SPIN-DEPENDENT CYCLOTRON DECAY RATES IN STRONG MAGNETIC FIELDS

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ABSTRACT

Cyclotron decay and absorption rates have been well studied in the literature, focusing primarily on spectral, angular and polarization dependence. Astrophysical applications usually do not require retention of information on the electron spin state, and these are normally averaged in obtaining the requisite rates. In magnetic fields, higher order quantum processes such as Compton scattering become resonant at the cyclotron frequency and its harmonics, with the resonances being formally divergent. Such divergences are usually eliminated by accounting for the finite lifetimes of excited Landau states. This practice requires the use of spin-dependent cyclotron rates in order to obtain accurate determinations of process rates very near cyclotronic resonances, the phase space domain most relevant for certain applications to pulsar models. This paper develops previous results in the literature to obtain compact analytic expressions for cyclotron decay rates/widths in terms of a series of Legendre functions of the second kind; these expressions can be expediently used in astrophysical models. The rates are derived using two popular eigenstate formalisms, namely that due to Sokolov and Ternov, and that due to Johnson and Lippmann. These constitute two sets of eigenfunctions of the Dirac equation that diagonalize different operators, and accordingly yield different spin-dependent cyclotron rates. This paper illustrates the attractive Lorentz transformation characteristics of the Sokolov and Ternov formulation, which is another reason why it is preferable when electron spin information must be explicitly retained.

Subject headings: radiation mechanisms: non-thermal — magnetic fields — relativity — stars: neutron — pulsars: general — gamma rays: theory

1. INTRODUCTION

The number of astrophysical sources thought to possess surface magnetic fields above the quantum critical field strength of $B_{crit} = m_e^2 c^3 / (e h) \approx 4.413 \times 10^{13} \text{G}$ has been steadily growing in recent years. The five presently known Soft Gamma-Ray Repeaters (SGRs) and six or seven Anomalous X-Ray Pulsars (AXPs) are believed to be magnetars, neutron stars having surface fields in the range $10^{14}$ $- 10^{15}$ G (Duncan & Thompson 1992). In addition, the Parkes Multibeam survey (Manchester et al. 2001) has discovered at least six new radio pulsars that have surface fields near or above critical, and several with fields comparable to those of some of the magnetars. The radiation processes that determine the observed X-ray and gamma-ray spectra in these sources are operating in the extreme relativistic and quantum regimes and thus require treatment that is accurate in such environments. One process that is particularly important in the source emission models is resonant Compton scattering, in which electrons scatter photons at the cyclotron resonance with a cross section much larger than at continuum energies. Relativistic electrons can blue-shift low energy photons into the resonance, upscattering the photons into a high-energy continuum (Daugherty & Harding 1989, Dermer 1990).

The quantum electrodynamical (QED) cross section for Compton scattering at the cyclotron fundamental (Herold 1979) and higher harmonics (Daugherty & Harding 1986, Bussard, Alexander & Mészáros 1986), accurate in arbitrarily high magnetic fields, has been known for some time. However, these derivations do not treat the natural line widths that render the cross section finite at the cyclotron resonances. In order to use such rates in spectral calculations in astrophysical models, the line widths that originate from the lifetimes of the excited Landau states must be included in the cross section (Wasserman & Salpeter 1980). The width of the nth resonance is equal to the cyclotron decay rate from that state (Pavlov et al. 1991), and the prescription for incorporating the widths in the QED cross section in the high-field regime has been discussed by Harding & Daugherty (1991) and Graziani (1993). Including resonant line widths in the scattering cross section necessarily requires spin-dependent decay rates, which appear in an infinite sum over Landau state $n$ and spin of the intermediate virtual states, even in the case of ground state-to-ground state scattering in the fundamental. Different spin states have different decay rates, and thus different resonant energy denominators. Since the electrons and positrons in intermediate states have non-zero momentum parallel to the magnetic field, it is important to use...
basis states that yield a spin dependence that is Lorentz invariant, i.e. boosts along the magnetic field do not lead to a mixing between the spin states. 

As is true for all quantum processes, the spin-dependent rates and cross sections depend on the choice of electron wavefunctions in a uniform magnetic field. Historically, several choices of wavefunctions have been used in calculations of the scattering cross section and cyclotron decay rates. The two most widely used wavefunctions are those of Johnson & Lippman (1949) and Sokolov & Ternov (1968). The Johnson & Lippman (JL) wavefunctions are derived in Cartesian coordinates and are eigenstates of the kinetic momentum operator. The Sokolov & Ternov (ST) wavefunctions, specifically their “transverse polarization” states, are derived in cylindrical coordinates and are eigenfunctions of the magnetic moment operator. Given the different spin dependence of the ST and JL eigenstates, one must use caution in making the appropriate choice when treating spin-dependent processes. Herold, Ruder & Wunner (1982) and Melrose and Parle (1983) have noted that the ST eigenstates have desirable properties that the JL states do not possess, such as being eigenfunctions of the Hamiltonian including radiation corrections, having symmetry between positron and electron states, and diagonalization of the self-energy shift operator. As found by Graziani (1993), the ST wavefunctions also diagonalize the Landau-Dirac operator, and are the physically correct choices for spin-dependent treatments and for incorporating widths in the scattering cross section. Although the spin-averaged ST and JL cyclotron decay rates are equal, the spin-dependent decay rates are not, except in the special case in which the initial momentum of the electron parallel to the field vanishes.

Perhaps the most fundamental argument against use of the JL wavefunctions is that radiative corrections cause excited JL Landau levels to become unstable to spin-flip transition within Landau states on timescales comparable to the timescale for decay to a lower Landau state. These radiative corrections to the magnetic moment break the spin degeneracy of both the ST and JL excited Landau states (Herold, Ruder & Wunner 1982), causing spin-dependent energy level shifts, and implying a dependence of the rates for spin-flip transitions within a Landau state on the choice of wavefunctions. The rates of such spin-flip transitions within split ST Landau states have been evaluated (Geprägs et al. 1994; see also Parle 1987), and are found to be of order of \( \alpha^6 (B/B_{cr})^4 \) for \( B < B_{cr} \), which is negligible compared to decay rates between Landau states. The existence of such relatively long-lived states is a premise of the S-matrix formalism and is also essential to any astrophysical calculations involving spin-dependent transitions between Landau states. In contrast, it is anticipated that since the JL eigenstates do not diagonalize the self-interaction Hamiltonian including radiation corrections (i.e. the mass operator), the associated mixing incurred in S-matrix evaluations will render spin-flip transition rates with fixed \( n \) comparable to ordinary cyclotron rates with changes in \( n \), a situation that is unphysical.

In this paper we discuss the Lorentz transformation characteristics of cyclotron decay rates for both ST and JL formulations, and derive simplified expressions for the decay rates from an arbitrary excited Landau state to the ground state. We show that the ST eigenstates preserve separability of the spin dependence under Lorentz boosts along the local magnetic field, a desirable property that does not extend to the JL formalism, for which such Lorentz boosts mix the corresponding spin states. In the ST formulation, by taking advantage of its spin-state preserving characteristics, our analytic simplifications can be elegantly applied to the spin-dependent decay rates; in the JL formalism the simplifications can be compactly applied only to the spin-averaged rates. The resulting expressions replace integrals over emergent photon angle with series of Legendre functions of the second kind, which correspond to sums of elementary functions, easily yielding simple asymptotic forms.

The expressions derived here should have wide applicability to modeling cyclotron emission, Compton scattering and other QED processes in super-critical fields. For such magnetar-type fields, resonant scattering takes place primarily at the fundamental, since the cyclotron energy exceeds 1 MeV. Although the resonant scattering line widths formally involve infinite sums over Landau states, in the case of the fundamental resonance the sum is dominated by the \( n = 1 \) state, whose width is equal to the \( n \rightarrow 0 \) cyclotron decay rate. For cyclotron scattering in higher harmonics \( l > 0 \), the intermediate sums in the vertex functions have the largest contributions from the \( n = l + 1 \) state. However, for \( B > B_{cr} \), cyclotron transitions to the ground Landau state dominate (Sokolov, Zhukovskii & Nikitina 1973, White 1974, Harding & Preece 1987), so that cyclotron decay rates for \( n \rightarrow 0 \) transitions treated in this paper should be good approximations to the widths of excited states, a circumstance pertinent to astrophysical models of magnetars.

