PP-waves with torsion and metric-affine gravity

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Abstract. A classical pp-wave is a 4-dimensional Lorentzian spacetime which admits a nonvanishing parallel spinor field; here the connection is assumed to be Levi-Civita. We generalise this definition to metric compatible spacetimes with torsion and describe basic properties of such spacetimes. We use our generalised pp-waves for constructing new explicit vacuum solutions of quadratic metric-affine gravity.

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1. Introduction

We consider spacetime to be a connected real 4-manifold $M$ equipped with a Lorentzian metric $g$ and an affine connection $\Gamma$. The $10$ independent components of the (symmetric) metric tensor $g_{\mu\nu}$ and the $64$ connection coefficients $\Gamma^\lambda_{\mu\nu}$ are the unknowns of our theory. This approach is known as metric-affine gravity [1].

We define our action as

$$S := \int q(R)$$

where $q$ is an $O(1,3)$-invariant quadratic form on curvature $R$. Independent variation of the metric $g$ and the connection $\Gamma$ produces Euler–Lagrange equations which we will write symbolically as

$$\frac{\partial S}{\partial g} = 0,$$  \hspace{1cm} \hspace{1cm} (2)

$$\frac{\partial S}{\partial \Gamma} = 0.$$ \hspace{1cm} \hspace{1cm} (3)

Our objective is the study of the combined system of field equations (2), (3). This is a system of $10+64$ real nonlinear partial differential equations with $10+64$ real unknowns.

The motivation for choosing a model of gravity which is purely quadratic in curvature is explained in Section 1 of [2]. Basically, we are hoping to describe physical phenomena whose characteristic wavelength is sufficiently small and curvature sufficiently large. Also, the choice of action which is homogeneous (of degree 2) in curvature means that we are looking for vacuum solutions.
The Yang–Mills action for the affine connection is a special case of (1) with
\[ q(R) = q_{\text{YM}}(R) := R^\kappa_{\lambda \mu \nu} R^\lambda_{\kappa \mu \nu}. \] (4)
With this choice of \( q \) equation (3) is the Yang–Mills equation for the affine connection.

The quadratic form \( q \) appearing in (1) is a generalisation of (4). The general formula for \( q \) contains 16 different \( R^2 \)-terms with 16 coupling constants. This formula is given in Appendix B of [2]. An equivalent formula can be found in [3, 4].

**Definition 1** We call a spacetime \( \{M, g, \Gamma\} \) Riemannian if the connection is Levi-Civita (i.e. \( \Gamma_{\lambda \mu \nu} = \left\{ \lambda_{\mu \nu} \right\} \)), and non-Riemannian otherwise.

The aim of this paper is to find new non-Riemannian solutions of the field equations (2), (3). These new solutions will be constructed explicitly and the construction will turn out to be very similar to the classical construction of a pp-wave, only with torsion. In fact, the generalisation of the concept of a pp-wave to spacetimes with torsion is the main tool in our analysis and a useful spin-off which might be of wider differential geometric interest.

The paper has the following structure. In Section 3 we recall basic facts concerning classical pp-waves (without torsion). In Section 4 we define the notion of a generalised pp-wave (with torsion) and list the main properties of such spacetimes. In Section 5 we write down explicitly our field equations (2), (3) and in Section 6 we present pp-wave solutions of these field equations. Theorem 1 of Section 6 is the main result of our paper. We discuss our results in Section 7. Finally, Appendix A and Appendix B contain some auxiliary mathematical facts.

2. Notation

Our notation follows [3, 6, 2]. In particular, we denote local coordinates by \( x^\mu \), \( \mu = 0, 1, 2, 3 \), and write \( \partial_\mu := \partial/\partial x^\mu \). We define the covariant derivative of a vector field as \( \nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda_{\mu \nu} v^\nu \) and torsion as \( T^\lambda_{\mu \nu} := \Gamma^\lambda_{\mu \nu} - \Gamma^\lambda_{\mu \nu} \). We say that our connection \( \Gamma \) is metric compatible if \( \nabla g \equiv 0 \). The Christoffel symbol is \( \left\{ \lambda_{\mu \nu} \right\} := \frac{1}{2} g^{\kappa \nu} (\partial_\mu g_{\kappa \nu} + \partial_\nu g_{\mu \kappa} - \partial_\kappa g_{\mu \nu}) \). The interval is \( ds^2 := g_{\mu \nu} dx^\mu dx^\nu \).

We define curvature as \( R^\kappa_{\lambda \mu \nu} := \partial_\mu \Gamma^\kappa_{\nu \lambda} - \partial_\nu \Gamma^\kappa_{\mu \lambda} + \Gamma^\kappa_{\mu \eta} \Gamma^\eta_{\nu \lambda} - \Gamma^\kappa_{\nu \eta} \Gamma^\eta_{\mu \lambda} \), Ricci curvature as \( Ric_{\lambda \nu} := R^\kappa_{\lambda \kappa \nu} \), scalar curvature as \( \mathcal{R} := Ric^\lambda_{\lambda} \), and trace-free Ricci curvature as \( Ric := Ric - \frac{1}{4} \mathcal{R} g \). We denote Weyl curvature by \( \mathcal{W} \); here, as in [6, 2], Weyl curvature is understood as the irreducible piece of curvature defined by conditions (20), (21) and \( Ric = 0 \).

We employ the standard convention of raising and lowering tensor indices by means of the metric tensor. Some care is, however, required when performing covariant differentiation: the operations of raising and lowering of indices do not commute with the operation of covariant differentiation unless the connection is metric compatible.

