gauge, a
shift symmetry on the anti-ghost prevents a ghost mass from being generated. This shift
symmetry is not present in the eBRST case, and only eBRST symmetry itself keeps the
ghost mass at zero. At one loop, a non-trivial cancellation between diagrams is required.
The one-loop calculation thus serves as a non-trivial check on the validity of the eBRST
approach, and as an example of the role of the ghost self-interaction.

While the main goal is to apply equivariant gauge fixing to the construction of a non-
perturbatively well-defined, gauge-fixed YM theory on the lattice, it can also be formulated
in the continuum. Since here we will be concerned with a perturbative investigation, we
will first deal with the continuum case, and then show that the same result is also obtained
on the lattice. We take $G = SU(N)$ and $H = U(1)^{N-1}$, the maximal abelian subgroup,
for which it was proven [5] that equivariant gauge fixing circumvents the no-go theorem
of Ref. [3]. We devote Sec. 2 to the equivariantly gauge-fixed theory in the continuum,
considering the ghost self-energy in dimensional regularization (DR). The fact that no ghost
mass is generated at one loop in DR results from the cancellation of poles in $1/(d-2)$
between individual diagrams. In Sec. 3, we extend our investigation to the lattice, showing
that quadratic divergences present in individual diagrams cancel, leading to the same result.
The final section contains our conclusions.

II. THE CONTINUUM CASE

Since only the coset $G/H$ will be fixed, we begin with splitting the gauge field

\begin{equation}
V_\mu = A_\mu + W_\mu, \quad A_\mu = A^i_\mu T^i, \quad W_\mu = W^\alpha T^\alpha.
\end{equation}

The generators $T^i$ span the algebra $\mathcal{H}$ for the subgroup $H$, while $T^\alpha$ span the coset space
$G/H$, with $G$ the algebra for the group $G$. The combined set $\{T^a\} = \{T^i, T^\alpha\}$ is normalized
by $\text{tr} (T^a T^b) = \frac{1}{2} \delta_{ab}$, and structure constants are defined by $[T^a, T^b] = if_{abc} T^c$. Indices
$i, j, k, \ldots (\alpha, \beta, \gamma, \ldots)$ will be used to indicate generators in $\mathcal{H}$ ($\mathcal{G}/\mathcal{H}$). The field strength
$F_{\mu\nu}$ is defined by $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + i[A_\mu, W_\nu]$, as usual.

Since we leave $H$ un-fixed, the gauge condition $\mathcal{F}(V)$ needs to be covariant under $H$, and
we choose

\begin{equation}
\mathcal{F}(V) = D_\mu (A) W_\mu \equiv \partial_\mu W_\mu + i[A_\mu, W_\mu].
\end{equation}

For the same reason, we introduce only $\mathcal{G}/\mathcal{H}$-valued ghost and anti-ghost fields

\begin{equation}
C = C^\alpha T^\alpha, \quad \overline{C} = \overline{C}^\alpha T^\alpha.
\end{equation}

With these ingredients, the continuum lagrangian for the equivariantly gauge-fixed YM
theory is\(^1\)

\[
\mathcal{L} = \frac{1}{2g^2} \text{tr} (F_{\mu\nu}^2) + \frac{1}{\xi g^2} \text{tr} (\mathcal{D}_\mu(A) W_\mu)^2 - 2 \text{tr} \left( \mathcal{C} \mathcal{D}_\mu(A) \mathcal{D}_\mu(A) \mathcal{C} \right) + 2 \text{tr} \left( \left[ W_\mu, \mathcal{C} \mathcal{H} \right] \left[ W_\mu, \mathcal{C} \mathcal{H} \right] \right) + i \text{tr} \left( \left( \mathcal{D}_\mu(A) \mathcal{C} \right) \left[ W_\mu, \mathcal{C} \right] + \left[ W_\mu, \mathcal{C} \right] \left( \mathcal{D}_\mu(A) \mathcal{C} \right) \right) + \xi g^2 \left( -\frac{1}{2} \text{tr} (\mathcal{C}^2 C^2) + \text{tr} (X X) - \frac{1}{4} \text{tr} (\tilde{X}^2) \right),
\]

where \(X, \bar{X} \) and \(\tilde{X} \) are defined by

\[
X \equiv \left( i C^2 \right)_{\mathcal{H}} = 2 i T^j \text{tr} (C^2 T^j),
\]

\[
\bar{X} \equiv \left( i \bar{C}^2 \right)_{\mathcal{H}} = 2 i T^j \text{tr} (\bar{C}^2 T^j),
\]

\[
\tilde{X} \equiv \left( i \{\bar{C}, C\} \right)_{\mathcal{H}} = 2 i T^j \text{tr} (\{\bar{C}, C\} T^j).
\]

This on-shell lagrangian is invariant under the on-shell eBRST algebra. For the construction of this lagrangian, and a discussion of eBRST and other symmetries, we refer to Sec. 2 of Ref. \([5]\). Here we only note that a ghost mass term is excluded by eBRST invariance, and not by any other symmetry of the gauge-fixed theory.

