Non-local Wess-Zumino Model on Nilpotent Noncommutative Superspace

Yoshishige Kobayashi* and Shin Sasaki†

Department of Physics, Faculty of Science,
Tokyo Metropolitan University,
1-1 Minami-osawa, Hachioji-shi,
Tokyo 192–0397, Japan

Abstract

We investigate the theory of the bosonic-fermionic noncommutativity, \([x^\mu, \theta^\alpha] = i \lambda^{\mu\alpha}\), and the Wess-Zumino model deformed by the noncommutativity. Such noncommutativity links well-known space-time noncommutativity to superspace non-anticommutativity. The deformation has the nilpotency. We can explicitly evaluate noncommutative effect in terms of new interactions between component fields. The interaction terms that have Grassmann couplings are induced. The noncommutativity does completely break full \(\mathcal{N} = 1\) supersymmetry to \(\mathcal{N} = 0\) theory in Minkowski signature. Similar to the space-time noncommutivity, this theory has higher derivative terms and becomes non-local theory. However this non-locality is milder than the space-time noncommutative field theory. Due to the nilpotent feature of the coupling constants, we find that there are only finite number of Feynman diagrams that give noncommutative corrections at each loop order.

*E-mail : yosh@phys.metro-u.ac.jp
†E-mail : shin-s@phys.metro-u.ac.jp
1 Introduction

The noncommutative field theory was originally suggested by Snyder [2] from the view point of regularizing ultraviolet (UV) divergence of ordinary quantum field theory (QFT). In recent years, quantum field theories on noncommutative (NC) space-time have been studied, since the relation to string theory was discovered [1].

In the string framework, noncommutativity among space-time coordinates

$$[x^\mu , x^\nu ] = i\theta ^{\mu \nu }$$

is realized on the D-brane world volume when there exists NS-NS B-field backgrounds [1]. This noncommutativity can be represented by Moyal-Weyl star product $$\star = \exp [\frac{i}{2} \theta ^{\mu \nu } \partial _{\mu } \partial _{\nu } ]$$ which is used to construct NC QFT on ordinary (commutative) space. NC QFT essentially contains infinite number of differential operators, therefore, becomes non-local theory. The quantization of the NC field theory in terms of higher derivative theory was studied in the literature [6]. Many interesting feature in this field has been investigated: UV/IR mixing, noncommutative instanton and relation to string and matrix theory, for example.

On the other hand, supersymmetric extension of ordinary field theories is itself very important for many reasons. Usually, supersymmetry (SUSY) makes a theory very tractable and sometimes completely solvable. Four-dimensional $$\mathcal{N} = 1$$ SUSY theory is the most efficiently formulated by use of the superspace $$z^M = (x^m , \theta ^{\alpha } , \bar{\theta } ^{\dot{\alpha } } )$$ and superfields on it.

Recently, non(anti)commutativity of fermionic coordinates

$$\{ \theta ^{\alpha } , \theta ^{\beta } \} = C^{\alpha \beta }$$

in four-dimensional Euclidean $$\mathcal{N} = 1$$ superspace was considered by turning on constant Ramond-Ramond five-form field strength $$F ^{\alpha \beta }$$ in Calabi-Yau compactification with wrapped D-brane [3, 4, 5]. Corresponding to this non(anti)commutativity, Seiberg considered Moyal-Weyl star product among superfields

$$f (\theta ) \star g (\theta ) = f (\theta ) \exp \left[ - \frac{C^{\alpha } }{2} \frac{\bar{\partial } }{\theta ^{\alpha } } \frac{\bar{\partial } }{\theta ^{\beta } } \right] g (\theta ).$$

This corresponds to Q-deformation (non supersymmetric deformation) of SUSY field theory [9]. It breaks half of original SUSY and gives $$\mathcal{N} = 1/2$$ SUSY in four-dimensional D-brane world volume. This star product is nilpotent thanks to their Grassmann property and adds finite number of terms to the ordinary Lagrangian. Because it has only fermionic differential, the additional component terms do not contain any extra space-time derivative. This is the difference between superspace non(anti)commutativity and space-time one. The latter case is inevitably non-local. Many aspects of this non(anti)commutative (NAC) supersymmetric field theory was studied [7].

In this paper, we propose another type of deformation such as bosonic-fermionic noncommutativity

$$[x^\mu , \theta ^{\alpha } ] = i\lambda ^{\mu \alpha }.$$  

1
This noncommutativity is the middle point of boson-boson (space-time) and fermion-fermion (superspace) noncommutativity. The QFT on this space links usual space-time noncommutative QFT to NAC supersymmetric theory. The Moyal-Weyl star product corresponding to this noncommutativity is also nilpotent and gives finite number of corrections to ordinary supersymmetric theories. Because of the space-time derivative in the star product the theory has higher derivative terms and becomes non-local, but the effect is milder than usual NC field theory. We find that this deformation breaks all of SUSY in Minkowski signature. A notable feature of this theory is that new interaction has Grassmann coupling. Therefore, induced vertices have nilpotent property in the perturbative calculus.

This paper is organized as follows. In section 2 we review the NAC $\mathcal{N} = 1/2$ theory in the framework of Seiberg’s work which is prerequisite to understand our work. In section 3 we introduce our setup and give explicit form of the Moyal-Weyl product. In section 4 we show that noncommutative deformation of the Wess-Zumino Lagrangian gives finite number of additional terms, only up to fourth order differential. In section 5 we discuss the quantum aspects of this deformed Wess-Zumino model. In section 6 we discuss the geometric origin of this noncommutativity, section 7 is devoted to the discussion.