Given a scalar function \( f : M \rightarrow \mathbb{R} \) we write for brevity
\[ \int f := \int_M f \sqrt{\det g} dx^0 dx^1 dx^2 dx^3, \quad \det g := \det(g_{\mu \nu}). \]
We define the action of the Hodge star on a rank $q$ antisymmetric tensor as

$$(\ast Q)_{\mu q+1...\mu 4} := (q!)^{-1} \sqrt{\det g} Q^{\mu_1...\mu_q} \varepsilon_{\mu_1...\mu_4}$$

where $\varepsilon$ is the totally antisymmetric quantity, $\varepsilon_{0123} := +1$. When we apply the Hodge star to curvature we have a choice between acting either on the first or the second pair of indices, so we introduce two different Hodge stars: the left Hodge star

$$(\ast R)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{\det g} R^\kappa_{\lambda\mu\nu} \varepsilon_{\kappa\lambda\kappa\lambda}$$

and the right Hodge star

$$(R^*)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{\det g} R_{\kappa\lambda}^{\mu\nu} \varepsilon_{\mu\nu\mu\nu}.$$ 

Note that in the general metric-affine setting curvature is not necessarily antisymmetric in the first pair of indices so use of the left Hodge star really makes sense only in metric compatible spacetimes.

We use the term “parallel” to describe the situation when the covariant derivative of some spinor or tensor field is identically zero.

We do not assume that our spacetime admits a (global) spin structure, cf. Section 11.6 of [7]. In fact, our only topological assumption is connectedness. This does not prevent us from defining and parallel transporting spinors or tensors locally.

3. Classical pp-waves

In this section spacetime is assumed to be Riemannian, see Definition 1.

**Definition 2** A pp-space is a Riemannian spacetime which admits a nonvanishing parallel spinor field.

We will call the metric of a pp-space metric of pp-wave or simply pp-metric. Such metrics were introduced by Peres [8, 9] who used the equivalent Definition 5 given further on in this section.

Throughout this paper we denote the nonvanishing parallel spinor field by $\chi = \chi^a$ and assume that this spinor field is fixed. Note that

- a nonvanishing parallel spinor can be scaled by a nonzero complex factor (there is no natural normalisation), and
- in flat space there are two linearly independent nonvanishing parallel spinor fields.

Fixation of the spinor field $\chi$ allows us to avoid ambiguity in subsequent arguments.

Put

$$l^a := \sigma^a_{ab} \chi^a \chi^b$$

where the $\sigma^a$ are Pauli matrices, see Appendix A for notation. Then $l$ is a nonvanishing parallel real null vector field. Define also the real scalar function

$$\varphi : M \rightarrow \mathbb{R}, \quad \varphi(x) := \int l \cdot dx.$$
This function is called the phase. It is defined uniquely up to the addition of a constant and possible multivaluedness resulting from a nontrivial topology of the manifold.

The 3-manifolds $\tilde{M} = \{ \phi = \text{const} \}$ are called wave fronts. Let us fix a particular wave front $\tilde{M}$, take a pair of points $\tilde{p}, \tilde{q} \in \tilde{M}$, and a curve $\tilde{c} \subset \tilde{M}$ connecting these points. Take a 4-vector tangent to $\tilde{M}$ at $\tilde{p}$ and parallel transport it in accordance with the Levi-Civita connection along $\tilde{c}$. It is easy to see that the resulting 4-vector will be tangent to $\tilde{M}$ at $\tilde{q}$. This means that the Levi-Civita connection $\Gamma$ over $TM$ admits a natural restriction to a connection $\tilde{\Gamma}$ over $T\tilde{M}$. (The latter cannot be interpreted as the Levi-Civita connection corresponding to the restriction of our Lorentzian 4-metric to the 3-manifold $\tilde{M}$ as this restricted metric is degenerate.) An important property of pp-spaces is that the connection $\tilde{\Gamma}$ is flat. This is why pp-spaces are often called “plane-fronted gravitational waves with parallel rays”.

The fact that the wave fronts are flat motivates the following definitions.

**Definition 3** We say that a complex vector field $u$ is transversal if $l_{\alpha} u^\alpha = 0$.

**Definition 4** We say that a complex vector field $v$ is a plane wave if $u^\alpha \nabla_\alpha v^\beta = 0$ for any transversal vector field $u$.

Of course, $l$ itself is transversal and a plane wave.

Put

$$F_{\alpha\beta} := \sigma_{\alpha\beta ab} \chi^a \chi^b$$

where the $\sigma_{\alpha\beta}$ are “second order Pauli matrices” (A.5). Then $F$ is a nonvanishing parallel complex 2-form with the additional properties $*F = \pm iF$ and $\det F = 0$. It can be written as

$$F = l \wedge a$$

where $a$ is a complex vector field satisfying $a_{\alpha} a^\alpha = l_{\alpha} a^\alpha = 0$, $a_{\alpha} \bar{a}^\alpha = -2$. The vector field $a$ is defined uniquely up to the addition of

$$\{ \text{arbitrary complex valued scalar function} \} \times l.$$

We can impose an additional restriction on our choice of $a$ requiring that $a$ be a plane wave. Under this restriction the vector field $a$ is defined uniquely up to the addition of

$$\{ \text{arbitrary complex valued scalar function of } \varphi \} \times l$$

and

$$\nabla_{\alpha} a_{\beta} = p l_{\alpha} l_{\beta}.$$  

where $p : M \rightarrow \mathbb{C}$ is some scalar function.

Throughout this paper our choice of the vector field $a$ is assumed to be fixed. This implies, in particular, that the function $p$ appearing in (9) is fixed.

It is known, see Section 4 in [10] or Section 3.2.2 in [11], that Definition 2 is equivalent to the following
**Definition 5** A pp-space is a Riemannian spacetime whose metric can be written locally in the form

$$ds^2 = 2dx^0 dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3)(dx^3)^2$$ (10)

in some local coordinates $(x^0, x^1, x^2, x^3)$.

We do all our practical calculations in coordinates (10) and with Pauli matrices (A.4). Of course, the choice of local coordinates in which the pp-metric assumes the form (10) is not unique. We will restrict our choice to those coordinates in which

$$\chi^a = (1, 0), \quad l^\mu = (1, 0, 0, 0), \quad a^\mu = (0, 1, \mp i, 0).$$ (11)

With such a choice formula (6) reads

$$\varphi(x) = x^3 + \text{const.}$$

The remarkable property of the metric (10) is that the corresponding curvature tensor $R$ is linear in $f$:

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2} (l \wedge \partial)_{\alpha\beta} (l \wedge \partial)_{\gamma\delta} f$$ (12)

where $(l \wedge \partial)_{\alpha\beta} := l_\alpha \partial_\beta - \partial_\alpha l_\beta$. Simplicity of the formula for curvature was the main motivation for Peres when he introduced [8, 9] the concept of a pp-space (pp-wave).