We will now sketch the calculation of the ghost self-energy. The relevant Feynman rules for the vertices follow from (after rescaling \(A_\mu^i \rightarrow g A_\mu^i, W_\mu^i \rightarrow g W_\mu^i\))

\[
\langle A_\mu^i(k) C^\alpha(p) \bar{C}^\beta(q) \rangle = -ig f_{i\alpha\beta}(p - q)_\mu, \quad (2.6)
\]

\[
\langle W_\mu^\rho(k) C^\alpha(p) \bar{C}^\beta(q) \rangle = -i g f_{\rho\alpha\beta}(p - q)_\mu,
\]

\[
\langle A_\mu^i(k) A_\nu^j(l) C^\alpha(p) \bar{C}^\beta(q) \rangle = 2 \left[ -\frac{1}{2} \xi g^2 (f_{i\alpha\gamma} f_{j\beta\gamma} + f_{j\alpha\gamma} f_{i\beta\gamma}) \delta_{\mu\nu} \right],
\]

\[
\langle W_\mu^\rho(k) W_\nu^\sigma(l) C^\alpha(p) \bar{C}^\beta(q) \rangle = 2 \left[ \frac{1}{2} \xi g^2 (f_{\rho\sigma i} f_{\sigma\beta i} + f_{\sigma\alpha i} f_{\rho\beta i}) \delta_{\mu\nu} \right],
\]

\[
\langle C^\alpha(p) \bar{C}^\beta(q) C^\rho(k) \bar{C}^\sigma(l) \rangle = 4 \left[ \xi g^2 \left( \frac{1}{16} f_{\alpha\gamma j} f_{\beta\gamma} + \frac{3}{16} f_{\alpha\beta i} f_{\beta i} \right) + \frac{1}{16} (f_{\alpha\beta i} f_{\alpha i} - f_{\alpha i} f_{\beta i}) \right] \delta(p + q),
\]

where on the left-hand side \(\langle \ldots \rangle\) denotes 1PI tree-level correlation functions, and the factors 2 and 4 outside the square brackets on the right-hand side are combinatorial factors. All momenta are in-going, and delta functions for momentum conservation are understood.

The ghost and \(W^i\) propagators are

\[
\langle C^\alpha(p) \bar{C}^\beta(q) \rangle = \frac{\delta_{\alpha\beta}}{p^2} \delta(p + q), \quad \langle W_\mu^\rho(p) W_\nu^\sigma(q) \rangle = \delta_{\alpha\beta} \left( \frac{\delta_{\mu\nu}}{p^2} + (\xi - 1) \frac{p_\mu p_\nu}{(p^2)^2} \right) \delta(p + q).
\]

\(^1\) This combines Eq. (2.1) and a rewriting of Eq. (2.24) in Ref. \([5]\).
In order to define the $A$ propagator, we have to gauge fix the remaining abelian group $H$ as well. We do this in Lorenz gauge, adding a term $\frac{1}{2\alpha}(\partial_{\mu}A_{\mu}^i)^2$ to the lagrangian, leading to the propagator

$$\langle A_{\mu}^i(p)A_{\mu}^j(q)\rangle = \delta_{ij} \left(\frac{\delta_{\mu\nu}}{p^2} + (\alpha - 1) \frac{p_{\mu}p_{\nu}}{(p^2)^2}\right) \delta(p + q).$$  \hfill (2.8)

No other modifications are needed in order to gauge fix $H$; in particular, no $H$-valued ghosts are needed.

The addition of $\frac{1}{2\alpha}(\partial_{\mu}A_{\mu}^i)^2$ to the lagrangian breaks eBRST symmetry. Reference 5 discussed how the Slavnov–Taylor identities are modified as a consequence, and also how these identities can still be used to prove perturbative unitarity of the theory after the two steps of $G/H$ and $H$ gauge fixing. Here we are only interested in whether a non-vanishing ghost mass is generated through radiative corrections, and the relevant observations are rather simple.