2. CYCLOTRON RATES: SOKOLOV AND TERNOV FORMALISM

This presentation of spin-dependent cyclotron rates appropriately focuses first on the formulation generated using the transverse polarization eigenstates of the Dirac equation derived by Sokolov & Ternov (1968) for electrons in a uniform magnetic field \( B \). Explicit forms for them can also be found in Herold, Ruder & Wunner (1982) and Harding & Preece (1987), whose exposition on cyclotron emission essentially forms the basis of results developed here. Herold et al. observed that these states diagonalize the operator that describes the electronic self-energy shift in an external magnetic field. Another attractive feature of the Sokolov & Ternov eigenstates is that they possess charge conjugation symmetries, i.e. between electron (positive energy) and positron (negative energy): see Eq. (12) of Herold, Ruder & Wunner (1982). An additional asset of these wavefunctions that is enunciated here is that they yield cyclotron rates whose spin dependence is effectively separable from Lorentz transformations along the field: i.e. such boosts do not mix spin states. This is a very useful characteristic that is not present in the Johnson & Lippmann formalism addressed in Section 3 below.

Herold, Ruder & Wunner (1982) obtained general expressions in their Eq. (17) for the spin-dependent cyclotron rate for transitions between arbitrary Landau levels, but for the case of zero initial momentum \( p \) parallel to the field. Latal (1986) independently obtained similar results for cyclotron transitions to the ground state, but retained arbitrary initial momenta \( p \) along \( B \); his Eq. (22) forms the starting point for the exposition here. Denoting \( \zeta = \pm 1 \) as the spin quantum number of the initial electron, ground state transitions are characterized by a single final spin \( \zeta' = -1 \). The energy level quantum numbers are initially \( n \) and \( n' = 0 \) after the transition. Latal’s spin-dependent total rates can be written in the form

\[
\Gamma_{n0}^\zeta = \left[ 1 - \frac{\zeta}{\sqrt{1 + 2nB}} \right] \Gamma_{n0}^0 ,
\]

where \( B \) is the magnetic field strength, expressed dimensionlessly hereafter in units of the quantum critical field \( B_{cr} = m^2 c^3/(\epsilon h) \approx 4.413 \times 10^{13} \) Gauss, and the spin-averaged, total
cycotron transition rate for \( n \to n' = 0 \) is given by
\[
\Gamma_{n0} = \frac{\alpha e}{\lambda} \frac{n^n}{(n-1)!} \frac{B}{E_n} \int_{-1}^{1} \frac{d\mu}{(E_n - p\mu)^2} \times \left( \frac{1}{1+\xi} \right) (2nB - 1) \exp \left(-n \frac{1-\xi}{1+\xi} \right) .
\]
(2)

Here \( \lambda = h/m_e c \) is the Compton wavelength of the electron over \( 2\pi \), and other quantities in this equation are defined as follows. For initial and final dimensionless momenta (i.e., in units of \( m_e c \)), \( p \) and \( q \) respectively, parallel to the field, the electron’s initial energy \( E_n \) and final energy \( E_0 \), both dimensionless (i.e., in units of \( m_e c^2 \)), are
\[
E_n = \sqrt{1+2nB+p^2} \quad \text{and} \quad E_0 = \sqrt{1+q^2} .
\]
(3)

This convention of using dimensionless energies and momenta will be adopted throughout the paper.

The integration variable \( \mu \) is the angle cosine of the emitted photon with respect the magnetic field direction, i.e. \( \mu = \cos \theta = \vec{k} \cdot \vec{B} / (|\vec{k}| \cdot |\vec{B}|) \) for photon wavenumber vector \( \vec{k} \). No integration by parts has been performed when obtaining Eq. (2), so that the cyclotron rate, differential in photon angles, \( d\Gamma_{n0}/d\mu \), again averaged over initial electron spins, corresponds directly to the integral of Eq. (2):
\[
\Gamma_{n0} = \int_{-1}^{1} \frac{d\Gamma_{n0}}{d\mu} d\mu .
\]
(4)

Corresponding spin-dependent differential cyclotron rates can be deduced from Eq. (1) in similar fashion.

The remaining variable, \( \xi \), emerges naturally from energy-momentum conservation in the interaction. Only momentum parallel to the field is conserved, since the system is not translationally-invariant orthogonal to \( \vec{B} \). Hence cyclotron emission satisfies two such conservation relations:
\[
E_n = E_0 + \omega , \quad p = q + \omega \mu
\]
(5)

for \( \mu = \cos \theta \). Here \( \omega \) is the dimensionless photon energy, having been scaled by \( m_e c^2 \). The elimination of \( q \) from the system in Eq. (5) leads to the identities
\[
\omega = \frac{2nB}{1+\xi} \frac{1}{E_n - p\mu} ,
\]
\[
\omega (1-\mu^2) = (1-\xi)(E_n - p\mu)
\]
(6)

that are simultaneously satisfied, with
\[
\xi = \sqrt{1 - \frac{2nB(1-\mu^2)}{(E_n - p\mu)^2}} ,
\]
(7)

being the \( n' \to 0 \) specialization of the definition in Eq. (11b) of Latal (1986). The angle integration in Eq. (2) for the cyclotron rate is non-trivial, due largely to the complicated dependence of \( \xi \) on \( \mu \), and the presence of the exponential.

This completes the definitions relevant to Eq. (2). However, we also note that the identities in Eq. (6) combine to yield
\[
\kappa = \frac{\omega^2(1-\mu^2)}{2B} = n \frac{1-\xi}{1+\xi} ,
\]
(8)

which is precisely the argument of the exponential in Eq. (2). This \( \exp(-\kappa) \) factor results from the \( n' = 0 \) specialization of associated Laguerre functions that are formed from the spatial integration (Fourier transform) of the eigenfunctions. Since \( \omega \sin \theta \) is Lorentz invariant under boosts along \( \vec{B} \), so also is \( \kappa \). Furthermore, forming \( E_n - p\mu \) using Eq. (1), and then eliminating \( \omega \) using the second identity in Eq. (6), quickly leads to the equivalence
\[
\xi = \frac{E_0 - q\mu}{E_n - p\mu} .
\]
(9)

When weighted by \( \omega \) top and bottom, this is a ratio of invariant products of four-momenta associated with the interaction; i.e. \( \xi \) is also a Lorentz invariant for boosts along the field, which can be inferred directly from Eq. (5).

2.1. Lorentz Transformation Properties

There is no need to compute Eq. (2) as it stands for arbitrary \( p \). Instead, one can appeal to a simple Lorentz transformation protocol. Consider boosts along the field between the inertial frame where the initial electron possesses momentum \( p \) along the field, and that frame where the initial parallel momentum is zero. In this latter “rest” frame let \( \mu_0 \) be the photon angle cosine with respect to \( B \), \( p_0 = 0 \) be the initial parallel electron momentum, and the initial electron energy be denoted
\[
\varepsilon_n = \sqrt{1+2nB} .
\]
(10)

Then, for a dimensionless boost velocity \( \beta \equiv v/c \) and Lorentz factor \( \gamma = 1/\sqrt{1-\beta^2} \) along \( \vec{B} \), the initial electron quantities in Eq. (2) satisfy
\[
E_n = \gamma(\varepsilon_n - \beta p_0) = \gamma \varepsilon_n ,
\]
\[
p = \gamma(p_0 - \beta n) = -\gamma \beta \varepsilon_n ,
\]
(11)

from which \( \beta = -p/E_n = -p/\sqrt{1+2nB+p^2} \) is established. The photon angle cosine satisfies the aberration formula
\[
\mu = \cos \theta = \frac{\mu_0 - \beta}{1-\beta \mu_0} .
\]
(12)

This latter relation identifies a suitable change of variables for the angle integration, for which \( d\mu = d\mu_0/|\gamma(1-\beta \mu_0)|^2 \). Other transformation identities include \( \omega \sin \theta = \omega_0 \sin \theta_0 \) and \( (1-\mu^2)/(E_n - p\mu)^2 = (1-\mu_0^2)/\varepsilon_n^2 \), from which the invariance in form of \( \xi \) is established via the definition in Eq. (7). It then follows, after a modicum of algebra, that this pure change of variables leads to an alternative form for the rate in Eq. (2):
\[
\Gamma_{n0} = \frac{\alpha e}{\lambda} \frac{n^n}{(n-1)!} \frac{B}{E_n} I_n(B) ,
\]
(13)

with
\[
I_n(B) = \int_{-1}^{1} \frac{d\mu_0}{\varepsilon_n} \frac{(1+\xi)^{n-1}}{(1+\xi)^{n+1}} (\varepsilon_n^2 - 1) \exp \left(-n \frac{1-\xi}{1+\xi} \right) .
\]
(14)

where, now \( \xi \) is obtained from Eq. (12) using the correspondences \( \mu \to \mu_0 \), \( p \to 0 \) and \( E_n \to \varepsilon_n \).