Observe now that in our special local coordinates $f$ satisfies the equations

$$l^\mu \partial_\mu f = 0, \quad a^\mu \partial_\mu f = \frac{p}{2}$$ (13)

where $p$ is the function from (9). Equations (13) are invariantly defined equations for a scalar function $f : M \to \mathbb{R}$. These equations allow us to give an invariant interpretation of our function $f$ as a potential for a pp-space. Equations (13) specify the gradient of $f$ along wave fronts, and, consequently, they define $f$ uniquely up to the addition of an arbitrary real valued scalar function of $\varphi$.

Formula (12) can now be rewritten in invariant form

$$R = -\frac{1}{2} (l \wedge \nabla) \otimes (l \wedge \nabla) f$$ (14)

where $l \wedge \nabla := l \otimes \nabla - \nabla \otimes l$. Indeed, in our special local coordinates all the terms with connection coefficients in the RHS of (14) cancel out and (14) turns into (12). As both sides of (14) are tensors formula (14) holds in any coordinate system.

It is easy to see that the curvature of a pp-space has only two irreducible pieces, trace-free Ricci and Weyl. Ricci curvature is proportional to $l \otimes l$ whereas Weyl curvature is a linear combination of $\text{Re} ((l \wedge a) \otimes (l \wedge a))$ and $\text{Im} ((l \wedge a) \otimes (l \wedge a))$.

4. PP-waves with torsion

The most natural way of generalising the concept of a classical pp-space is simply to extend Definition 2 to general metric compatible spacetimes, i.e. spacetimes whose connection is not necessarily Levi-Civita. However, this gives a class of spacetimes which is too wide and difficult to work with. We choose to extend the classical definition in a more special way better suited to the study of the system of field equations (2), (3).
Consider the polarized Maxwell equation
\[ \star dA = \pm idA \] (15)
in a classical pp-space, see Section 3. Here \( A \) is the unknown complex vector field. We seek plane wave solutions of (15), see Definition 4. These can be written down explicitly:
\[ A = h(\varphi)a + k(\varphi)l. \] (16)
Here \( l \) and \( a \) are the vector fields defined in Section 3. \( h, k : \mathbb{R} \to \mathbb{C} \) are arbitrary functions, and \( \varphi \) is the phase (6).

**Definition 6** A generalised pp-space is a metric compatible spacetime with pp-metric and torsion
\[ T := \frac{1}{2} \text{Re}(A \otimes dA) \] (17)
where \( A \) is a vector field of the form (17).

We list below the main properties of generalised pp-spaces. Here and further on we denote by \( \{\nabla\} \) the covariant derivative with respect to the Levi-Civita connection which should not be confused with the full covariant derivative \( \nabla \) incorporating torsion.

In the beginning of Section 3 we introduced the spinor field \( \chi \) satisfying \( \{\nabla\}\chi = 0 \). It turns out that this spinor field also satisfies \( \nabla\chi = 0 \). In other words, the generalised pp-space and underlying classical pp-space admit the same nonvanishing parallel spinor field. Consequently, both admit the same nonvanishing parallel real null vector field \( l \) and the same nonvanishing parallel complex 2-form \( l \wedge a \).

The torsion of a generalised pp-space is purely tensor, i.e.
\[ T^\alpha_{\alpha\gamma} = 0, \quad \varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} = 0. \] (18)

The curvature of a generalised pp-space is
\[ R = -\frac{1}{2} (l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f + \frac{1}{4} \text{Re} \left((h^2)'' (l \wedge a) \otimes (l \wedge a)\right). \] (19)
Examination of formula (19) reveals the following remarkable properties of generalised pp-spaces.

- The curvatures generated by the Levi-Civita connection and torsion simply add up (compare formulae (14) and (19)).

- The second term in the RHS of (19) is purely Weyl. Consequently, the Ricci curvature of a generalised pp-space is completely determined by the pp-metric.

- The curvature of a generalised pp-space has all the usual symmetries of curvature in the Riemannian case (see Definition 1), that is,
\[ R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \] (20)
\[ \varepsilon^{\kappa\lambda\mu}\nu R_{\kappa\lambda\mu\nu} = 0, \] (21)
\[ R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}, \] (22)
\[ R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}. \] (23)

Of course, (23) is true for any curvature whereas (22) is a consequence of metric compatibility. Also, (22) follows from (20) and (23).
The second term in the RHS of (16) is pure gauge in the sense that it does not affect curvature (19). It does, however, affect torsion (17).

The Ricci curvature of a generalised pp-space is zero if and only if
\[ f_{11} + f_{22} = 0 \]  
and the Weyl curvature is zero if and only if
\[ f_{11} - f_{22} = \text{Re} \left((h^2)^\nu\right), \quad f_{12} = -\frac{1}{2} \text{Im} \left((h^2)^\nu\right). \]

Here we use special local coordinates (10), (11) and denote \( f_{\alpha\beta} := \partial_\alpha \partial_\beta f \).

The curvature of a generalised pp-space is zero if and only if we have both (24) and (25). Clearly, for any given function \( h \) we can choose a function \( f \) such that \( R = 0 \); this \( f \) is a quadratic polynomial in \( x^1, x^2 \) with coefficients depending on \( x^3 \). Thus, as a spin-off, we get a class of examples of Weitzenböck spaces \((T \neq 0, R = 0)\).

5. Explicit representation of our field equations

We write down explicitly our field equations (2), (3) under the following assumptions.

• Our spacetime is metric compatible.
• Torsion is purely tensor, see (18).
• Curvature has symmetries (20), (21).
• Scalar curvature is zero.

Note that a generalised pp-space automatically possesses these properties.