2 Non(anti)commutative $\mathcal{N} = 1/2$ supersymmetric theory

In [3], Seiberg considered $\mathcal{N} = 1/2$ supersymmetric theory based on the non(anti)commutative setup

$$\{\theta^\alpha, \theta^\beta\} = C_{\alpha\beta}, \quad \{\bar{\theta}^\dot{\alpha}, \bar{\theta}^\dot{\beta}\} = 0.$$  \hspace{1cm} (5)

This setup is possible in the Euclidean space-time. In that case $\theta$ and $\bar{\theta}$ are independent variables. Corresponding to this relation, we use Moyal-Weyl star product among superfields. In [3] he considered a simple situation such that chiral coordinates $y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha}$ do commute with all of coordinates, i.e.

$$[y^\mu, y^\nu] = [y^\mu, \theta^\alpha] = [y^\mu, \bar{\theta}^\dot{\alpha}] = 0.$$  \hspace{1cm} (6)

Though these relations imply space-time noncommutativity $[x^\mu, x^\nu] \neq 0$. This is because original world sheet coordinates $x^\mu$ are affected by the graviphoton background field (vertex operator), while chiral coordinates $y^\mu$ is free from introduced graviphoton background.

In the chiral base, the Moyal-Weyl star product can be written by supercharge $Q_\alpha$, namely

$$\star = \exp \left[ -\frac{C_{\alpha\beta}}{2} \overleftrightarrow{Q_\alpha Q_\beta} \right].$$  \hspace{1cm} (7)

This star product gives non-supersymmetric deformation and $\mathcal{N} = 1/2$ SUSY is broken to $\mathcal{N} = 1/2$ due to the relation $\{Q_\alpha, \overrightarrow{Q}\beta\} \neq 0$. On the other hand, by the help of the properties $\{D_\alpha, Q_\beta\} = \{\overrightarrow{D}_\dot{\alpha}, \overrightarrow{Q}_\beta\} = \{D_\alpha, \overleftarrow{Q}_\beta\} = \{\overrightarrow{D}_\dot{\alpha}, \overleftarrow{Q}_\beta\} = 0$, we can easily construct the deformed
version of supersymmetric model preserving chirality. From these facts, Seiberg showed the non(anti)commutative effect gives only one induced interaction term to the original Wess-Zumino Lagrangian:

\[
\mathcal{L} = \int d^4 \theta \overline{\Phi} \star \Phi + \int d^2 \theta \left( \frac{1}{2} m \Phi \star \Phi + \frac{1}{3} g \Phi \star \Phi \star \Phi \right) + \int d^2 \theta \left( \frac{1}{2} \overline{m} \overline{\Phi} \star \overline{\Phi} + \frac{1}{3} \overline{g} \overline{\Phi} \star \overline{\Phi} \star \overline{\Phi} \right) - \frac{1}{3} g \det C F^3 + \text{(total derivative)}. (8)
\]

This apparently preserves $Q$ SUSY but does not $\overline{Q}$. We need to consider only the deformation part to grasp the physical meanings of this non(anti)commutative $\mathcal{N} = 1/2$ supersymmetric theory.

The $\mathcal{N} = 1/2$ supersymmetric Yang-Mills theory can also be constructed in the same manner, i.e. by replacing each product with the star product:

\[
W_\alpha = -\frac{1}{4} DDe^{-V} D_\alpha e^V, \quad \overline{W}_\dot{\alpha} = \frac{1}{4} DDe^V \overline{D}_\dot{\alpha} e^{-V}. (9)
\]

The gauge transformation is also achieved by the star product and Seiberg found the field redefinition that allows component fields to transform in the canonical way. The deformed Lagrangian of the gauge fields in the modified Wess-Zumino gauge is

\[
\int d^2 \theta \text{Tr} W \star W = \int d^2 \theta \text{Tr} W W - i C^{\mu\nu} \text{Tr} F_{\mu\nu} \overline{\lambda} \lambda + \frac{|C|^2}{4} \text{Tr}(\overline{\lambda} \lambda)^2, \\
\int d^2 \theta \text{Tr} \overline{W} \star \overline{W} = \int d^2 \theta \text{Tr} \overline{W} \overline{W} - i C^{\mu\nu} \text{Tr} F_{\mu\nu} \lambda \overline{\lambda} + \frac{|C|^2}{4} \text{Tr}(\lambda \overline{\lambda})^2 + \text{(total derivative)}. 
\]

3 Bosonic-Fermionic noncommutativity

In this section, we consider mixed noncommutativity between bosonic and fermionic coordinates in the $\mathcal{N} = 1$ superspace,

\[
[x^\mu, \theta^\alpha] = i \lambda^{\mu\alpha}. (10)
\]

Note that the way of noncommutative deformation of the Lagrangian depends on which signature we choose, namely, Minkowski or Euclidean. Because in the Euclidean space we can choose $\theta$ and $\bar{\theta}$ independently but in the Minkowski space they are related with each other by Hermitian conjugate. When we are in the Minkowski space the relation (10) induces $[x^\mu, \bar{\theta}^\alpha] = i \lambda^{\mu\dot{\alpha}}$ by Hermitian conjugation. On the other hand, the Euclidean deformation can keep “conjugation” piece commutative in spite of noncommutative relation (10).

\[\text{1} \text{The Minkowski formulations of NAC superspace can be found in [10] [11].}\]
In the Minkowski situation we should carefully formulate the deformation of the theory. We require that chiral coordinates \( y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_\alpha \bar{\theta}^\beta \) to be commutative; \([y^\mu, y^\nu] = [y^\mu, \bar{\theta}^\beta] = 0\), other fermionic coordinates hold Grassmann property \( \{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}^\beta, \bar{\theta}^\alpha\} = 0\). The corresponding Moyal-Weyl star product is written as

\[
 f(y, \theta) \star g(y, \theta) = f(y, \theta) \exp \left[ \frac{i}{2} \lambda^{\mu\alpha} \left( \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial \theta^\alpha} - \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial y^\mu} \right) \right] g(y, \theta). \tag{11}
\]

Our notation and convention are standard \cite{17} and the other useful formulae are available in appendix. We adopt non-Hermitian star product, i.e. \( (\star) = \bar{\star} \). The \( \star \) is formal Hermite conjugate of \( \star \) product

\[
 \bar{\star} = \exp \left[ \frac{i}{2} \lambda^{\mu\alpha} \left( \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial \theta^\alpha} - \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial y^\mu} \right) \right], \tag{12}
\]

and has property \( (f(y, \theta) \star g(y, \theta))^\dagger = \bar{g}(\bar{y}, \bar{\theta}) \bar{\star} \bar{f}(\bar{y}, \bar{\theta}) \) \cite{10}. We should stress that, the star product \( \star \) breaks \( Q \)-SUSY but preserves \( \bar{Q} \)-SUSY while the conjugate star product \( \bar{\star} \) breaks \( Q \)-SUSY but preserves \( \bar{Q} \)-SUSY.