Accordingly, this identifies attractive Lorentz transformation behavior, with the rate reduced (i.e. lifetime dilated) by just the Lorentz factor \( \gamma \) of the boost. Moreover, such boosts along the field keep the spin states “separated,” i.e. there is no implied mixing of states incurred by such transformations. This
inherent simplicity is an appealing characteristic of the Sokolov & Ternov states, and was identified by Graziani (1993); it is not exhibited in the Johnson & Lipman formulism explored in Sec. 3. Clearly, this extraction of \( p > 0 \) case via a simple modification factor outside the integral is expedient for the subsequent analytic developments.

2.2. Analytic Developments

While the integral expressions for the cyclotron rates can be routinely evaluated numerically, they can also be represented by compact analytic series in terms of elementary functions that are readily amenable to computation. The integration variable \( \mu_0 \) for \( I_n(B) \) in Eq. (13) is not the most convenient; a more expedient choice for the purposes of analytic reduction is

\[
\phi \equiv \frac{k}{n} = \frac{1 - \xi}{1 + \xi} , \quad \xi = \sqrt{1 + 2nB \mu_0^2} .
\]

The integration is even in \( \mu_0 \), and since \( 1/\varepsilon_n \leq \xi \leq 1 \), the integration range maps over to

\[
0 \leq \phi \leq \phi_n \equiv \frac{\varepsilon_n - 1}{\varepsilon_n + 1} .
\]

After a modest amount of algebra, the change of variables leads to the form

\[
I_n(B) = \int_0^{\phi_n} \frac{d\phi e^{-n\phi} \phi^{n-1}}{\sqrt{(\phi_n - \phi)(1/\phi_n - \phi)}} \left[ 1 - \frac{\phi}{2} \left( \phi_n + \frac{1}{\phi_n} \right) \right] .
\]

This integral can be evaluated in terms of Appell functions, the degenerate, two-dimensional hypergeometric functions, using Eq. 3.385 of Gradshteyn & Ryzhik (1980). Such a step does not facilitate evaluation, since Appell function expansions usually develop double, infinite power series representations (e.g. see Burchnall & Chaundy 1941; Exton 1976).

Analytic progress is quickly made via Taylor series expansion of the exponential \( e^{-n\phi} \) around \( \phi = 0 \). The order of integration and the infinite summation can then be interchanged because the integration is uniformly convergent on the interval \( 0 \leq \phi \leq \phi_n \), since \( \phi_n \) is strictly less than unity. Another change of variables \( \phi = e^{-t} \), with the definition

\[
z_n = \cosh t_n = \frac{1}{2} \left( \phi_n + \frac{1}{\phi_n} \right) \equiv 1 + \frac{1}{nB} ,
\]

then establishes

\[
I_n(B) = \sum_{k=0}^{\infty} \frac{(-n)^k}{k!} J_{n+k}(z_n) ,
\]

with

\[
J_\nu(z_n) = \frac{1}{\sqrt{2}} \int_{t_n}^{\infty} \frac{dt e^{-(\nu-1/2)t}}{\sqrt{\cosh t - \cosh t_n}} \left[ 1 - e^{-t} \cosh t_n \right] .
\]

Using the identity 8.715.2 of Gradshteyn & Ryzhik (1980) gives an integral representation of the Legendre function \( Q_\nu(z_n) \) of the second kind:

\[
Q_\nu(z_n) = \frac{1}{\sqrt{2}} \int_{t_n}^{\infty} \frac{dt e^{-(\nu+1/2)t}}{\sqrt{\cosh t - \cosh t_n}} , \quad z_n > 1 .
\]

These special functions are finite sums of elementary logarithmic and polynomial functions of \( z_n \) (e.g. see Abramowitz & Stegun 1965). Note that the \( Q_\nu(z) \) are generalizable to associated Legendre functions \( Q_\nu^m(z) \), defined in 8.703 of Gradshteyn & Ryzhik (1980), from which \( Q_\nu(z) \equiv Q_\nu^0(z) \). It follows from Eqs. (20) and (21) that

\[
J_{n+k}(z_n) = Q_{n+k-1}(z_n) - z_n Q_{n+k}(z_n) ,
\]

and relevant properties of the \( Q_\nu \) and \( J_\nu \) functions are listed in the Appendix. Computational issues for the \( J_m \) are discussed there also, and an expedient approach for integer \( m = n + k \) is to compute the \( J_m \) using the recurrence relation in Eq. (A4).

The final result of these analytic reductions is

\[
\Gamma_{n0} = \frac{\alpha c}{\lambda} n^n \frac{B}{\gamma \varepsilon_n} \sum_{k=0}^{\infty} \frac{(-n)^k}{k!} J_{n+k}(z_n) ,
\]

which expresses the spin-averaged cyclotron rate as a comparatively simple series of elementary functions. For low values of \( n \), this series is rapidly convergent for all \( z_n \geq 1 \), typically requiring 3–4 terms to achieve 0.01% accuracy. Such a rate of convergence renders the series evaluation computationally much more efficient than a numerical integration of Eq. (2) by Simpson’s rule or a quadrature technique. The appearance of the Legendre functions of the second kind, \( Q_\nu(z) \), in the series expansion for the rate is a result of the cylindrical symmetry imposed by the presence of the external magnetic field.

![Spin-Averaged Cyclotron Rates](image)
The result, for general arguments \( n \), can then be invoked to obtain the and \( z \) in Eqs. (18) and (21), respectively, of Melrose & Russell (2002), since here \( n' > 0 \) contributions are omitted from consideration. However, in the particular case of \( n = 1 \), there is only a single \( n' < n \) final state, namely \( n' = 0 \), and with the prescription \( \beta_1^2 \rightarrow 2B \), Eq. (6.19) of Bekefi (1966) specializes to yield \( \Gamma_1 = \frac{e \alpha c B_1^2}{3 \lambda} \), which is precisely the value obtained in Eq. (24). Derivation of the \( nB \gg 1 \), ultra-quantum limit is slightly more involved. In this situation, \( \phi_n \rightarrow 1 \) and \( z_n \rightarrow 1 \) so that the entire \( k \) series in Eq. (23) is retained. The key useful result is derived in the Appendix, in Eq. (A12), which establishes \( J_{n+k}(z) \rightarrow 1/(n+k) \) as \( z \rightarrow 1^+ \). Including the next order logarithmic contribution then leads to an approximate form for \( I_n(B) \) in Eq. (A10), so that it then follows that

\[
\Gamma_{n0}^{-1} = \frac{1}{1 + 2nB + 1} \frac{1}{1 + 2nB - 1} \quad (24)
\]

of the spin-dependent rates for \( n = 1 \); higher \( n \) cases display similar behavior. This curve illustrates how the rate is approximately independent of initial spin when \( nB \gg 1 \), whereas \( \zeta = -1 \) initial states lead to much faster transition rates (by order \( (nB)^{-1} \)) when \( nB \ll 1 \), reflecting the well-known dominance of non-spin-flip cyclotron transitions by non-relativistic electrons (e.g. see Melrose & Zheleznyakov 1981).