It turns out that under the above assumptions the field equations are

\begin{align*}
    d_1 \mathcal{W}^{\kappa\mu\nu} R_{\kappa\mu\nu} + d_3 \left( R_{\kappa\lambda} R_{\kappa\nu} - \frac{1}{4} g^{\kappa\nu} R_{\kappa\mu\lambda} R^{\kappa\mu} \right) &= 0, \quad (26) \\
    d_6 \nabla_{\lambda} R_{\kappa\mu\nu} - d_7 \nabla_{\kappa} R_{\lambda\mu\nu} &= 0, \quad (27)
\end{align*}

where

\begin{align*}
    d_1 &= b_{912} - b_{921} + b_{10}, \\
    d_3 &= b_{922} - b_{911}, \\
    d_6 &= b_{912} - b_{911} + b_{10}, \\
    d_7 &= b_{912} - b_{922} + b_{10},
\end{align*}

the \( b \)'s being coefficients from formula (51) of [2]. The LHS's of equations (26) and (27) are the components of tensors \( A \) and \( B \) from the formula

\[ \delta S = \int \left( 2 A^{\lambda\nu} \delta g_{\lambda\nu} + 2 B^{\kappa\mu\lambda} \delta \Gamma_{\mu\kappa}^{\lambda} \right). \]
Here $\delta g$ and $\delta \Gamma$ are the (independent) variations of the metric and the connection, and $\delta S$ is the resulting variation of the action. In (27) the first two indices of $B$ have been lowered to make the expression easier to read.

Equation (26) is equation (12) of [2] but with $R = 0$. This is not surprising because when we vary the metric it does not matter whether the curvature tensor $R^{\kappa}_{\lambda\mu\nu}$ was generated by a Levi-Civita connection or a general affine connection. What matters are the symmetries (20), (21) which in our case are the same as in the Riemannian case. In fact, our case is simpler because scalar curvature is zero.

Equation (27) is similar to equation (13) of [2] but is not exactly the same. Namely,

- the first line of (27) coincides with the LHS of equation (13) of [2] with $R \equiv 0$,
- the remaining lines of (27) contain extra algebraic terms generated by torsion.

Note also that the covariant derivatives in (27) and in equation (13) of [2] are different: we use the notation $\nabla$ for the full covariant derivative, so the $\nabla$ in (27) itself incorporates torsion. The arguments which produce (27) are outlined in Appendix B.

6. PP-wave type solutions of our field equations

The main result of this paper is the following

**Theorem 1** Generalised pp-spaces of parallel Ricci curvature are solutions of the system of field equations (2), (3).

**Proof** The theorem is proved by direct substitution of formulae for torsion, Ricci curvature and Weyl curvature of a generalised pp-space into the field equations (26), (27). The $\nabla Ric$ terms in the LHS of (27) vanish as Ricci curvature is assumed to be parallel, so it remains to check the vanishing of the remaining purely algebraic terms in the LHS’s of (26), (27).

According to Section 4, torsion, Ricci curvature and Weyl curvature of a generalised pp-space are of the form

\[ T = \sum_{j,k=1}^{2} t_{jk} a_j \otimes (l \wedge a_k) + \sum_{j=1}^{2} t_j l \otimes (l \wedge a_j), \]
\[ Ric = s l \otimes l, \]
\[ W = \sum_{j,k=1}^{2} w_{jk} (l \wedge a_j) \otimes (l \wedge a_k), \]

where $t_{jk}$, $t_j$, $s$, $w_{jk}$ are some real scalars satisfying

\[ t_{jk} = t_{kj}, \quad w_{jk} = w_{kj}, \quad t_{11} + t_{22} = w_{11} + w_{22} = 0, \]

$l$ and $a$ are vectors introduced in Section 3 and $a_1 = \text{Re}a$, $a_2 = \text{Im}a$. Note that the real vectors $l$, $a_1$, $a_2$ satisfy

\[ l \cdot l = l \cdot a_1 = l \cdot a_2 = a_1 \cdot a_2 = 0, \quad a_1 \cdot a_1 = a_2 \cdot a_2 = -1. \]
All the algebraic terms containing $Ric$ in the LHS’s of (26), (27) vanish because they involve contractions with at least one of the indices of $Ric$, the latter being of the form (29) with vector $l$ orthogonal to all other vectors appearing in (28)–(30). It remains to consider the $\mathcal{W} \times T$ terms in the LHS of (27). The terms with 3 contractions vanish because in view of (28) at least one of the contractions involves the vector $l$. The term $\mathcal{W}^{\kappa \lambda \mu \nu \xi \eta} l_{\kappa} l_{\lambda} l_{\mu}$ also vanishes because in view of (30) at least one of the contractions involves the vector $l$. Thus, the proof of Theorem 1 reduces to checking that

$$\mathcal{W}^{\eta}_{\mu \nu \xi} \left( T_{\eta}^{\xi} \lambda - T_{\lambda}^{\xi} \eta \right) + \mathcal{W}^{\eta}_{\mu \lambda \xi} \left( T_{\kappa}^{\xi} \eta - T_{\eta}^{\xi} \kappa \right) = 0.$$  \hspace{1cm} (31)

The tensor in the LHS of (31) is proportional to $l_{\lambda} l_{\mu} l_{\kappa}$ and is antisymmetric in $\kappa$, $\lambda$, hence it is zero. □

Let $\{Ric\}$ denote the Ricci curvature generated by the Levi-Civita connection and let $\{\nabla\}$ denote, as usual, the covariant derivative with respect to the Levi-Civita connection. We know (see list of properties at the end of Section 4) that in a generalised pp-space $Ric = \{Ric\}$. Moreover, it is easy to see that in a generalised pp-space $\nabla Ric = \{\nabla\} Ric$. This means that when using Theorem 1 it does not really matter whether the condition “parallel Ricci curvature” is understood in the non-Riemannian sense $\nabla Ric = 0$, the Riemannian sense $\{\nabla\} \{Ric\} = 0$, or any combination of the two ($\{\nabla\} Ric = 0$ or $\nabla \{Ric\} = 0$). In special local coordinates (10), (11) the condition that Ricci curvature is parallel is written as $f_{11} + f_{22} = \text{const}$ (compare with (24)).

7. Discussion

7.1. Interpretation of our solutions

Our interest in pp-spaces, classical and generalised, stems from our previous publication [2]. It contained a comprehensive study of Riemannian (see Definition 1) solutions of the field equations (2), (3). It was shown in [2] that the following two classes of Riemannian spacetimes are solutions:

- Einstein spaces ($Ric = \Lambda g$), and
- classical pp-spaces of parallel Ricci curvature.