On the lattice no gauge fixing of the abelian subgroup $H$ is required, and eBRST symmetry does indeed preclude a non-zero ghost mass term from appearing in the $G/H$ gauge-fixed theory 1. 5. The equivariantly gauge-fixed lattice partition function is finite and gauge invariant under $H$, and the ghost fields are just a set of minimally-coupled “matter” fields from the point of view of $H$ symmetry. Once the $H$ symmetry is also fixed in order to set up perturbation theory, the pole mass of the ghost propagator has to be $H$-gauge independent. A proof of this can be given by adapting a similar proof of the gauge independence of the perturbative pole mass of quarks in QCD given in Ref. 6. Combining these two facts leads to the conclusion that the ghost mass vanishes to all orders in perturbation theory, independent of the gauge parameter $\alpha$. (For a very similar argument about the gauge independence of $S$-matrix elements in supersymmetric gauge theories in the Wess–Zumino gauge, see Ref. 6. 7.) Our explicit calculation of the ghost self-energy at one loop will serve to confirm these general observations.

At one loop, there are five diagrams contributing to the ghost self-energy. Two “sunset” diagrams involve the three-point vertices in Eq. (2.6), and have either an internal $A$ or $W$ line in addition to an internal ghost line. There are three tadpole diagrams involving the four-point vertices in Eq. (2.6), with an internal $A$, $W$ or ghost line. The sunset diagram with an $A$ propagator is

$$\Sigma_{ACC}^A(p) = g^2 f_{\alpha\gamma i} f_{\beta\gamma i} \int \frac{d^d k}{(2\pi)^d} \frac{(2p - k)_\mu(2p - k)_\nu}{(k - p)^2} \times \frac{1}{k^2} \left(\delta_{\mu\nu} + (\alpha - 1) \frac{k_\mu k_\nu}{k^2}\right).$$  \hfill (2.9)

The quadratic divergence present in this diagram, which in $d$ dimensions shows up as a pole in $1/(d - 2)$, can be made more explicit by writing

$$\Sigma_{ACC}^{pole}(p) = \alpha g^2 f_{\alpha\gamma i} f_{\beta\gamma i} I(p),$$ \hfill (2.10)

$$I(p) = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{2(1 - x)(k^2)^2}{(k^2 + \Delta)^2},$$

where $\Delta = x(1 - x)p^2$. The $A$ tadpole is

$$\Sigma_{A\bar{C}C}^A(p) = -(d + \alpha - 1) g^2 f_{\alpha\gamma i} f_{\beta\gamma i} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}. $$  \hfill (2.11)
By inserting a factor $[k^2(k-p)^2]/[k^2(k-p)^2]$ into the integrand, the $d=2$ pole part can be written as

$$\Sigma_{A^2\overline{C}}^{\text{pole}}(p) = -(d + \alpha - 1)g^2 f_{\alpha\gamma i} f_{\beta\gamma i} I(p),$$

and we see that the $\alpha$-dependent part already cancels between the two diagrams containing internal $A$ lines.

A very similar calculation leads to the following contribution to the $d=2$ pole part from the two diagrams containing $W$ lines:

$$\Sigma_{W^2\overline{C}}^{\text{pole}}(p) + \Sigma_{W^2\overline{C}}^{\text{pole}}(p) = g^2 \left( \frac{1}{4} \xi f_{\alpha\gamma\rho} f_{\beta\gamma\rho} + (d + \xi - 1) f_{\alpha\gamma i} f_{\beta\gamma i} \right) I(p).$$

Finally, the contribution of the ghost tadpole is

$$\Sigma_{(\overline{C}C)^2}^{\text{pole}}(p) = -\xi \left( \frac{1}{4} f_{\alpha\gamma\rho} f_{\beta\gamma\rho} + f_{\alpha\gamma i} f_{\beta\gamma i} \right) g^2 I(p).$$

Clearly, all contributions to the pole near $d = 2$ cancel, thus proving that no ghost mass is generated at one loop.

### III. THE LATTICE CASE

In this section, we repeat the one-loop calculation of the previous section, but we use the lattice instead of DR as a regulator. There are two reasons for doing this: First, equivariant gauge fixing was designed to construct a lattice gauge-fixed gauge theory with a form of BRST symmetry. Second, since the lattice regulator involves a physical cutoff $\propto 1/a$, with $a$ the lattice spacing, the fact that no ghost mass is generated follows directly from the vanishing of the ghost self-energy at zero momentum. Unlike in DR, there is no need to go to two dimensions to see the cancellation. We will only highlight the differences between the lattice and continuum calculations.