Alternative suggestion was proposed. In \cite{11} they used Hermitian star product \( \star = \star^\dagger \). But their star product is not associative and needs some reordering of the fields. Here, we require associativity of product. This formulation is natural since non-associative deformation may be related to the curved background setup \cite{13} and in that case the effective world volume theory is described by the Kontsevich type deformation \cite{14}.

Explicit expansion of our star product is

\[
 \star = \exp \left[ \frac{i}{2} \lambda^{\mu\alpha} \left( \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial \theta^\alpha} - \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial y^\mu} \right) \right] = 1 + \frac{i}{2} \lambda^{\mu\alpha} \left( \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial \theta^\alpha} - \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial y^\mu} \right) + \frac{1}{8} \lambda^{\mu\alpha} \lambda^{\nu\beta} \left( \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} + 2 \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\gamma} \frac{\partial}{\partial \theta^\beta} \frac{\partial}{\partial y^\nu} + \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \frac{\partial}{\partial \theta^\gamma} \frac{\partial}{\partial y^\nu} \right) + \frac{i}{16} \lambda^{\mu\alpha} \lambda^{\nu\beta} \lambda^{\rho\gamma} \left( \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \frac{\partial}{\partial \theta^\gamma} \frac{\partial}{\partial \theta^\delta} \frac{\partial}{\partial \theta^\epsilon} \frac{\partial}{\partial \theta^\sigma} \frac{\partial}{\partial y^\nu} \right) + \frac{1}{64} \lambda^{\mu\alpha} \lambda^{\nu\beta} \lambda^{\rho\gamma} \lambda^{\sigma\delta} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \frac{\partial}{\partial \theta^\gamma} \frac{\partial}{\partial \theta^\delta} \frac{\partial}{\partial \theta^\epsilon} \frac{\partial}{\partial \theta^\sigma} \frac{\partial}{\partial y^\nu}. \tag{13}
\]

Here we denote second derivatives as \( \frac{\partial^2}{\partial y^\mu \partial y^\nu} \equiv \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} \) and \( \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \equiv \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \). From this we see the relation

\[
 [y^\mu, \theta^\alpha]_\star = i\lambda^{\mu\alpha}, \quad [y^\mu, \bar{\theta}^\beta]_\star = 0, \quad [\bar{y}^\mu, \theta^\alpha]_\star = i\lambda^{\mu\alpha},
\]

\[
 [y^\mu, y^\nu]_\star = 0, \quad [y^\mu, \bar{\theta}^\beta]_\star = 0. \tag{14}
\]
and
\[ [y^\mu, \bar{y}^\nu]_* = 2\lambda^{\mu\alpha} (\sigma^\nu)_{\alpha\beta} \bar{\theta}^\beta, \]
\[ [x^\mu, x^\nu]_* = \lambda^{\mu\alpha} (\sigma^\nu)_{\alpha\beta} \bar{\theta}^\beta - \lambda^{\nu\alpha} (\sigma^\mu)_{\alpha\beta} \bar{\theta}^\beta. \]  
(15)

The Hermitian conjugate of these relations leads to consistent results — for example \([\bar{y}^\mu, \bar{\theta}^\alpha]_* = i\bar{\lambda}^{\mu\dot{\alpha}}.\]

Contrary to the Minkowski case, in our case we need only one star product \(\star\) to formulate the deformation theory in the Euclidean case because we don’t require the Hermite Lagrangian.

4 Wess-Zumino model on the noncommutative superspace

We can treat the chiral superfields appropriately, because our star product is the Q-deformation which preserves chirality, i.e. the products of chiral superfields are again chiral superfields. The chiral superfield \(\Phi\) is defined by the ordinary condition \(\overline{D}_\alpha \Phi = 0\) and the anti-chiral superfield \(\overline{\Phi}\) is \(D_\alpha \overline{\Phi} = 0\). The general solution of this constraint for the chiral superfield is the function of \(y^m\) and \(\theta^\alpha\),

\[ \Phi(y, \theta) = A(y) + \sqrt{2} \theta^\alpha \psi^\alpha(y) + \theta^2 F(y). \]  
(16)

One may consider that equation (16) should be expanded by using the star product. But in our case the star product doesn’t introduce \(\bar{\theta}\) so we can redefine the component fields after explicit use of equation (13). The anti-chiral superfield expansion is Hermitian conjugate of the chiral superfield expansion (16) in the Minkowski space.

4.1 Lagrangian in the Euclidean space

In the Euclidean space, we can consider the noncommutative relation only in the chiral sector
\[ [y^\mu, \theta^\alpha]_* = i\lambda^{\mu\alpha}, \]
\[ [\bar{y}^\mu, \bar{\theta}^\alpha] = 0. \]  
(17)

The Wess-Zumino action which is deformed on this noncommutative superspace is
\[ \mathcal{L} = \int d^4\theta \overline{\Phi} \star \Phi + \int d^2\theta \left( \frac{m}{2} \Phi \star \Phi + \frac{g}{3} \Phi \star \Phi \star \Phi \right) + \int d^2\bar{\theta} \left( \frac{\bar{m}}{2} \overline{\Phi} \star \overline{\Phi} + \frac{\bar{g}}{3} \overline{\Phi} \star \overline{\Phi} \star \overline{\Phi} \right) \]  
(18)