### 2.2.1. Asymptotic Limits

Now consider the two pertinent asymptotic limits for the spin-averaged rate in Eq. (23). When \( nB \ll 1 \), \( \phi_n \approx nB/2 \ll 1 \) and \( z_n \approx 1 \). Using the first identity in Eq. (A11), Eq. 8.771.2 of Gradshteyn & Ryzhik (1980) can then be invoked to obtain the leading order dependence of \( J_n(z_n) \), noting the specialization \( 2F(\alpha, \beta; \gamma; 0) = 1 \) of the common hypergeometric function. The result, for general arguments \( z \), is

\[
J_n(z) \approx \frac{(n+1)!}{(2n+1)!} \frac{\Gamma(1/2)}{2^{1/2}} (2z)^{-n} \quad , \quad z \gg 1 \quad . \quad (25)
\]

Here, \( \Gamma(x) \) represents the Gamma function, which is the leading order \((k = 0)\) term in the series evaluation of \( I_n(B) \), and hence this result is readily checked by taking the \( \phi_n \rightarrow 0 \) limit of Eq. (17). The doublet formula for the Gamma function, in Eq. 8.335.1 of Gradshteyn & Ryzhik (1980), can now be used to yield an asymptotic form

\[
\Gamma_{n0} \approx \frac{\alpha c}{\lambda} \frac{(2n)^n}{n!} \frac{(n+1)!}{(2n+1)!} \frac{1}{2^{1/2}} B^{n+1} \quad , \quad nB \ll 1 \quad , \quad (26)
\]

which is just Eq. (31d) of Latal (1986). This form illustrates the strong dependence of the cyclotron rates on both \( n \) and \( B \). A sample \( n = 3 \) case of the limit in Eq. (26) is displayed in Fig. 4. This limit can only be compared with the well-studied classical cyclotron limit in a restricted fashion, since it is still an essentially quantum result. Moreover, it incorporates the intrinsically relativistic effect of spin-orbit coupling. Observe that Eq. (26), in conjunction with Eq. (1), reproduces both the non-spin-flip \((\zeta = -1)\) and spin-flip \((\zeta = 1)\) ST rates encapsulated in Eqs. (18) and (21), respectively, of Melrose & Russell (2002), or equivalently, the \( n' = 0 \) specialization of Eq. (18) of Herold, Ruder & Wunner (1982).

Classical formulations of cyclotron emissivities (e.g. see Eq. (6.19) of Bekefi 1966) yield rates that depend on components \( \beta_1 c \) of the electron velocity perpendicular to the field: \( \Gamma_n \propto B \beta_1^2 \). Here the classical rate \( \Gamma_n \) effectively represents the \( nB \ll 1 \) limit of a sum over various \( n \rightarrow n' \) transitions, with \( n' < n \). By invoking the correspondence \( 2nB \rightarrow p_1^2 \beta_1^2 \), it is quickly established that \( \Gamma_n \propto B^{n+1} \), the same dependence as in Eq. (26). Yet in general, the classical \( \Gamma_n \) does not equal the quantum \( \Gamma_{n0} \), since here \( n' > 0 \) contributions are omitted from consideration. However, in the particular case of \( n = 1 \), there is only a single \( n' < n \) final state, namely \( n' = 0 \), and with the prescription \( \beta_1^2 \rightarrow 2B \), Eq. (6.19) of Bekefi (1966) specializes to yield \( \Gamma_1 = \frac{e \alpha c B_1^2}{3 \lambda} \), which is precisely the value obtained in Eq. (24). Derivation of the \( nB \gg 1 \), ultra-quantum limit is slightly more involved. In this situation, \( \phi_n \rightarrow 1 \) and \( z_n \rightarrow 1 \) so that the entire \( k \) series in Eq. (23) is retained. The key useful result is derived in the Appendix, in Eq. (A12), which establishes \( J_{n+k}(z) \rightarrow 1/(n+k) \) as \( z \rightarrow 1^+ \). Including the next order logarithmic contribution then leads to an approximate form for \( I_n(B) \) in Eq. (A10), so that it then follows that

\[
\Gamma_{n0} \approx \frac{\alpha c}{\lambda} \sqrt{\frac{B}{2n}} \left\{ \frac{\gamma(n, n)}{\Gamma(n)} - \frac{n^n e^{-n}}{\Gamma(n)} \frac{\log_2 n}{2nB} \right\} \quad , \quad nB \gg 1 \quad . \quad (27)
\]

Here, \( \gamma(n, z) \) is the incomplete Gamma function in 8.350.1 of Gradshteyn & Ryzhik (1980). Note that the same result can be derived directly from the \( \phi_n \rightarrow 1 \) limit of Eq. (17) using the integral definition of the incomplete Gamma function. The weak \( \sqrt{B} \) dependence of the rate on \( B \) in this limit is now apparent, and is evinced in Fig. 4 by the crowding of curves for supercritical fields. In the limit of \( n \gg 1 \), further specialization is possible using the limit \( \gamma(n, n)/\Gamma(n) \rightarrow 1/2 \) as \( n \rightarrow \infty \), derived in the Appendix, to yield the approximation

\[
\Gamma_{n0} \approx \frac{\alpha c}{\lambda} \sqrt{\frac{B}{8n}} \quad , \quad n \gg 1 \quad , \quad B \gg 1 \quad . \quad (28)
\]

This ultra-quantum result corresponds to the limit given in Eq. (32a) of Latal (1986), though the numerical coefficient of Latal’s does not agree exactly with the \( 1/\sqrt{8} \) factor in Eq. (28), the difference being around 10%. However, the expressions for \( n = 0 \) quantum cyclotron transition rates derived in Eq. (9) of Sokolov, Zhukovskii & Nikitina (1973) and Eq. (2.56) of White (1974) reduce exactly to Eq. (28) in the limit \( B \gg 1 \), confirming the asymptotic analysis here. A sample \( n = 30 \) case of this limit is displayed in Fig. 4 the precision of which is around 2% relative to the exact result, and better than Eq. (32a) of Latal (1986). The first order correction to this asymptotic limit that would follow from the logarithmic term in Eq. (27) is \( O(\log_n n/\sqrt{n}) \) relative to the leading order contribution in Eq. (28).

### 2.2.2. Differential Photon Spectra

In the \( p = 0 \) frame, the spin-averaged differential cyclotron rates \( d\Gamma_{n0}/d\omega_0 \) can be immediately extracted from Eq. (19) using the prescription implied by Eq. (4). However, given that Eqs. (4) and (4) define a one-to-one correspondence between the angle cosine \( \mu_0 \) and the energy \( \omega \) of the emitted cyclotron photon, the differential rates can suitably be presented as differential photon spectra, \( d\Gamma_{n0}/d\omega \). The photon energy is \( \omega = (\varepsilon_n - 1/\varepsilon_n)/(1 + \xi) \), so that the kinematic bound \( 1/\varepsilon_n \leq \xi \leq 1 \) translates to

\[
\frac{nB}{\sqrt{1 + 2nB}} \equiv \omega_- \leq \omega \leq \omega_+ \equiv \sqrt{1 + 2nB} - 1 \quad . \quad (29)
\]

The change of variables is then defined by two branches,

\[
\mu_0 = \pm \frac{1}{\omega} \sqrt{(\varepsilon_n - 1 - \omega)(\varepsilon_n + 1 - \omega)} \quad , \quad (30)
\]
with the factors inside the square root reflecting the kinematics associated with both the cyclotron radiation and the one-photon pair creation interactions. This has a Jacobian $d\Gamma_0/d\omega = -\xi(1 + \xi)\xi/[(\xi^2 - 1)]$, leading to the spin-averaged differential spectrum

$$
\frac{d\Gamma_0}{d\omega} = \frac{\alpha e}{\lambda} \frac{1}{B} \frac{n^{-1}}{n+1} \left( \frac{\omega}{\omega_0} - 1 \right)^{n-1} \left( \xi - \omega \right) \exp \left( -\frac{n \omega}{\omega_0} \right).
$$

This spectral rate is summed over the polarizations of the radiated photons. Note that a factor of 2 has been included to account for the two branches of the relation between $\mu_0$ and $\omega$. Observe also that this spectrum can be obtained directly from Eq. (13) and the form for $I_n(B)$ in Eq. (17) by using the correspondence $\phi = \omega/\omega_0 - 1$.

Key characteristics of the differential spectrum include its broadening and shifting to higher energies as $nB$ increases, behavior that is largely governed by the kinematic result in Eq. (29). An exponential decline is present in $n = 1$ cases until a sharp kinematic peaking at $\omega \sim \omega_0 = \xi_0 - 1$ overtakes it. In $n \gg 1$ cases, this exponential contribution is dominated by the power-law factor $(\omega/\omega_0 - 1)^{n-1}$, leading to a steeply-rising spectrum. When $nB \gg 1$, nearly all of the radiative power in the spectrum is confined to energies $\omega \approx \sqrt{2nB}$. In the opposite domain, $nB \ll 1$, the spectra are extremely narrow, and little insight is gained by attempting an involved comparison with the classical limit; most classical results are rendered for sums over a multitude of transitions $n \rightarrow n'$.