The ghost part of the lattice lagrangian is

$$\mathcal{L}_{\text{ghost}} = 2 \text{tr} \left( [T^i, D^+ D^-] \right) U_{x,\mu} U^+_{x,\mu} D^+ \overline{C} \right)$$

$$- i \text{tr} \left( 2 \left[D^+ D^- \overline{C} \right] \mathcal{W}_{x,\mu} - \mathcal{W}_{x,\mu} D^+ \overline{C} \right) + \xi g^2 \left( \frac{1}{2} \text{tr} (\overline{C}^2 C^2) + \text{tr}(\nabla X) - \frac{1}{4} \text{tr}(\nabla^2 X) \right),$$

in which

$$D^+ \Phi_x = U_{x,\mu} \Phi_{x+\mu} U^+_{x,\mu} - \Phi_x.$$

We note that this covariant difference can be split into a (lattice) derivative term $\partial^+ \Phi_x = \Phi_{x+\mu} - \Phi_x$ and a term at least quadratic in the fields if one expands $U_{x,\mu} = \exp(iV_{x,\mu})$:

$$D^+ \Phi_x = \partial^+ \Phi_x + (U_{x,\mu} \Phi_{x+\mu} U^+_{x,\mu} - \Phi_{x+\mu})$$

$$= \partial^+ \Phi_x + i[V_{x,\mu}, \Phi_{x+\mu}] + \ldots.$$
are only interested in the ghost self-energy at zero momentum, four-point vertices should not involve any (lattice) derivatives of ghost or anti-ghost fields, and three-point vertices should involve at most one derivative of a ghost or anti-ghost field. This implies in particular that we may replace $U_{x,\mu} T^i U_{x,\mu}^\dagger \rightarrow T^i$ in the first term of Eq. (3.1), because vertices coming from expanding the link variables in this expression would either be three-point vertices involving both derivatives on $C$ and on $\overline{C}$, or four-point vertices involving at least one such derivative. It follows that the list of relevant vertices is similar to that in Eq. (2.6). The ghost, $W$ and $A$ propagators are as given in Eqs. (2.7,2.8), but with $k_\mu$ replaced by $\hat{k}_\mu \equiv 2 \sin (k_\mu/2)$.

At zero momentum, it follows that one finds contributions to the ghost self-energy in one-to-one correspondence to the continuum contributions discussed in the previous section. The results for the various contributions are those given in Eqs. (2.10,2.12,2.13,2.14), if one makes the replacements $d \rightarrow 4$, and

$$I(p) \rightarrow \frac{1}{a^2} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^2} \frac{1}{\hat{k}^2},$$

where $\hat{k}^2 = 4 \sum_\mu \sin^2 (k_\mu/2)$. Adding up the contributions, we find that $\Sigma(p = 0) = 0$, and we conclude that also on the lattice no ghost mass is generated at one loop, consistent with the DR calculation. At non-zero momentum the calculation of the ghost self-energy is more complicated than in the continuum, because for non-zero momentum there are one-loop diagrams involving “lattice-artifact” vertices (irrelevant vertices of order $a^n$ with $n \geq 1$) which contribute to the ghost self-energy.

**IV. CONCLUSION**

We have shown by explicit calculation that no ghost mass is generated in an equivariantly gauge-fixed YM theory for the case that the subgroup $H$ is chosen to be abelian. This is expected, because eBRST symmetry forbids such a term in the lagrangian. Nevertheless, the explicit calculation is interesting for two reasons. First, it provides an example of the role of the ghost self-interaction which necessarily appears in the equivariantly gauge-fixed theory \[4,5\]. Second, in order to investigate the equivariantly gauge-fixed theory perturbatively, a further gauge fixing of the subgroup $H$ is required. Since this further gauge fixing should not change particle masses in the equivariantly gauge-fixed theory, one expects that masses are not affected by this additional gauge fixing, even if it leads to a modification of the Slavnov–Taylor identities for eBRST symmetry \[5\]. This general argument is confirmed by our result.

We would like to comment on the extension of our results to higher orders in perturbation theory. In Sec. 2, we already gave a general argument as to why one expects no ghost mass to be generated at any order in perturbation theory. The explicit extension of the lattice perturbative calculation to higher loops involves some technicalities. The Lorenz gauge employed here to gauge fix the abelian symmetry $H$ is non-linear on the lattice, because the relation between $A_\mu$ and $U_\mu$ is non-linear. Thus, even though $H$ is abelian, the Lorenz gauge on the lattice requires another set of $H$-valued ghosts in order to maintain BRST invariance for $H$ in lattice perturbation theory. (The no-go theorem of Ref. \[3\] is not relevant here, just as it is not relevant when setting up standard perturbation theory for YM theory.) No such $H$-ghosts are required in the continuum of course, so this is a pure lattice artifact. The resulting new vertices on the lattice therefore correspond to irrelevant operators. These
new vertices do not show up in the one-loop diagrams contributing to the self-energy of the $\mathcal{G}/\mathcal{H}$-valued ghosts, which is why they did not appear in our lattice calculation of the self-energy in Sec. 3.

Acknowledgements

We thank Yigal Shamir for useful discussions and comments. MG was supported in part by the US Department of Energy, and LZ was supported by the National Science Foundation funded GK-12 fellowship program under grant number DGE-0337949.