After expanding the Moyal-Weyl products and doing partial integrations, we find the component action of the deformed theory;
\[ S = \int d^4x \mathcal{L}_0 + \frac{i}{\sqrt{2}} \lambda^{\mu\alpha} \psi^\alpha (\partial_\mu \overline{A}) + \frac{1}{\sqrt{2}} \lambda^{\mu\alpha} F (\sigma^\nu)_{\alpha\beta} (\partial_\mu \partial_\nu \overline{\psi}^\beta) - \frac{1}{2} \lambda^{\mu\nu} F (\partial_\mu \partial_\nu \overline{A}) + \frac{\bar{g}}{2} \lambda^{\mu\nu} \overline{\lambda}^{\nu\beta} F (\partial_\mu \psi^\beta) (\partial_\nu \psi^\alpha) + g \lambda^{\mu\nu} F (\partial_\mu F) (\partial_\nu A) + \frac{\bar{g}}{3} \lambda^{\mu\nu\rho} F (\partial_\mu \partial_\nu F) (\partial_\rho A) + \frac{\bar{g}}{3} \lambda^{\mu\nu} \left[ -(\partial_\mu \overline{A}) (\partial_\nu A) (\overline{\square} A) + (\partial_\nu \overline{A}) (\partial_\rho A) (\overline{\partial} \rho \overline{A}) \right]. \]  
(19)
Here $\mathcal{L}_0$ is the undeformed (original) Wess-Zumino Lagrangian. Because of the spacetime-superspace-mixed noncommutativity, the higher derivative terms are induced. In this Euclidean setup the deformation breaks Hermicity. In the next section we will show that Hermite deformation can be obtained in the Minkowski space.

### 4.2 Lagrangian in the Minkowski space

Noncommutative deformed Minkowski Wess-Zumino model is also obtained by replacing the ordinary products with the Moyal-Weyl star products. But in this case we should be careful because we want to keep the theory Hermite.

The deformation of the interaction parts is straightforward. We can simply replace ordinary product with star

$$\frac{m}{2} \Phi \Phi + \frac{g}{3} \Phi \Phi \Phi + \text{(h.c.)} \rightarrow \frac{m}{2} \Phi \star \Phi + \frac{g}{3} \Phi \star \Phi \star \Phi + \text{(h.c.)}. \quad (20)$$

Here (h.c.) in the RHS represents Hermitian conjugated parts, namely, $\bar{m} \Phi \star \bar{\Phi}$ and $\frac{g}{3} \Phi \star \Phi \star \Phi$.

The kinetic term is little bit different. Since such term contains both chiral and anti-chiral superfield. We simply choose the kinetic term as $\frac{1}{2} \Phi \star \Phi + \frac{1}{2} \bar{\Phi} \star \bar{\Phi}$. Due to the property $(\Phi \star \Phi)^\dagger = \bar{\Phi} \star \bar{\Phi}$, the kinetic term is Hermite. The resulting deformed Wess-Zumino action is

$$\mathcal{L} = \frac{1}{2} \int d^4 \theta \bar{\Phi} \star \Phi + \frac{1}{2} \int d^4 \bar{\theta} \bar{\Phi} \star \bar{\Phi}$$

$$+ \int d^2 \theta \left( \frac{m}{2} \Phi \star \Phi + \frac{g}{3} \Phi \star \Phi \star \Phi \right) + \int d^2 \bar{\theta} \left( \frac{m}{2} \bar{\Phi} \star \bar{\Phi} + \frac{g}{3} \bar{\Phi} \star \bar{\Phi} \star \bar{\Phi} \right). \quad (21)$$

The explicit calculation of the star product and the Grassmann integration show the component Lagrangian of the deformed Wess-Zumino model;

$$S = \int d^4 x \mathcal{L}_0 + \int d^4 x \left[ \lambda^{\alpha \beta} \chi^{\mu \nu} \left( \chi_\alpha (\partial_\mu \psi_\beta)(\partial_\nu F) + \frac{1}{2} F(\partial_\mu \psi_\beta)(\partial_\nu \psi_\alpha) \right) 
+ \lambda^{\alpha \beta} (F \psi_\alpha (\partial_\mu \partial_\nu \psi_\alpha) - A F(\partial_\mu \partial_\nu F) - A(\partial_\mu F)(\partial_\nu F) - F^2(\partial_\mu \partial_\nu A)) 
+ \lambda^{\mu \nu \rho \sigma} F(\partial_\mu \partial_\nu F)(\partial_\rho \partial_\sigma F) \right] + \text{[h.c.]}. \quad (22)$$

Here $\mathcal{L}_0$ is the undeformed original Wess-Zumino action in the Minkowski space;

$$\mathcal{L}_0 = \bar{A} \square A + i \partial_\mu \bar{\psi}_\alpha (\bar{\sigma}_\mu)^{\hat{\alpha} \hat{\beta}} \psi_\beta + \bar{F} F$$

$$+ \left[ m(A F - \frac{1}{2} \psi \psi) + g(A^2 F - \psi \psi A) + \text{[h.c.]} \right]. \quad (23)$$

But this is not necessary because free part of Lagrangian is not affected by noncommutativity in the integral by general property of star product. Here, we define component expansion of anti-chiral superfield as formal complex conjugate of chiral one.
The generalized (including higher derivative terms) equation of motion \[15\] can be applied. The equations of motion of the auxiliary fields are

\[ F = \bar{m} \bar{A} + \bar{g} \bar{A}^2 \]

\[ + \bar{g} \lambda^{\mu\nu} \bar{F} \partial_\mu \partial_\nu \bar{A} - \frac{1}{2} \bar{g} \bar{\lambda}^{\alpha\beta} \bar{\lambda}^{\nu\beta} (\partial_\mu \bar{\psi}_\beta \partial_\nu \bar{\psi}_\alpha) \]

\[ - \frac{1}{3} \bar{g} \bar{\lambda}^{\mu\nu\rho\sigma} \left\{ (\partial_\mu \partial_\nu \bar{F}) (\partial_\rho \partial_\sigma \bar{F}) + 2 \partial_\mu F \partial_\rho F \right\}, \] \hspace{1cm} (24)

and Hermitian conjugate one.