Moreover, it was shown in [2] that for a generic quadratic action the above two classes of spacetimes are the only Riemannian solutions.

In General Relativity Einstein spaces are an accepted mathematical model for vacuum. However, classical pp-spaces of parallel Ricci curvature do not have an obvious physical interpretation. Our current paper is an attempt at understanding whether such spacetimes are of mathematical or physical significance.

Our analysis of vacuum solutions of quadratic metric-affine gravity shows (Theorem 1) that classical pp-spaces of parallel Ricci curvature should not be viewed on their own. They are a particular (degenerate) representative of a wider class of solutions, namely, generalised pp-spaces of parallel Ricci curvature. The latter appear to admit
a sensible physical interpretation. Indeed, according to formula (19) the curvature of a generalised pp-space is a sum of two curvatures: the curvature
\[ -\frac{1}{2}(l \land \{\nabla\}) \otimes (l \land \{\nabla\})f \]
of the underlying classical pp-space and the curvature
\[ \frac{1}{4}\text{Re}\left((h^2)'' (l \land a) \otimes (l \land a)\right) \]
generated by a torsion wave travelling over this classical pp-space. Our torsion (16) and corresponding curvature (33) are waves travelling at speed of light because \( h \) and \( k \) are functions of the phase \( \varphi \) which plays the role of a null coordinate, \( g^{\mu\nu}\nabla_\mu\varphi \nabla_\nu\varphi = 0 \), see formula (6). The underlying classical pp-space of parallel Ricci curvature can now be viewed as the “gravitational imprint” created by a wave of some massless matter field. Such a situation occurs in Einstein–Maxwell theory (classical model describing the interaction of gravitational and electromagnetic fields) and Einstein–Weyl theory (classical model describing the interaction of gravitational and neutrino fields). The difference with our model is that Einstein–Maxwell and Einstein–Weyl theories contain the gravitational constant which dictates a particular relationship between the strengths of the fields in question, whereas our model is conformally invariant and the amplitudes of the two curvatures (32) and (33) are totally independent.

The fundamental question is whether torsion is a matter field, and, if it is, which matter field. In the remainder of this subsection we outline a (highly speculative) argument in favour of interpreting our torsion wave (16) as a mathematical model for a neutrino field.

We base our interpretation on the analysis of the curvature (33) generated by our torsion wave. Examination of formula (33) indicates that it is more convenient to deal with the complexified curvature
\[ \mathcal{R} := r (l \land a) \otimes (l \land a) \]
where \( r := \frac{1}{4}(h^2)'' \) (this \( r \) is a function of the phase \( \varphi \)); note also that complexification is in line with the traditions of quantum mechanics. Our complex curvature is polarized,
\[ ^*\mathcal{R} = \mathcal{R}^* = \pm i\mathcal{R}, \]
and purely Weyl, hence it is equivalent to a (symmetric) rank 4 spinor \( \omega \). The relationship between \( \mathcal{R} \) and \( \omega \) is given by the formula
\[ \mathcal{R}_{\alpha\beta\gamma\delta} = \sigma_{\alpha\beta\gamma\delta} \omega^{abcd} \sigma_{\gamma\delta cd} \]
where the \( \sigma_{\alpha\beta} \) are “second order Pauli matrices” (A.5). Resolving (36) with respect to \( \omega \) we get, in view of (34), (8), (7),
\[ \omega = \xi \otimes \xi \otimes \xi \otimes \xi \]
where
\[ \xi := r^{1/4} \chi \]
and \( \chi \) is the spinor field introduced in the beginning of Section 8.
Formula (37) shows that our rank 4 spinor $\omega$ has additional algebraic structure: it is the 4th tensor power of a rank 1 spinor $\xi$. Consequently, the complexified curvature generated by our torsion wave is completely determined by the rank 1 spinor field $\xi$.

We claim that the spinor field (38) satisfies Weyl’s equation, see (A.10) or (A.11). Indeed, as $\chi$ is parallel checking that $\xi$ satisfies Weyl’s equation reduces to checking that $(r^{1/4})' \sigma'^{\mu}_{\nu a} l_{\mu} \chi^a = 0$. The latter is established by direct substitution of the explicit formula for $l$, see (5).

7.2. Comparison with existing literature

There are a number of publications in which authors suggested various generalisations of the concept of a classical pp-space. These generalisations were performed within the Riemannian setting (see Definition 1) and usually involved the incorporation of a constant nonzero scalar curvature; see [12] and extensive further references therein. Our construction described in Section 4 generalises the concept of a classical pp-space in a different direction: we add torsion while retaining zero scalar curvature.

A powerful method which in the past has been used for the construction of vacuum solutions of quadratic metric-affine gravity is the so-called double duality ansatz [13, 14, 15, 6, 2, 16]. Its basic version [6] is as follows. For certain types of quadratic actions (see item (b) below) the following is known to be true: if the spacetime is metric compatible and curvature is irreducible (i.e. all irreducible pieces except one are identically zero) then this spacetime is a solution of (2), (3). This fact is referred to as the double duality ansatz because the proof is based on the use of the double duality transform $R \mapsto R^*$ (this idea is due to Mielke [13]) and because the above conditions imply $R^* = \pm R$. However, solutions presented in Theorem 1 do not fit into the double duality scheme. This is due to the following reasons.

(a) The curvature of a pp-space, classical or generalised, contains trace-free Ricci and Weyl pieces, hence this curvature is not necessarily irreducible and not necessarily an eigenvector of the double duality operator. Namely, for a pp-space the following statements are equivalent:

\[ R \text{ is purely trace-free Ricci} \iff \text{condition (25) is satisfied} \iff *R^* = +R, \]
\[ R \text{ is purely Weyl} \iff \text{condition (24) is satisfied} \iff *R^* = -R. \]

Furthermore, the curvature of a pp-space, classical or generalised, does not necessarily satisfy the conditions of the modified double duality ansatz [14, 15, 16].