### 2.2.3. Cyclotron Power

To conclude the formalism based on Sokolov & Ternov states, expressions for the cyclotron energy loss rate, or power, $P_{n0} = dE_{n0}/dt$ can be expeditiously presented. The development of such rates parallels that for $\Gamma_{n0}$, with the inclusion of an extra (dimensionless) weighting factor $\omega$ to the differential decay rate $d\Gamma_{n0}/d\omega$. Hence, one can quickly write down

$$
\mathcal{T}_{n0} = m_e c^2 \int_{-1}^{1} \omega \frac{d\Gamma_{n0}}{d\omega} d\mu
$$

$$
= \frac{\alpha e m_e c^3}{\lambda} \frac{B_n}{E_n} \int_{-1}^{1} \frac{\omega d\mu}{(E_n - \mu_0)^2}
$$

$$
\times \left( \frac{1 - \xi}{(1 + \xi)^{n+1}} + 2nB - \frac{1}{\xi} \right) \exp \left( -n \frac{1 - \xi}{1 + \xi} \right).
$$

as the spin-averaged power, using Eq. (32). This can be conveniently transformed to the integration variable $\mu_0$, representing the $p = 0$ frame, using an analysis similar to that presented in Sec. 2.2.1. The only change is the manipulation of the additional $\omega = 2nB/(1 + \xi)/(E_n - \mu_0)$ factor inside the integration, for which the Lorentz transformation relations in Eqs. (11) and (12) yield $1/(E_n - \mu_0) = \gamma(1 - \beta_0)/\xi_0$. Since $d\Gamma_{n0}/d\mu_0$ is even in $\mu_0$, it follows that the term proportional to $\mu_0 d\Gamma_{n0}/d\mu_0$ in the transformed integrand of Eq. (32a) does not contribute. Therefore, comparison with Eqs. (13) and (14) then indicates that

$$
\mathcal{T}_{n0} = \frac{\alpha e m_e c^3}{\lambda} \frac{n^{n+1}}{(n-1)!} \frac{B^2}{\xi^2} K_n(B),
$$

is the form for the cyclotron power for general $p \geq 0$, with

$$
K_n(B) = 2 \int_{-1}^{1} \frac{d\mu_0}{\xi^2} \left( \frac{1 - \xi}{1 + \xi} \right)^{n-1} \left( \xi_0 - 1 \right)^{n-1} \exp \left( -n \frac{1 - \xi}{1 + \xi} \right),
$$

where $\xi$ is given by Eq. (15). This expression for the power is identical to that in Eq. (28) of Latal (1986), who illustrated the magnetic field and $n$ dependence in his Fig. 1. The spin-dependent powers $P_{n0}$ are, of course, just given by additional multiplicative factors: $P_{n0}^{\perp} = (1 - \xi^2/\xi_0)\mathcal{T}_{n0}$ for $\xi = \pm 1$.

Observe that the $\gamma$ dependence has now disappeared from the analysis, highlighting the fact that $\mathcal{T}_{n0}$ is a Lorentz invariant for boosts along $B$. This follows from the exact compensation of time dilation and photon redshifting effects when integrating over all angles; such an attractive invariance property for the total power clearly does not extend to its differential counterpart $d\mathcal{T}_{n0}/d\mu_0$.

The integral for $K_n(B)$ can be manipulated in similar fashion to that for $I_n(B)$. Since the only difference is an extra $(1 + \xi)$ factor, which translates to a $(1 + \phi)$ factor in the light of Eq. (13), it follows that $K_n(B)$ can be worked into the form of Eq. (17), but with the extra $(1 + \phi)$ factor. The subsequent developments in Sec. 2.2.2 lead quickly to the identity

$$
K_n(B) = \sum_{k=0}^{\infty} \left( \frac{-n}{k} \right) \left( J_{n+k}(z_n) + J_{n+k+1}(z_n) \right)
$$

$$
= \sum_{k=0}^{\infty} \left( \frac{-n}{k} \right) \left( 1 - \frac{k}{n} \right) J_{n+k}(z_n),
$$

for insertion into Eq. (32a). Given this form, asymptotic limits can be routinely obtained by replicating the approaches of Sec. 2.2.1. When $nB \ll 1$, it is evident that $K_n(B) \approx I_n(B)$, since then $J_{n+1}(z_n) \ll J_{n}(z_n)$, as can be inferred for $z_n \gg 1$ from Eq. (29). Furthermore, for $nB \gg 1$, the result in Eq. (43) quickly yields a leading order contribution $K_n(B) \approx \gamma(n, n)/n^n + \gamma(n + 1, n)/n^{n+1}$. Then the identity $\gamma(n + 1, n) = n \gamma(n, n) - n^n e^{-n}$ and Eq. (43) can be used to derive the limiting form as $n \rightarrow \infty$. The results are

$$
\mathcal{T}_{n0} \approx \frac{\alpha e m_e c^3}{\lambda} \left( \frac{2n^{2n+1}}{2n+1} B_n^{n+2} \right), \quad nB \ll 1,
$$

$$
\mathcal{T}_{n0} \approx \frac{\alpha e m_e c^3}{\lambda} \left( \frac{2n^{2n+1}}{2n+1} B_n^{n+2} \right), \quad nB \gg 1.
$$

While the $nB \ll 1$ case here replicates that inferred from Eq. (31) of Latal (1986), the $nB \gg 1$ result differs from the numerical result in Eq. (32a) of Latal (1986) by around 10%.

The mean cyclotron photon energy ($\langle \omega \rangle$) is just the ratio of $\mathcal{T}_{n0}$ to $\mathcal{T}_{n0}$, so that one quickly deduces from a comparison of Eq. (32) with Eqs. (26) and (28) that $\langle \omega \rangle \approx nB$ when $nB \ll 1$, as expected from classical cyclotron theory, and that $\langle \omega \rangle \approx \sqrt{2nB}$ when $nB \gg 1$. Note that Latal (1986) misses the $\sqrt{2}$ factor (see his Eq. (32b) for the $nB \gg 1$ domain) due to the improper nature of his estimates of the pertinent numerical factors in $\mathcal{T}_{n0}$ and $\mathcal{T}_{n0}$. These behaviors of $\langle \omega \rangle$ are reflected by the differential spectral form in Eq. (41), where for $nB \ll 1$ the spectrum is essentially a delta function pinned at the cyclotron harmonic energy $nB$, and where for $n \gg 1$ and $B \gg 1$ the spectrum is somewhat broad but markedly asymmetric, skewed strongly towards $\omega \approx \omega_0 \approx \sqrt{2nB}$. 
3. THE JOHNSON AND LIPPMANN FORMULATION

A complementary formulation of the cyclotron problem was provided by Daugherty & Ventura (1978, hereafter DV78) using wavefunctions derived by Johnson & Lippmann (1949). This is essentially the foremost presentation in the literature of cyclotron radiation calculations that specifically uses the Johnson & Lippmann eigenstates, hereafter referred to as JL states. These states are eigenvalues simultaneously of the Dirac Hamiltonian and the kinetic momentum operator \( \hat{\pi} = \hat{p} + e \hat{A}/c \), where \( \hat{A} \) is the vector potential, and therefore are not eigenstates of \( \hat{\pi} \). These states are eigenvalues simultaneously of the Dirac Hamiltonian eigenstates, hereafter referred to as JL states. These states are eigenvalues simultaneously of the Dirac Hamiltonian eigenstates, hereafter referred to as JL states.

First, note that the Fourier transform of the spatial portion of the interaction amplitude, basically the vertex function of Melrose and Parle (1983), but with the temporal phase factors already extracted. The initial electron wavefunction is \( u_n^{(i)}(p, a, \vec{x}) \), and can possess either \( \zeta = \pm 1 \) spin designation (note that this spin notation differs from the \( s = \pm 1 \) notation adopted by DV78); the final electron wavefunction, \( u_n^{(f)}(q, b, \vec{x}) \), is an \( n = 0 , \zeta = -1 \) ground state, so that an explicit spin label is suppressed. Expressions for these spatial parts of the Johnson & Lippmann eigenfunctions of the Dirac equation are given Eq. (10) of DV78; these do not exhibit the same charge conjugation symmetry that the Sokolov & Ternov “transverse polarization” states do. In Eq. (10), \( M_\sigma \) is one of the two “polarization matrices” in Eq. (15) of DV78, representing contracted products of polarization vectors and the Dirac \( \gamma \) matrices. Since the \( u_n^{(f)} \) and \( u_n^{(i)} \) functions possess the dimensions of (length\(^{-3/2} \)), it follows that \( V_{fi} \) and consequently \( S_{fi} \) and \( \Phi_\sigma \) are dimensionless. Observe also that the temporal integration over the wavefunction products has already been performed leading to \( \delta \) function in Eq. (39) that expresses energy conservation. Finally, note that \( a \) defines the mean location of the initial electron wavefunction perpendicular to \( \mathbf{B} \), i.e. is a counterpart to \( b \).