Eq. (24) looks unusual because the auxiliary field \( \bar{F} \) couples to other components or even to the derivatives of itself. But in fact eq. (24) can be solved for \( F \) and \( \bar{F} \) exactly as follows: Using the conjugate equation of (24) to eliminate \( \bar{F} \) in eq. (24), one obtains the expression of \( F \) in terms of other dynamical fields and \( F \) itself. After that, replacing \( F \) in the RHS of (24) with the expression of the whole the RHS of \( F \) – like the successive approximation. This recursive replacement is done iteratively but will stop thanks to the nilpotency of couplings \( \lambda \), since every \( F \) in the RHS appears with \( \lambda \). In the end one get \( F \) written by other dynamical components only. It is not difficult to see that no auxiliary fields become dynamical by the deformation, in other words, new degrees of freedom on shell never appear.

Let us check the SUSY transformation of the action. The undeformed part is invariant under the \( Q \)-supersymmetry transformation

\[ \delta_\xi A = \sqrt{2} \xi \psi, \]

\[ \delta_\xi \psi = \sqrt{2} \xi F, \]

\[ \delta_\xi F = 0. \] \hspace{1cm} (25)

The \( \lambda \)-dependent parts \( \mathcal{L}_\lambda \) in (22) is also invariant under the \( Q \)-supersymmetry transformation

\[ \frac{3}{g} \delta \mathcal{L}_\lambda = \sqrt{2} \lambda^{\mu\alpha} \lambda^{\nu\beta} \left( \xi_\alpha F \partial_\mu \psi_\beta \partial_\nu F + \psi_\alpha \partial_\mu \xi_\beta F \partial_\nu F + \frac{1}{2} F \partial_\mu \xi_\beta \partial_\nu \psi_\alpha + \frac{1}{2} F \partial_\mu \psi_\beta \partial_\nu \xi_\alpha F \right) \]

\[ - \sqrt{2} \lambda^{\mu\nu} \left( - F \xi_\alpha \bar{F} \partial_\mu \partial_\nu \psi_\alpha - F \psi_\alpha \partial_\mu \partial_\nu \xi_\alpha F + \xi_\alpha \psi_\alpha \partial_\mu \partial_\nu F \right) \]

\[ + \xi_\alpha \psi_\alpha \partial_\mu \partial_\nu F \partial_\rho F + F^2 \partial_\mu \partial_\nu \xi_\alpha \psi_\alpha \right) \]

\[ = \sqrt{2} \lambda^{\mu\alpha} \lambda^{\nu\beta} \psi_\alpha \xi_\beta \partial_\mu F \partial_\nu F - \sqrt{2} \lambda^{\mu\nu} \xi_\alpha \psi_\alpha \partial_\mu F \partial_\nu F \]

\[ = 0. \] \hspace{1cm} (26)

We used the identity \( \lambda^{\mu\alpha} \partial_\mu F \lambda^{\nu\beta} \partial_\nu F = - \varepsilon^{\alpha\beta} \lambda^{\mu\nu} \partial_\mu F \partial_\nu F \). But \( \mathcal{L}_\lambda \) is not invariant under the \( \bar{Q} \)-supersymmetry. The \( \bar{\lambda} \)-dependent parts is invariant under the \( \bar{Q} \)-supersymmetry but not \( Q \)-supersymmetry. Then all of the supersymmetry is broken as a whole. We notice that there is remaining symmetry

\[ \delta_\xi A = \sqrt{2} \xi \psi, \quad \delta_\xi \bar{A} = \sqrt{2} \xi \bar{\psi}, \]

\[ \delta_\xi \psi = \sqrt{2} \xi F, \quad \delta_\xi \bar{\psi} = \sqrt{2} \xi \bar{F}, \]

\[ \delta_\xi F = 0, \quad \delta_\xi \bar{F} = 0. \] \hspace{1cm} (27)

7
5 Quantum property of the deformed Wess-Zumino model

In this section, we consider quantum aspects of the deformed Wess-Zumino model in the Minkowski space. As we mentioned in the previous section, $\mathcal{N} = 1$ supersymmetry is completely broken to $\mathcal{N} = 0$ in the Minkowski space. Therefore we use the component formalism rather than the supergraph to study quantum effects. The component formalism of the Wess-Zumino model was studied in the literature \[16, 18\].

The non-local terms which are induced by the deformation have at least one Grassmann (even) coupling constant. These terms give the new vertices in the Feynman diagrams but their additional contributions to the quantum effects are nilpotent.

Let us separate the Lagrangian into the two parts, namely, undeformed part $L_0$ and the induced part $L_\lambda$. To simplify the quantum calculation, we do partial integrations and dropping total derivative terms. We get the $\lambda(\bar{\lambda})$-dependent interaction terms;

$$L_\lambda \equiv \frac{g}{2} \chi^{\mu a} \chi^{\nu b} F(\partial_\mu \psi_\beta)(\partial_\nu \psi_\alpha) + g \chi^{\mu a} F(\partial_\mu F)(\partial_\nu A) + \frac{g}{3} \chi^{\mu a} \chi^{\nu \rho} F(\partial_\mu \partial_\nu F)(\partial_\rho \partial_\sigma F) + (\text{h.c.}).$$ (28)

We can write the generating functional following the standard procedure,

$$Z[J, \eta] = N \exp \left[ i \int d^4x L_{\text{int}} (\delta/\delta J(x), \delta/\delta \eta(x)) \right] \int \mathcal{D}\phi \exp \left[ i \int d^4x (L_{\text{free}} + L_{\text{source}}) \right].$$ (29)

Here $N$ is normalization constant, $L_{\text{free}}$ and $L_{\text{int}}$ are free and interaction part of the Lagrangian respectively:

$$L_{\text{int}} = \left[ g(A^2F - \psi F\psi) + \frac{g}{2} \chi^{\mu a} \chi^{\nu b} F(\partial_\mu \psi_\beta)(\partial_\nu \psi_\alpha) + g \chi^{\mu a} F(\partial_\mu F)(\partial_\nu A) + \frac{g}{3} \chi^{\mu a} \chi^{\nu \rho} F(\partial_\mu \partial_\nu F)(\partial_\rho \partial_\sigma F) \right] + (\text{h.c.}).$$ (30)