(b) The double duality ansatz in its basic [6] or modified [14, 15, 16] forms does not work for the most general 16-parameter actions introduced in [3, 4, 2] and considered in our current paper. It works only for more special actions with up to 11 coupling constants. The fundamental difference between the 11-parameter and 16-parameter models is best seen if one considers the specialisation of the field equation (3) to the Levi-Civita connection:

\[ \frac{\partial S}{\partial \Gamma}|_{\text{L-C}} = 0. \]
Equation (39) arises when one looks for Riemannian solutions of (3). Here it is important to understand the logical sequence involved in the derivation of (39): we set \( \Gamma_\lambda^{\mu\nu} = \left\{ \lambda^{\mu\nu} \right\} \) after the variation of the connection has been carried out. It is known [6] that for a generic 11-parameter action equation (39) reduces to
\[
\nabla_\lambda \mathrm{Ric}_{\kappa\mu} - \nabla_\kappa \mathrm{Ric}_{\lambda\mu} = 0,
\]
whereas according to [2] for a generic 16-parameter action equation (39) reduces to
\[
\nabla \mathrm{Ric} = 0.
\]
The field equations (40) and (41) are very much different, with (41) being by far more restrictive. In particular, Nordström–Thompson spacetimes (Riemannian spacetimes with \( R^* = +R \)) satisfy (40) but do not necessarily satisfy (41).

(c) The basic double duality ansatz [6] can be reformulated in a way that makes it applicable to 16-parameter actions: one has to impose the additional condition that curvature is simple, i.e. the given irreducible subspace of the vector space of curvatures is not isomorphic to any other irreducible subspace. See Section 6 of [2] for details. According to formula (44) of [2] the (symmetric) trace-free Ricci piece of curvature is not simple, hence the version of the double duality ansatz from [2] works for a pp-space, classical or generalised, only when curvature is purely Weyl.

The new vacuum solutions of quadratic metric-affine gravity presented in Theorem 1 are similar to those of Singh and Griffiths [17]. The main differences are as follows.

- The solutions in [17] satisfy the condition \( \{ \mathrm{Ric} \} = 0 \) whereas our solutions satisfy the weaker condition \( \nabla \{ \mathrm{Ric} \} = 0 \) (see also last paragraph of Section 6).
- The solutions in [17] were obtained for the Yang–Mills case (4) whereas we deal with a general O(1, 3)-invariant quadratic form \( q \) with 16 coupling constants.

The observation that one can construct vacuum solutions of quadratic metric-affine gravity in terms of pp-waves is a recent development. The fact that classical pp-spaces of parallel Ricci curvature are solutions was first pointed out in [18, 19, 2].

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Appendix A. Spinor formalism for generalised pp-spaces

In this appendix, unless otherwise stated, we work in a general metric compatible spacetime with torsion. We start by recalling basic facts about spinors.

Define the “metric spinor”
\[
\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
with the first index enumerating rows and the second enumerating columns. We raise and lower spinor indices according to the formulae
\[ \xi^a = \epsilon^{ab} \xi_b, \quad \xi_a = \epsilon_{ab} \xi^b, \quad \eta^a = \epsilon^{ab} \eta_b, \quad \eta_a = \epsilon_{ab} \eta^b. \] (A.2)

Our definition (A.1), (A.2) has the following advantages.

- The spinor inner product is invariant under the operation of raising and lowering of indices, i.e. \((\epsilon_{ac} \xi^c)(\epsilon^d \eta_d) = \xi^a \eta_b.\)
- The “contravariant” and “covariant” metric spinors are “raised” and “lowered” versions of each other, i.e. \(\epsilon^{ab} = \epsilon^{ac} \epsilon_{cd} \epsilon^{bd}\) and \(\epsilon_{ab} = \epsilon_{ac} \epsilon^{cd} \epsilon_{bd}.\)

The disadvantage of our definition (A.1), (A.2) is that the consecutive raising and lowering of a single spinor index leads to a change of sign, i.e. \(\epsilon_{ab} \epsilon^{bc} \xi_c = -\xi_a.\) This inconsistency is related to the well known fact that a spinor does not have a particular sign (say, a spatial rotation of the coordinate system by \(2\pi\) leads to a change of sign).

In formulae where the sign is important we will be careful in specifying our choice of sign; see, for example, (A.3), (A.4).

Let \(v\) be the real vector space of Hermitian \(2 \times 2\) matrices \(\sigma_{ab}\). Pauli matrices \(\sigma_{\alpha ab}\), \(\alpha = 0, 1, 2, 3,\) are a basis in \(v\) satisfying \(\sigma_{\alpha ab} \sigma_{\beta cd} + \sigma_{\beta ab} \sigma_{\alpha cd} = 2 \eta^{a\beta} \delta_\alpha^c\) where
\[ \sigma_{\alpha ab} := \epsilon^{ac} \sigma_{\alpha cd} \epsilon_{bd}. \] (A.3)

At every point of the manifold \(M\) Pauli matrices are defined uniquely up to a Lorentz transformation. For the pp-metric (10) we choose Pauli matrices
\[ \sigma_{0 ab} = \begin{pmatrix} 1 & 0 \\ 0 & -f \end{pmatrix}, \quad \sigma_{1 ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2 ab} = \begin{pmatrix} 0 & \mp i \\ \pm i & 0 \end{pmatrix}, \quad \sigma_{3 ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \] (A.4)

Our two choices of Pauli matrices differ by orientation. When dealing with a classical pp-space the choice of orientation of Pauli matrices does not really matter, however in a generalised pp-space it is convenient to choose orientation of Pauli matrices in agreement with the signs in (15) and (35) as this simplifies the resulting formulae.

Define
\[ \sigma_{\alpha\beta ac} := \frac{1}{2} (\sigma_{ac bd} \epsilon_{\beta \gamma} - \sigma_{\beta \gamma ac} \epsilon_{bd}). \] (A.5)

These “second order Pauli matrices” are polarized, i.e.
\[ * \sigma = \pm i \sigma \] (A.6)
depending on the orientation of “basic” Pauli matrices \(\sigma_{\alpha ab}\), \(\alpha = 0, 1, 2, 3.\) Note that with our choice of Pauli matrices the signs in formulae (A.3) and (A.6) agree.