The squaring of the matrix elements in Eq. (38) is performed using the approach of, for example, Bjorken & Drell (1964) for handling the \( \delta \) function. The result is

\[
\Phi_\sigma \rightarrow \frac{\alpha}{2\omega} \frac{2\pi^3}{L^2} \frac{\delta(E_n - E_0 - \omega)}{E_n E_0} |V_{fi}|^2 \quad (41)
\]

in parallel with the developments leading to Eq. (19) of DV78. Integrations analogous to those in Eq. (40) have been performed analytically in DV78 for the cyclotron absorption case, resulting in the appearance of associated Laguerre functions that are characteristic of magnetized QED calculations involving either free electrons or electron propagators. Here, since specialization to the \( n' = 0 \) case is made, these functions reduce to simple combinations of power-laws and exponentials, viz.,

\[
|\Omega_{n0}|^2 = \frac{\kappa^2}{n!} e^{-\kappa} \quad , \quad \kappa = \frac{\omega^2 \sin^2 \theta}{2B} \quad . (42)
\]

in the notation of Eq. (18) of DV78. As these spatial integrations embody translational invariance along the field, they yield a \( \delta \) function expressing conservation of momentum parallel to \( \mathbf{B} \), i.e. establishing \( p = q + \omega \cos \theta \), and completing the conservation laws in Eq. (41).

A simple crossing symmetry manipulation of Eq. (16) of DV78 facilitates the evaluation of Eq. (40), and also Eq. (38), which can be compared closely with Eq. (19) of DV78. Using the identity in Eq. (42), and reverting to the \( \mu = \cos \theta \) notation, a modest amount of algebra then yields relatively compact expressions for the spin-dependent cyclotron emission rate in the Johnson & Lippmann formalism:

\[
\Gamma_{n0}^\zeta = \frac{\alpha c}{4\lambda} \int_{-1}^{1} d\mu \frac{(E_n - E_0)}{E_n (E_n - \mu E_0)} e^{-\kappa \kappa^{n-1}} \frac{d\mu}{(n-1)!} \times (E_n + 1)(E_0 + 1) \left\{ \lambda_\uparrow + \lambda_\downarrow \right\} . \quad (43)
\]
The different photon polarization and electron spin combinations yield the following contributions:

\[
\Lambda_{\parallel}^{+1} = \left\{ \left( \frac{q}{E_0 + 1} - \frac{p}{E_n + 1} \right) \mu - \frac{\omega(1 - \mu^2)}{E_n + 1} \right\}^2
\]

\[
\Lambda_{\parallel}^{-1} = 2nB \left\{ \left( \frac{q}{E_0 + 1} + \frac{p}{E_n + 1} \right) \frac{\omega(1 - \mu^2)}{2nB} - \frac{\mu}{E_n + 1} \right\}^2
\]

(44)

for the \( \parallel \) polarization, and

\[
\Lambda_{\perp}^{+1} = \left( \frac{q}{E_0 + 1} - \frac{p}{E_n + 1} \right)^2
\]

\[
\Lambda_{\perp}^{-1} = \frac{2nB}{(E_n + 1)^2}
\]

(45)

for photons of \( \perp \) polarization. It is now almost trivial to demonstrate that for the \( n = 1 \) case, Eq. (43) in concert with Eqs. (44) and (45) reduces to Eqs. (A2) and (A3) of Daugherty & Ventura (1978), when \( \zeta = +1 \) and \( \zeta = -1 \), respectively. Accordingly, Eqs. (43)-(45) serve as a generalization of the cyclotron emission derivations presented in DVT78 to arbitrary \( n \to 0 \) transitions.

The isolation of the photon polarization contributions in this development is an attribute that is absent from the exposition in the Appendix of Daugherty & Ventura (1978). Such an isolation expedites the algebraic developments that reduce this form into expressions that much more closely resemble those in the Sokolov & Ternov formulation, i.e., Eqs. 11 and 22. The analytic reduction is routine, but lengthy, involving the expedient use of several identities. Among these are Eqs. (50)-(53), which can be used to quickly derive

\[
E_n\mu - p = E_0\mu - q \quad , \quad \xi(E_n - p\mu) = E_0 - q\mu
\]

(46)

the latter of which is just a rearrangement of Eq. (40). For the \( \zeta = +1 \) case, it is just a short path to yield the equivalences

\[
\Lambda_{\parallel}^{+1} = \left\{ -\frac{\omega}{E_n + 1} \left( E_0 - 1 + q\mu \right) \right\}^2
\]

\[
\Lambda_{\perp}^{+1} = \left\{ -\frac{\omega}{E_n + 1} \left( E_0 + 1 - q\mu \right) \right\}^2
\]

(47)

and for \( \zeta = -1 \), a moderate amount of work to yield

\[
\Lambda_{\parallel}^{-1} = 2nB \left\{ \left( 1 + \frac{nB}{E_n + 1} \right) \frac{q}{E_0 + 1} - \frac{p}{E_n + 1} \right\}^2
\]

(48)

With this assembly of tools, and various algebraic manipulations, one is lead to the following final forms for the spin-dependent cyclotron rates in the Johnson & Lippmann formulation:

\[
\Gamma_{n_0}^{\zeta} = \frac{\alpha_i c}{\lambda} \frac{n^n}{(n - 1)!} \frac{B}{E_n} \int_{-1}^{1} e^{-\kappa} d\mu (1 - \xi)^{n-1} \left( 1 - \xi \right)^{n+1} X_{\zeta}
\]

(49)

where Eq. (49) gives \( \kappa \) in terms of \( \xi \), and the spin-dependent factors in the integrand are

\[
X_{+1} = \frac{2nB}{\xi(E_n + 1)} \left\{ \frac{2nB}{E_n - p\mu} - E_n (1 - \xi) \right\}
\]

\[
X_{-1} = 2 \left( 1 + 2nB - \frac{1}{\xi} \right) - X_{+1}
\]

(50)

These are clearly forms resembling the integral that Latal (1986) derived, though indicating a slightly more complicated differential distribution \( d\Gamma_{n_0}^{\zeta} / d\mu \) than for the ST case. The average of the \( X_\zeta \) factors is obviously

\[
\overline{X} = \frac{1}{2} (X_{+1} + X_{-1}) = 1 + 2nB - \frac{1}{\xi}
\]

(51)

from which it is evident that the spin-averaged cyclotron rate for this Johnson & Lippmann formulation is identical to that of the Sokolov & Ternov formalism, i.e., Eqs. (11) and (22), as should be the situation. Such degeneracy between formalisms also prevails for the power \( \Gamma_{n_0}^{\zeta \mu} \) radiated when averaged over the initial electron spins.

Note that while the spin-dependent rates are generally different for the two sets of wavefunctions, in the limit \( p \to 0 \), it is clear from Eq. (50) that \( X_\zeta \to (1 - \zeta / \varepsilon_n) \overline{X} \) so that the two formalisms become degenerate in this limit. This is a consequence of the mathematical identity of the Sokolov & Ternov and Johnson & Lippmann eigenstates in the special case of \( p = 0 \).