$L_{\text{source}}$ is the source terms

$$L_{\text{source}} = J_A A + \bar{J}_A \bar{A} + J_F F + \bar{J}_F \bar{F} + \bar{\eta}^a \psi_\alpha + \eta_\alpha \bar{\psi}^\alpha.$$ (31)

Changing the variables of path integral and using some technical identities we find

$$Z[J, \eta] = N' \exp \left[ i \int d^4x L_{\text{int}} (\delta/\delta J(x), \delta/\delta \eta(x)) \right] \exp \left[ -iJ_A \cdot \Delta_F J_A + i\bar{m} J_F \cdot \Delta_F J_A + i m \bar{J}_A \cdot \Delta_F \bar{J}_F - iJ_F \cdot \square \Delta_F \bar{J}_F - \frac{i}{2} m \eta_\alpha \cdot \Delta_F \bar{\eta}^\alpha - \frac{i}{2} \bar{m} \bar{\eta}^\alpha \cdot \Delta_F \eta_\alpha + \bar{\eta}^a \cdot \partial_\mu \Delta_F (\sigma^{\mu}_{\alpha \alpha}) \eta^\alpha \right],$$ (32)

dot stands for space-time integration. $\Delta_F$ is an ordinary inverse operator

$$\Delta_F = \frac{1}{\square - |m|^2}.$$ (33)

It should be emphasized that only finite number of additional vertices appear in the diagrams because of the nilpotency. This fact enables one to get the exact expansion of the exponential of
The finite number of the corrections to the diagrams at each order of perturbation appear. To see this explicitly we expand the exponential of $\mathcal{L}_\lambda$ part,

$$
\begin{align*}
Z[J, \eta] &= \mathcal{N}' \exp \left[ i \int d^4x \mathcal{L}^\lambda_{\text{int}} (\delta/\delta J, \delta/\delta \eta) \right] \left[ 1 + \int d^4x \mathcal{L}^\lambda_{\text{int}} (\delta/\delta J, \delta/\delta \eta) + (\text{terms up to } \mathcal{O}(\lambda^8, \bar{\lambda}^8)) \right] \\
&\times \exp \left[ -i \bar{J}_A \cdot \Delta_F J_A + i \bar{m} J_F \cdot \Delta_F J_A + i m \bar{J}_A \cdot \Delta_F \bar{J}_F \\
&-i J_F \cdot \square \Delta_F \bar{J}_F - \frac{i}{2} m \bar{\eta}_\alpha \cdot \Delta_F \bar{\eta}_\bar{\alpha} + \bar{\eta}_\bar{\alpha} \cdot \Delta_F \bar{\eta}_\alpha + \partial_\mu \Delta_F (\sigma^\mu)_{\alpha\bar{\alpha}} \eta^\bar{\alpha} \right].
\end{align*}
$$

(34)

We consider the vacuum energy to grasp some quantum properties of this model. One loop order contributions vanish due to their original $\mathcal{N} = 1$ supersymmetry. This fact agrees with the planar and non-planar superfield calculation\(^3\). The first non-trivial contributions come from two-loop order. Now, for simplicity, we are going to consider only the $\mathcal{O}(\lambda^2, g^2)$ terms. There are eight additional Feynman diagrams [fig.1]. Explicit calculation shows that all contributions cancel out exactly at least at this order, like in the $\mathcal{N} = 1/2$ case in spite of the completely broken SUSY nature of this model. Contrary to the $\mathcal{N} = 1/2$ model, we have no mechanism which guarantee such cancellations. This is very suggestive fact. The divergence structure of this model may be milder than the usual non-SUSY models.

\(^3\)There is only planar contribution to the vacuum diagram at one-loop order. This gives exactly zero contribution because of $\mathcal{N} = 1$ SUSY property of the original Wess-Zumino model.
We stress that the nilpotent property of additional Feynman diagrams helps us to investigate the detailed structures of higher order quantum effects. For example, when we focus on only $\lambda$ sector (not $\bar{\lambda}$), and consider the contributions from $\lambda^{\mu\nu\rho\sigma} F(\partial_\mu \partial_\nu F)(\partial_\rho \partial_\sigma F)$ term, the diagrams which contain at least one $\lambda^{\mu\nu\rho\sigma}$ vertex are depicted in [fig2]. The shaded regions are $\lambda = 0$ contributions to the diagrams which are already known from the $\mathcal{N} = 1$ Wess-Zumino model. We need to evaluate only finite number (seven in this case) of the diagrams to study the

Figure 2: F cubic contribution to the diagram. Diagram (I) is $\mathcal{O}(\lambda^4)$, (II) is $\mathcal{O}(\lambda^6)$ and (III) is $\mathcal{O}(\lambda^8)$ contributions.

deformation effect.

6 Geometric origin

It is well known that NS-NS Kalb-Ramond field $B_{\mu\nu}$ induces the space-time noncommutativity on the open string endpoint, i.e. on the D-brane world volume. Recently, Ooguri and Vafa showed that when we consider the superstring in the presence of the graviphoton (R-R) background with the compactified six dimensional Calabi-Yau manifold, we can realize four-dimensional $\mathcal{N} = 1/2$ NAC superspace on the string world sheet boundary [4]. More clear derivation can be seen in [3]. On the other hand, in [19] they showed the generalized
non(anti)commutativities among supercoordinates in the ten dimension by considering the superstring in the constant supergravity backgrounds. They realized the bosonic-fermionic noncommutativity $[x^\mu, \theta^\alpha] = i\lambda^{\mu\alpha}$ on the open string endpoint by turning on the constant gravitino background $\psi_\alpha^\mu$.

The backgrounds have to be consistent with the supergravity equations of motion. Consider the type IIB supergravity action. For simplicity we keep the R-R field strength, dilaton $\phi$ and dilatino $\psi$ to be zero. We take here the constant NS-NS B-field and constant gravitino. The Einstein equation for IIB supergravity is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = T^{\mu\nu}_{\text{gravitino}}.$$  (35)

Here $H = dB$ is zero for the constant B-field and gives no gravitational back-reaction. Taking flat limit in both sides in the Einstein equation gives the consistent zero limit because the stress tensor of the constant spin 3/2 Rarita-Schwinger field on the flat space vanishes$^4$. Then the constant gravitino allows flat metric, so we can consider appropriate setup for the quantized superstring.