We define the covariant derivatives of spinor fields as
\[ \nabla_\mu \xi^a = \partial_\mu \xi^a + \Gamma^a_{\mu \rho} \xi^b, \quad \nabla_\mu \xi_a = \partial_\mu \xi_a - \Gamma^b_{\mu a} \xi_b, \]
\[ \nabla_\mu \eta^a = \partial_\mu \eta^a + \Gamma^a_{\mu b} \eta^b, \quad \nabla_\mu \eta_a = \partial_\mu \eta_a - \Gamma^b_{\mu a} \eta_b, \]
where \(\Gamma^a_{\mu b} = \overline{\Gamma^a_{\mu b}}.\) The explicit formula for the spinor connection coefficients \(\Gamma^a_{\mu b}\) can be derived from the following two conditions:
\[ \nabla_\mu \xi^{ab} = 0, \] (A.7)
\[
\n\nabla_\mu \dot j^\alpha = \sigma_{ab}^{\alpha} \nabla_\mu \zeta_{ab}^{\alpha} \tag{A.8}
\]

where \( \zeta \) is an arbitrary rank 2 mixed spinor field and \( j^\alpha := \sigma_{ab}^{\alpha} \zeta_{ab}^{\alpha} \) is the corresponding vector field (current). Conditions (A.7), (A.8) give a system of linear algebraic equations for \( \Re \Gamma_{a\mu b} \), \( \Im \Gamma_{a\mu b} \) the unique solution of which is

\[
\Gamma_{a\mu b} = \frac{1}{4} \sigma_{ac}^{\alpha} (\partial_\mu \sigma_{b\delta}^{\alpha} + \Gamma_{\alpha\mu \beta} \sigma_{b\delta}^{\beta}) \tag{A.9}
\]

In a generalised pp-space formula (A.9) reads as follows: the nonzero coefficients are

\[
\Gamma_{1\,12} = \frac{1}{2} h h', \quad \Gamma_{1\,22} = \mp \frac{i}{2} h h', \quad \Gamma_{1\,32} = \frac{1}{2} \left( \frac{\partial f}{\partial x^1} \pm i \frac{\partial f}{\partial x^2} \right) - \frac{1}{2} k h'.
\]

Here we use special local coordinates (10), (11) and Pauli matrices (A.4).

The generally accepted point of view [20, 21, 22, 23, 24] is that a neutrino field in a metric compatible spacetime with or without torsion is described by the action

\[
S_{\text{neutrino}} := 2i \int (\xi_{\alpha} \sigma_{ab}^{\mu} (\nabla_\mu \bar{\xi}_{\alpha}) - (\nabla_\mu \xi_{\alpha}) \sigma_{ab}^{\mu} \bar{\xi}_{\alpha}),
\]

see formula (11) of [24]. Variation in \( \xi \) produces Weyl's equation

\[
\sigma_{ab}^{\mu} \nabla_\mu \xi_{\alpha} = \frac{1}{2} T^n_{\eta \mu} \sigma_{ab}^{\mu} \xi_{\alpha} = 0
\]

which can be equivalently rewritten as

\[
\sigma_{ab}^{\mu} (\nabla_\mu \xi_{\alpha}) = \frac{1}{4} \varepsilon_{\alpha \beta \gamma \delta} T^{\alpha \beta \gamma} \sigma_{ab}^{\delta} \xi_{\alpha} = 0
\]

where \( \{ \nabla \} \) is the covariant derivative with respect to the Levi-Civita connection. In a generalised pp-space torsion is purely tensor, see (18), so Weyl's equation takes the form

\[
\sigma_{ab}^{\mu} \nabla_\mu \xi_{\alpha} = 0 \tag{A.10}
\]

or, equivalently,

\[
\sigma_{ab}^{\mu} (\nabla_\mu \xi_{\alpha}) = 0 \tag{A.11}
\]

Appendix B. Derivation of the second field equation

In this Appendix we outline the arguments which produce (27). Throughout this Appendix the metric is assumed to be fixed and the connection is being varied. We also assume that we start variation from a spacetime satisfying the four conditions listed in the beginning of Section 5.

Following the reasoning of Section 3 of [2], we rewrite our quadratic form as

\[
q(R) = c_1 (R^{(1)}, R^{(1)})_{\text{YM}} + c_3 (R^{(3)}, R^{(3)})_{\text{YM}} + 2(b_{911} - b_{922}) (P_-, P_+) + \ldots \tag{B.1}
\]

where \((\cdot, \cdot)_{\text{YM}}\) is the Yang–Mills inner product on curvatures \((R, Q)_{\text{YM}} := R^\kappa_{\lambda \mu \nu} Q^{\lambda \kappa \mu \nu}\) and \(\ldots\) denote terms which do not contribute to \(\delta S\). Here the \(R^{(j)}\)s are the irreducible pieces of curvature labelled in accordance with [6]. The tensors \(P_\pm\) are defined by
stands for $\delta R$ with the other

now, in our case $R$'s being zero. Substituting these expressions into (B.3) we get

\[ \delta \int (R^{(1)}, R^{(1)})_{YM} = 2 \int (\nabla_\lambda \text{Ric}_{\kappa\mu} - \nabla_\kappa \text{Ric}_{\lambda\mu} + g_{\kappa\mu} \nabla_\eta \text{Ric}_\lambda^{\eta} - g_{\lambda\mu} \nabla_\eta \text{Ric}_\kappa^{\eta}) \]

\[ + \text{Ric}_\kappa^{\eta}(T_{\eta\lambda\mu} - T_{\eta\lambda\mu}) + \text{Ric}_\lambda^{\eta}(T_{\kappa\mu\eta} - T_{\kappa\mu\eta})) \delta \Gamma^{\lambda\mu\kappa}, \]  

(B.5)

\[ \delta \int (R^{(3)}, R^{(3)})_{YM} = 4 \int (\nabla_\eta \mathcal{W}_{\kappa\lambda\mu}^{\eta} + \mathcal{W}_{\kappa\lambda}^{\eta \xi} T_{\mu\eta\xi}) \delta \Gamma^{\lambda\mu\kappa}. \]  