3.2. Lorentz Boost Characteristics and Analytics

As with the analysis in Sec. 2, it is instructive to discern the Lorentz transformation behavior of the Johnson & Lippmann rates. Such a development closely parallels that in Sec. 2.1, again writing the relevant integrals in terms of variables in the frame in which \( p = 0 \). With the identities \( E_n = \gamma \varepsilon_n \) and \( 1 / (E_n - p\mu) = (\gamma - \beta \mu_0) / \varepsilon_n \), it can be quickly established that

\[
X_{+1} = \frac{2nB}{\gamma \varepsilon_n + 1} \frac{\gamma \varepsilon_n}{\gamma \varepsilon_n (1 - \gamma^2 \varepsilon_n)} I_n(B)
\]

(52)

which can be used also for \( X_{-1} = 2X - X_{+1} \). As for the cyclotron power considerations of Sec. 2.2, since the spin-averaged integrand or \( d\Gamma_{n_0}^{\zeta \mu} / d\mu_0 \) is even in \( \mu_0 \), the term proportional to \( \mu_0 \) in Eq. (52) contributes exactly zero to the integral in Eq. (49). This simplification clearly also applies to \( X_{-1} \). It then follows that an alternative form for the rate in Eq. (49) is

\[
\Gamma_{n_0}^{\zeta} = \frac{\alpha_i c}{\lambda} \frac{n^n}{(n - 1)!} \frac{B}{\gamma \varepsilon_n} \left[ 1 - \frac{1}{\gamma^2 \varepsilon_n (1 + \gamma^2 \varepsilon_n)} \right] I_n(B)
\]

(53)

This form conveniently avails itself of the analytic developments of Sec. 2.2 so that no further reduction of integrals is necessary: the evaluation of \( I_n(B) \) as a series of Legendre functions of the second kind in Eq. (49) is immediately applicable. Such a simplification cannot be applied to the spin-dependent cyclotron powers, a point that is addressed below.

The Lorentz-transformed expression for the cyclotron rates in Eq. (50) is attractively simple, but is not as elegant as its equivalent for the Sokolov & Ternov formulation, as embodied in Eqs. (11) and (13). Here, boosts along the field do not introduce simple time dilution factors of \( 1 / \gamma \), essentially amounting to a mixing of JL spin states induced by such transformations. This less than ideal characteristic is a consequence of the fact that the JL wavefunctions are eigenstates of a somewhat convoluted spin operator that is not symmetric under charge conjugation (e.g., see Melrose & Parle 1983), nor is it manifestly covariant under boosts along the field. In contrast, the ST “transverse polarization” wavefunctions are eigenstates of the component \( \mu_z \) along \( B \) of the magnetic moment operator \( \mu = \hat{\sigma} - i \gamma \times (\hat{p} + e \hat{A}) \).
(here $\vec{\sigma}$ and $\vec{\gamma}$ are matrix vectors for the Dirac algebra), and such eigenstates exhibit both appropriate symmetry between positrons and electrons, and simple transformation properties for boosts parallel to the field.

The sensitivity of the spin-dependent cyclotron rates, and in particular their Lorentz transformation characteristics, to the choice of spin eigenfunctions is naturally expected, since the inherently relativistic effect of spin-orbit coupling is explicitly incorporated in the Dirac equation in an external field, thereby emphasizing the interpretative importance of the choice of wavefunctions. The greater elegance of the results in Sec. 2 is yet another argument in favor of usage of Sokolov & Ternov states for the cyclotron problem, and therefore for other QED processes where spin-dependent cyclotron rates are required.

Note that the differential spectrum for spin-dependent considerations can be routinely obtained in a manner similar to the derivation of Eq. (61), observing that $\sum_{\lambda}$ can be replaced by $X$ in Eq. (62), or $X_{-1} = 2X - X_{+1}$, as desired, and then using the substitution $\xi = (\varepsilon_n - 1/\varepsilon_n)/\omega - 1$.

The ratios of the JL rates in Eq. (53) to their ST counterparts are plotted in Fig. 2 being specifically given by

$$\frac{\Gamma_{n0}^\gamma_{\text{JL}}}{\Gamma_{n0}^\gamma_{\text{ST}}} = \left[ 1 - \frac{\xi(\varepsilon_n + 1)}{\varepsilon_n} \right] / \left[ 1 - \frac{\xi}{\varepsilon_n} \right].$$

In general, this ratio is less than unity for $\xi = -1$ and greater than unity for $\xi = 1$; it is also clearly a monotonically increasing function of $\xi$ for $\xi = +1$, and decreases monotonically with $\xi$ for $\xi = -1$. This ratio approaches unity for either $\gamma = 1$ (dotted line) or for $2nB \gg 1$, and in the case of $\xi = -1$ also for $2nB \ll 1$. In the limit $2nB \ll 1$, the ratio approaches $2\gamma/(\gamma + 1)$ when $\xi = 1$, the maximum departure from unity in this case. In the $2nB \sim 1$ domain, as $\gamma \rightarrow \infty$, for $\xi = -1$ the minimum in the ratio is $(4 + 2\sqrt{2})/(4 + 3\sqrt{2}) \approx 0.83$ and occurs for $nB = \frac{1}{\gamma^2}$.

The significant deviations from unity underline the importance of the choice of wavefunctions when retaining spin dependence. This becomes a profound issue for resonant Compton scattering problems (e.g. see Graziani 1993), where, as discussed above, the spin-dependent cyclotron lifetime is required to render the intermediate electron states metastable and thereby truncate the divergent resonance at the cyclotron frequency.

To conclude this section, a brief exposition of the spin-dependent powers is offered. The spin-averaged power is, of course, that in Eq. (55). To retain spin dependence in the JL formalism, one proceeds by weighting the integrand of Eq. (49) with a $\omega = \frac{2nB}{(1 + \xi)/(\varepsilon_n - \mu_0)}$ factor. Guided by the analysis in Sec. 2.2.3, terms that are odd in $\mu_0$ contribute zero, and the by now routine manipulations yield the form

$$P_{n0}^\gamma = \frac{\alpha_e m_e c^3}{\lambda} \frac{n^{n+1}}{(n-1)!} B^2 \varepsilon_n^2$$

$$\times \left\{ 1 - \frac{\xi(\varepsilon_n + 1)}{\varepsilon_n(\gamma \varepsilon_n + 1)} \right\} K_n(B) + \xi \frac{2\gamma^2(2nB)^3}{(\varepsilon_n(\gamma \varepsilon_n + 1)) L_n(B)},$$

where $K_n(B)$ is the integral in Eq. (54), or equivalently, the series in Eq. (55), and the $L_n(B)$ term emerges from a $\mu_0$ term in the integrand resulting from the product $\omega X^\gamma$, and corresponds to

$$L_n(B) = \sum_{k=0}^{\infty} \frac{(-n)^k}{n+k+1} J_{n+k}(\varepsilon_n).$$

Eq. (55) is clearly not as elegant as the corresponding ST result, namely $P_{n0}^\gamma = (1 - \xi/\varepsilon_n)P_{n0}^\gamma$, but does reduce to this simpler form in the limit $\beta \rightarrow 0$, as expected. Furthermore, averaging Eq. (55) over spins $\xi = \pm 1$ then yields exactly $P_{n0}^\gamma$, so that degeneracy between the JL and ST formalisms is realized when spin information is not explicitly retained. Note that asymptotic limits of $P_{n0}^\gamma$ for small and large $nB$ can be readily obtained but are not particularly elucidating for the purposes of this exposition.

4. CONCLUSION

This paper presents useful analytic developments of spin-dependent cyclotron decay rates/widths for general $n \rightarrow 0$ transitions between Landau states, expressing these in terms of various series of Legendre functions $Q_n^\pm(z)$ of the second kind. The rates and the radiated cyclotron powers are derived using two popular eigenstate formalisms appropriate for the Dirac equation in a uniform magnetic field, namely that due to Sokolov & Ternov (1968), and that due to Johnson & Lippmann (1949). The resulting expressions are Eq. (63) for the cyclotron rate and Eq. (65) in conjunction with Eq. (55) for the Sokolov & Ternov formulation, and Eq. (66) for the rate and Eq. (65) for the power.
for the Johnson & Lippmann case. These compact forms can be expeditiously used in astrophysical models, since they generally amount to rapidly convergent series of elementary functions. These spin-dependent results are particularly important for addressing resonant cyclotron divergences that appear in various higher-order QED mechanisms in external magnetic fields. The Johnson & Lippmann developments extend the work of Daugherty & Ventura (1978) to arbitrary \( n \to 0 \) transitions, while the series development permits a more accurate determination of \( nB \gg 1 \) asymptotics of the Sokolov & Ternov rates than that offered in Latal (1986).

In addition, the Lorentz transformation characteristics of the rates are derived for boosts along the magnetic field, yielding a simple time dilation multiplicative factor in the ST case, but a slightly more complicated and less ideal dependence on boost Lorentz factor for the JL analysis, where spin-state mixing is implied by such boosts. Such comparative complexity of the JL formalism is more pronounced for the radiated power. This comparison again underlines the superior attributes of Sokolov & Ternov eigenstates for use in evaluating rates and cross sections of magnetized processes in quantum electrodynamics.