The OPE between $x$ and $\theta$ shows

$$[x^\mu, x^\nu] = 2\pi\alpha' \left( \frac{1}{g + 2\pi\alpha' B} \right)^{\mu\nu}_A,$$

$$[x^\mu, \theta^\alpha] = -i(2\pi\alpha')^2 \left( \frac{1}{g + 2\pi\alpha' B} \right)^{\mu\nu}_A \psi^\alpha_{\nu},$$

$$\{\theta^\alpha, \theta^\beta\} = i(2\pi\alpha')^3 \psi^\alpha_{\mu} \left( \frac{1}{g + 2\pi\alpha' B} \right)^{\mu\nu}_A \psi^\beta_{\nu}.$$  (36)

Here we suppose the boundary condition $\theta^\alpha = \tilde{\theta}^\alpha$ and $\psi^\alpha_m = \tilde{\psi}^\alpha_m$. Next we take zero-slope limit $\alpha' \to 0$ to decouple massive mode and obtain the effective field theory. It is not possible that only $x-\theta$ commutator have non-zero value by the scaling itself. Note that we have to consider the special configurations of the fields to cancel the gravitino by the NS-NS sector in the $\theta-\theta$ anti-commutator. At the same time to keep the $x-x$ commutator zero, we need to scale the fields carefully. Our scaling is $\alpha' \to 0$ with $(g + 2\pi\alpha' B)^{-1}_A = \text{constant}$ and $\psi \sim \alpha'^{-2}$. Then we can realize the appropriate condition for our purpose on the D3-brane world volume. But to reduce the supersymmetry needs further investigation.

7 Discussion

In this paper, we studied bosonic-fermionic noncommutative superspace and the field theory on it. There are two possibilities to deform the theory, namely, Euclidean or Minkowski. In case of the theory in Minkowski signature, we have to adopt non-Hermitian Moyal star product to

$^4$But this result is acceptable up to quadratic order of $\psi^\mu_\alpha$ because no complete actions of the IIB supergravity are found.
keep the associativity, and the resulting higher derivative structure of space-time gives non-local nature in the theory. Our deformation is non-local in space-time as in the bosonic (space-time) noncommutative field theory, and the deformation is nilpotent and additional terms to the ordinary Lagrangian have Grassmann coupling constant. Due to this nilpotency, there are only finite number of additional Feynman diagrams at each loop order, and we found that the vacuum energy vanishes at the two loop $O(\lambda^2)$ in spite of the violation of SUSY. This unexpected result tempt us to think quantum effect is more controllable than in $\mathcal{N} = 0$ theory, though further investigations are required.

The fact that we have to introduce two types of star products $\star$ and $\bar{\star}$ in the Lagrangian in case of Minkowski signature gives technical difficulties to construct the gauge theories. We have to extract Hermitian sector from the gauge kinetic term and gauge transformation. On the other hand in case of the Euclidean signature, we may use only one star product by virtue of the relation $[y^\mu, \theta^\alpha] = i\lambda^{\mu\alpha}$ with $[\bar{y}^\mu, \bar{\theta}^\dot{\alpha}] = 0$. This case is more closely related to the Seiberg’s Euclidean $\mathcal{N} = 1/2$ theory$^5$. In this case, it is easy to construct gauge theory on the $[y^\mu, \theta^\alpha] = i\lambda^{\mu\alpha}$ space because there are only one deformation of product $\star$ which has to be used to define noncommutative gauge transformation.

### Acknowledgments

We would like to thank Y. Tanii and N. Kitazawa for useful comments. We especially would like to thank S. Saito for discussion about non-locality.

### A Notation

We follow the standard notation and convention [17] in both Euclidean and Minkowski space. The fermionic differentiations are

$$\frac{\bar{\partial}}{\partial \bar{\theta}^\alpha} \theta^\beta = \delta^\beta_\alpha, \quad \theta^\beta \frac{\overleftarrow{\partial}}{\overleftarrow{\partial} \theta^\alpha} = -\delta^\beta_\alpha,$$

$$\frac{\overleftarrow{\partial}}{\overleftarrow{\partial} \theta^\alpha} \theta^2 = 2\theta^\alpha, \quad \theta^2 \frac{\bar{\partial}}{\bar{\partial} \bar{\theta}^\alpha} = 2\bar{\theta}^\alpha,$$

$$\frac{\overleftarrow{\partial}}{\overleftarrow{\partial} \theta^\alpha \theta^\beta} \theta^2 = 2\varepsilon_{\alpha\beta}, \quad \theta^2 \frac{\bar{\partial}}{\bar{\partial} \bar{\theta}^\alpha \bar{\theta}^\beta} = -2\varepsilon_{\alpha\beta},$$

and the deformation parameters are defined as follows;

$$\lambda^{\mu\nu} \equiv \frac{1}{2} \lambda^{\mu\alpha} \lambda^\nu_\alpha, \quad \bar{\lambda}^{\mu\nu} \equiv (\lambda^\mu)^\dagger = -\frac{1}{2} \bar{\lambda}^{\mu\dot{\alpha}} \bar{\lambda}^{\nu\dot{\beta}} \varepsilon_{\dot{\alpha}\dot{\beta}},$$

$$\lambda^{\mu\nu\rho\sigma} \equiv \frac{1}{4} \lambda^{\mu\nu} \lambda^{\rho\sigma} = \frac{1}{16} \lambda^{\mu\alpha} \lambda^\nu_\alpha \lambda^{\rho\beta} \lambda^\sigma_\beta, \quad \bar{\lambda}^{\mu\nu\rho\sigma} \equiv (\lambda^{\mu\nu\rho\sigma})^\dagger = \frac{1}{4} \bar{\lambda}^{\mu\nu} \bar{\lambda}^{\rho\sigma},$$

here bar denotes formal complex conjugate $(\bar{\lambda}^{\mu\alpha}) = \bar{\lambda}^{\mu\dot{\alpha}}$.