(B.6)

The variation of $\int (P_-, P_+)$ turns out to be

\[ \delta \int (P_-, P_+) = \int (\text{Ric}, \delta \text{P}_+) + \frac{1}{2} \int (\text{Ric}, \delta \text{Ric}) + \frac{1}{2} \int (\text{Ric}, \delta \text{Ric}^{(2)}) \]

\[ = -\frac{1}{2} \int [\nabla_\lambda \text{Ric}_{\kappa\mu} + \nabla_\kappa \text{Ric}_{\lambda\mu} - g_{\mu\nu} \nabla_\eta \text{Ric}_\lambda^{\eta} - g_{\lambda\mu} \nabla_\eta \text{Ric}_\kappa^{\eta}) \]

\[ + \text{Ric}_\kappa^{\eta}(T_{\eta\lambda\mu} - T_{\eta\lambda\mu}) + \text{Ric}_\lambda^{\eta}(T_{\eta\lambda\mu} - T_{\kappa\mu\eta})] \delta \Gamma^{\lambda\mu\kappa} \]  

(B.7)

(compare with the corresponding formula in Section 3 of [2]).

Combining formulae (B.1), (B.2), (B.5)–(B.7), we arrive at the explicit form of the field equation [3]:

\[ d'_6(\nabla_\lambda \text{Ric}_{\kappa\mu} - g_{\mu\nu} \nabla_\eta \text{Ric}_\lambda^{\eta} - T_{\lambda\mu\eta} \text{Ric}_\kappa^{\eta} + T_{\lambda\mu\eta} \text{Ric}_\kappa^{\eta}) \]

\[ - d'_7(\nabla_\kappa \text{Ric}_{\lambda\mu} - g_{\kappa\mu} \nabla_\eta \text{Ric}_\lambda^{\eta} - T_{\kappa\mu\eta} \text{Ric}_\lambda^{\eta} + T_{\kappa\mu\eta} \text{Ric}_\kappa^{\eta}) \]

\[ + 2b_{10}(\nabla_\eta \mathcal{W}_{\mu\lambda\kappa}^{\eta} - \mathcal{W}_{\mu\lambda\kappa}^{\eta \xi} T_{\xi\mu\eta}) = 0 \]  

(B.8)

where

\[ d'_6 = b_{912} - b_{911}, \quad d'_7 = b_{912} - b_{922}. \]

Let us now make use of the Bianchi identity for curvature

\[ (\partial_\xi + [\Gamma_\xi, \cdot]) R_{\mu\nu} + (\partial_\nu + [\Gamma_\nu, \cdot]) R_{\xi\mu} + (\partial_\mu + [\Gamma_\mu, \cdot]) R_{\nu\xi} = 0 \]  

(B.9)
where we hide the Lie algebra indices of curvature by using matrix notation as in (B.4). Making one contraction in (B.9) and using the four assumptions listed in the beginning of Section 5 we get

\begin{align*}
\frac{1}{2} \left[ \nabla_\kappa \mathcal{R}ic_{\mu \lambda} - \nabla_\lambda \mathcal{R}ic_{\mu \kappa} + g_{\mu \kappa} \nabla_\eta \mathcal{R}ic^\eta_{\lambda} - g_{\mu \lambda} \nabla_\eta \mathcal{R}ic^\eta_{\kappa} \\
+ \mathcal{R}ic^\eta_{\xi} (g_{\mu \kappa} T_{\eta \xi \lambda} - g_{\mu \lambda} T_{\eta \xi \kappa}) + \mathcal{R}ic^\eta_{\kappa} (T_{\eta \lambda \mu} - T_{\lambda \eta \mu}) + \mathcal{R}ic^\eta_{\lambda} (T_{\eta \kappa \mu} - T_{\kappa \eta \mu}) \right] \\
+ \nabla_\eta \mathcal{W}^\eta_{\mu \lambda \kappa} + \mathcal{W}^\eta_{\mu \kappa \xi} (T_{\lambda \eta \xi} - T_{\eta \lambda \xi}) + \mathcal{W}^\eta_{\mu \lambda \xi} (T_{\eta \kappa \xi} - T_{\kappa \eta \xi}) = 0. \tag{B.10}
\end{align*}

Another contraction in (B.10) yields

\begin{align*}
\nabla_\eta \mathcal{R}ic^\eta_{\lambda} &= - \mathcal{R}ic^\eta_{\xi} T_{\eta \xi \lambda} - \frac{1}{2} \mathcal{W}^{\eta \kappa \xi} (T_{\xi \eta \zeta} - T_{\xi \zeta \eta}). \tag{B.11}
\end{align*}

Substitution of (B.11) into (B.10) gives

\begin{align*}
\nabla_\eta \mathcal{W}^\eta_{\mu \lambda \kappa} &= \mathcal{W}^\eta_{\mu \kappa \xi} (T_{\lambda \eta \xi} - T_{\eta \lambda \xi}) \tag{B.12} \\
&\quad + \frac{1}{4} (T_{\xi \eta \zeta} - T_{\xi \zeta \eta}) (g_{\mu \lambda} \mathcal{W}^{\eta \kappa \xi} - g_{\mu \kappa} \mathcal{W}^{\eta \xi \lambda}) \\
&\quad + \frac{1}{2} \left[ \nabla_\lambda \mathcal{R}ic_{\mu \kappa} - \nabla_\kappa \mathcal{R}ic_{\mu \lambda} + \mathcal{R}ic^\eta_{\kappa} (T_{\lambda \eta \mu} - T_{\eta \lambda \mu}) + \mathcal{R}ic^\eta_{\lambda} (T_{\eta \kappa \mu} - T_{\kappa \eta \mu}) \right].
\end{align*}

Formulae (B.11) and (B.12) allow us to exclude the terms with \( \nabla_\eta \mathcal{R}ic^\eta_{\lambda} \), \( \nabla_\eta \mathcal{R}ic^\eta_{\kappa} \) and \( \nabla_\eta \mathcal{W}^\eta_{\mu \lambda \kappa} \) from equation (B.8) reducing the latter to (27).

References

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