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APPENDIX

PROPERTIES INVOLVING THE ASSOCIATED LEGENDRE FUNCTIONS OF THE SECOND KIND, \( J_\nu(z) \)

This Appendix provides several identities germane to the developments in the text, principally in relation to the Legendre functions of the second kind, \( Q_\nu(z) \), which form the basis of the series expansions for the cyclotron rates. They can be defined in terms of hypergeometric functions (e.g. see Eq. 8.820.2 of Gradshteyn & Ryzhik 1980), or alternatively the integral representation in Eq. (21). For integer indices \( \nu = n \), the specialization of interest here, they are simple combinations of logarithmic and polynomial functions (e.g. see Abramowitz & Stegun 1965), that lead to similar simplicity for the \( J_\nu(z) \) using Eq. (22). Observe that \( Q_\nu(z) \equiv Q_\nu^0(z) \), where the associated Legendre functions \( Q_\nu^0(z) \) are defined in 8.703 of Gradshteyn & Ryzhik (1980). Note also the alternative identities

\[
J_\nu(z) = -(\nu + 1) \sqrt{z^2 - 1} Q_\nu^{-1}(z) = (\nu + 1) \int_z^\infty Q_\nu(t) \, dt , \quad z > 1 ,
\]

(A1)

embodied in the results 8.734.3 and 8.752.5 in Gradshteyn & Ryzhik (1980). Explicit functional forms for some of the \( Q_n(z) \) are given in Abramowitz & Stegun (1965) and Eq. 8.827 of Gradshteyn & Ryzhik (1980), from which the following identities for the first few \( J_n(z) \) can be formed:

\[
\begin{align*}
J_1(z) &= z - \frac{z^2 - 1}{2} \log|\frac{z + 1}{z - 1}| , \\
J_2(z) &= -1 + \frac{3z^2}{2} - \frac{3z(z^2 - 1)}{4} \log|\frac{z + 1}{z - 1}| , \\
J_3(z) &= -\frac{13z}{6} + \frac{5z^3}{2} - \frac{(5z^2 - 1)(z^2 - 1)}{4} \log|\frac{z + 1}{z - 1}| .
\end{align*}
\]

(A2)

An effective way to derive expressions for the \( J_n(z) \) in terms of elementary functions, at least for low \( n \), is to use the Rodrigues formula in 8.836.1 of Gradshteyn & Ryzhik (1980), and then invoke Eq. (22). An alternative approach is to use identities 8.831.2 and 8.831.3 of Gradshteyn & Ryzhik (1980), to arrive at a relation between \( Q_n(z) \) and a sum of Legendre polynomials \( P_n(z) \) and a logarithmic term. This quickly yields

\[
J_n(z) = Q_n(z) - z Q_n(z)
\]

(A3)

\[
= \frac{1}{2} \left[P_{n-1}(z) - z P_n(z)\right] \log\left|\frac{z + 1}{z - 1}\right| - \sum_{k=1}^{n-1} \frac{1}{k} P_{k-1}(z) P_{n-1-k}(z) + z \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(z) P_{n-k}(z) ,
\]

a form that clearly displays the combined logarithmic and polynomial character of these functions, and which is useful for asymptotic results, explored just below.

There are various possible approaches to numerical calculations of the \( J_n \) functions for integer indices. Options include direct use of Eq. (A3), combined with an efficient algorithm for computing the Legendre polynomials for arguments \( z > 1 \), such as invoking recurrence relations (e.g. see Eq. 8.5.3 of Abramowitz & Stegun 1965), which are numerically accurate for upward recurrence in the polynomial index \( n \). Alternatively, direct recurrence of the \( Q_n(z) \) functions, which satisfy the same relations as the \( P_n(z) \), is viable, though this is only stable for downward iterations in \( n \) when \( z > 1 \). This approach therefore requires evaluation of the \( P_n(z) \) for two particular initial values \( n \), perhaps via the hypergeometric function series using the identity in Eq. 8.1.5 of
Abramowitz & Stegun (1965). However, probably the most expedient numerical technique is to use the recurrence relation for the $J_n(z)$ functions directly:

$$n(n + 1)J_{n+1}(z) - n(2n + 1)z J_n(z) + (n^2 - 1)J_{n-1}(z) = 0.$$  

(A4)

This identity can be deduced after a modicum of algebra from Eq. 8.5.3 of Abramowitz & Stegun (1965). Note that strong cancellation often arises between the last two terms of this equation, so that for large $n$ and $z > 1$, the values of $J_n(z)$ decline exponentially with $n$ according to Eq. (A2). Starting with initial values prescribed by Eq. (A2), this technique is stable to upward recurrence in $n$ for any $z > 1$, the range of arguments appropriate for the cyclotron problem. This quickly yields suitably precise values (numerically tested for $n \leq 100$) for a range of indices required in the series summations that appear in the cyclotron rates and powers, for example in Eqs. (28) and (35). The use of this recurrence becomes superfluous typically for $n > 30$, since such indices are generally required only in the classical cyclotron limit when $z_n \gg 1$, and the asymptotic limit in Eq. (25) is then the preferred computational tool.

Now focusing on asymptotic issues, two limiting values of $J_n(z)$ are required for the analysis in Sec. 2.2. The $z \gg 1$ limit is routinely derived, and is listed in Eq. (25). Since $z \geq 1$, the other limiting domain is $z \rightarrow 1$. This is a somewhat more involved case. By inspection of Eq. (A3), as $z \rightarrow 1$, the logarithmic terms approach zero since $P_n(z) \rightarrow 1 + n(n+1)(z-1)/2 + O(z-1)^2$. The Legendre polynomials thus simplify to approximately unity, to leading order, and it follows that in the neighborhood of $z = 1$,

$$J_n(z) = \frac{1}{n} - \frac{n+1}{2}(z-1) \log \frac{2}{z-1} + O(z-1).$$  

(A5)

defines the leading order terms to the Taylor series expansion. Insertion in Eq. (19) then yields

$$I_n(B) \approx \sum_{k=0}^{\infty} \frac{(-n)^k}{k!} \left\{ \frac{1}{n+k} - \frac{n+k+1}{2} (z_n - 1) \log \frac{2}{z_n - 1} \right\} = \frac{\gamma(n, n)}{n^k} - e^{-n} \left( \frac{z_n - 1}{2} \right) \log \frac{2}{z_n - 1},$$

(A6)
correct to order $O((|z-1| \log |z-1|))$. Here, $\gamma(n, x)$ is the incomplete Gamma function, and can be introduced using the series identity 8.354.1 of Gradshteyn & Ryzhik (1980).

Another identity required for the development of Eq. (28) pertains to the evaluation of $\gamma(n, n)$ for large $n$. For low values of $n$, say $n \leq 20$, one can use 8.352.1 of Gradshteyn & Ryzhik (1980), namely

$$\gamma(n, n) = 1 - \frac{\Gamma(n, n)}{\Gamma(n)} = 1 - e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!},$$

(A7)
as an efficient series evaluation for either the $\gamma(n, n)$ or $\Gamma(n, n)$ incomplete Gamma functions. For larger $n$, one can appeal to the identity $\Gamma(n+1, n) = n \Gamma(n, n) + n^2 e^{-n}$, together with Stirling’s expansion $\Gamma(n) \approx e^{-n} n^{n-1/2} \sqrt{2\pi}$ and identity 6.5.25 of Abramowitz & Stegun (1965), namely $\Gamma(n+1, n) \approx e^{-n} n^{n\sqrt{n}/2}$, to establish the results

$$\lim_{n \rightarrow \infty} \frac{\gamma(n, n)}{\Gamma(n)} = \frac{1}{2} = \lim_{n \rightarrow \infty} \frac{\Gamma(n, n)}{\Gamma(n)}.$$  

(A8)

Retaining the next order contributions to $\gamma(n, n)$ and $\Gamma(n, n)$ then yields corrections of approximately $\pm 1/(3\sqrt{2\pi})$ to the two ratios in Eq. (A8).

REFERENCES

Exton, H. 1976, Multiple Hypergeometric Functions and Applications, (Ellis Horwood Ltd., Chichester).