$^5$Actually, this deformation breaks half of SUSY.
Properties of the Moyal-Weyl star product

The Fourier transformation of chiral superfield \( f(y, \theta) \) is

\[
\tilde{f}(k, \pi) = \int \frac{d^4y}{(2\pi)^4} \int d^2\theta \, e^{ik\mu y_\mu + i\pi^\alpha \theta_\alpha} f(y, \theta), \tag{39}
\]

\[
f(y, \theta) = \int \frac{d^4k}{(2\pi)^4} \int d^2\pi \, e^{-ik\mu y_\mu - i\pi^\alpha \theta_\alpha} \tilde{f}(k, \pi). \tag{40}
\]

In our convention,

\[
\int d^2\pi \, e^{i\pi^\alpha (\theta - \theta')_\alpha} = \int d^2\pi \left( 1 + i\pi^\alpha (\theta - \theta')_\alpha + \frac{1}{4} \pi^2 (\theta - \theta')^2 \right) = \frac{1}{4} \delta^{(2)}(\theta - \theta'). \tag{41}
\]

The product of superfields on the noncommutative superspace \([\hat{y}^\mu, \hat{\theta}^\alpha] = i\lambda^{\mu\alpha}\) is

\[
\hat{f}(\hat{y}, \hat{\theta}) \hat{g}(\hat{y}, \hat{\theta}) = \int \frac{d^4y_1}{(2\pi)^4} \int \frac{d^4y_2}{(2\pi)^4} \int d^2\pi_1 \int d^2\pi_2 e^{-i(k_1^\mu y_\mu + i\pi_1^\alpha \theta_\alpha)} e^{-i(k_2^\mu y_\mu + i\pi_2^\alpha \theta_\alpha)} \tilde{f}(k_1, \pi_1) \tilde{g}(k_2, \pi_2). \tag{42}
\]

We can use the BCH formula to combine the exponential sector,

\[
e^{-i(k_1 + k_2)^\mu y_\mu - i(\pi_1 + \pi_2)^\alpha \theta_\alpha + i\Delta}, \tag{43}
\]

here the factor \(i\Delta\) is

\[
i\Delta = \frac{i}{2} \lambda^{\mu\alpha} (k_1^\mu \pi_2^\alpha - \pi_1^\alpha k_2^\mu). \tag{44}
\]

We then define the noncommutative star product between ordinary chiral superfields

\[
f(y, \theta) \star g(y, \theta) \equiv \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^2\pi_1 \int d^2\pi_2 e^{-i(k_1 + k_2)^\mu y_\mu - i(\pi_1 + \pi_2)^\alpha \theta_\alpha} e^{i\Delta} \tilde{f}(k_1, \pi_1) \tilde{g}(k_2, \pi_2), \tag{45}
\]

from which we can find explicit form of the Moyal-Weyl star product. The parameter transformation

\[
k_1 = \frac{k}{2} + k', \quad k_2 = \frac{k}{2} - k', \quad \pi_1 = \frac{\pi}{2} + \pi', \quad \pi_2 = \frac{\pi}{2} - \pi', \tag{46}
\]

gives Jacobian = 1 and

\[
\tilde{f} \star g(0) = \int \frac{d^4k'}{(2\pi)^4} \int d^2\pi' \, \tilde{f}(k', \pi') \tilde{g}(-k', -\pi'). \tag{47}
\]
By taking $k = 0$, the extra phase factor vanish, and we find

$$\int d^4 y d^2 \theta \ f(y, \theta) \ast g(y, \theta) = \int d^4 y d^2 \theta \ f(y, \theta) g(y, \theta). \quad (48)$$

So, free part of the theory is not deformed by the star product.

Next, let us check the associativity $f(y, \theta) \ast (g(y, \theta) \ast h(y, \theta)) = (f(y, \theta) \ast g(y, \theta)) \ast h(y, \theta)$. Fourier transformation of the chiral superfields are

$$f(y, \theta) = \int \frac{d^4 k}{\sqrt{(2\pi)^4}} \int 2d^2 \pi e^{-ik \cdot y - i\pi \cdot \theta} f(k, \pi), \quad g(y, \theta) = \int \frac{d^4 p}{\sqrt{(2\pi)^4}} \int 2d^2 \rho e^{-ip \cdot y - i\rho \cdot \theta} g(p, \rho), \quad h(y, \theta) = \int \frac{d^4 q}{\sqrt{(2\pi)^4}} \int 2d^2 \sigma e^{-iq \cdot y - i\sigma \cdot \theta} h(q, \sigma). \quad (49)$$

It is enough to check

$$e^{-ik \cdot y - i\pi \cdot \theta} \ast (e^{-ip \cdot y - i\rho \cdot \theta} \ast e^{-iq \cdot y - i\sigma \cdot \theta}) = (e^{-ik \cdot y - i\pi \cdot \theta} \ast e^{-ip \cdot y - i\rho \cdot \theta}) \ast e^{-iq \cdot y - i\sigma \cdot \theta}. \quad (50)$$

Calculation is straightforward by the use of BCH formula,

$$\text{LHS} = e^{-i(k+p+q) \cdot y - i(\pi+\rho+\sigma) \cdot \theta} e^{\frac{i}{2} \lambda_{\mu\nu} [k^{\mu} (\rho+\sigma)^{\nu} -(p+q)^{\mu} \pi^{\nu} + p^{\mu} \sigma^{\nu} - q^{\mu} \rho^{\nu}]}, \quad (51)$$
$$\text{RHS} = e^{-i(k+p+q) \cdot y - i(\pi+\rho+\sigma) \cdot \theta} e^{\frac{i}{2} \lambda_{\mu\nu} [k^{\mu} \rho^{\nu} - p^{\mu} \pi^{\nu} + (k+p)^{\mu} \sigma^{\nu} - q^{\mu} (\pi+\rho)^{\nu}]}. \quad (52)$$

It is obvious that LHS = RHS.

References


