It has been claimed in the literature that impossibility of faster-than-light quantum communication has an origin of indistinguishability of ensembles with the same density matrix. We show that the two concepts are not related. We argue that: 1) even with an ideal single-atom-precision measurement, it is generally impossible to produce two ensembles with exactly the same density matrix; or 2) to produce ensembles with the same density matrix, classical communication is necessary. Hence the impossibility of faster-than-light communication does not imply the indistinguishability of ensembles with the same density matrix.

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If Alice could tell the difference between these two ensemble preparations, then she would be able to read Bob’s information immediately, accomplishing a faster-than-light communication.

However, this is impossible. Preskill stressed in Ref. [5] that “though the two preparation methods are surely different, both ensembles are described by precisely the same density matrix $\rho_A$. Thus there is no conceivable measurement Alice can make that will distinguish the two ensembles, and no way for Alice to tell what action Bob performed. The ‘message’ is unreadable”.

Preskill then provided a method for Alice and Bob to distinguish the two ensemble preparations with some additional help. Alice can choose a measuring device, say $\sigma_x$, to measure each of her particles and compare with the result of Bob which was transmitted to him through a telephone line. If her results have perfect agreement with Bob, then she knows that Bob’s action is $\sigma_x$, otherwise Bob’s action is $\sigma_z$. Here ideal conditions are assumed for simplicity. Apparently, the two ensemble preparations can be distinguished. However, Preskill stressed that faster-than-light communication is not possible because telephone calls are needed for the distinction, and signal in a telephone line travels at the speed of light.

Preskill has related the impossibility of faster-than-light communication to the indistinguishability of ensembles having the same density matrix.

There is a flaw in the Preskill analysis. Apparently Preskill’s analysis ignored the fluctuation in the measured result. It is true that in both measurements, each particle has 1/2 probability to collapse into $|\uparrow_Z\rangle$ ($|\uparrow_X\rangle$) or $|\downarrow_Z\rangle$ ($|\downarrow_X\rangle$), but the number of particles in the $|\uparrow_Z\rangle$ ($|\uparrow_X\rangle$) direction is not exactly the same as that in the
FIG. 3: By measuring $\sigma_z$ on each qubit, each pair collapses into $|\uparrow_z\rangle$ or $|\downarrow_z\rangle$. This action, the $\sigma_z$ measurement, represents the value 1.

\[ |\downarrow_Z\rangle = (|\downarrow_X\rangle) \text{ direction.} \]

Hence the density matrix of one ensemble is

\[ \rho_A = \begin{pmatrix} \frac{1}{2} - \frac{N\delta}{N} & 0 \\ 0 & \frac{1}{2} + \frac{N\delta}{N} \end{pmatrix}, \quad (3) \]

where $N\delta$ is a random number that could be positive or negative and is proportional to $\sqrt{N}$. $N\delta$ indicates the difference between the number of particles in the $|\uparrow_z\rangle$ ($|\uparrow_x\rangle$) state and the $N/2$. We stress that the two different ensemble preparations in general leads to different density matrices if the number of particles is finite, which is usually true in real physical circumstances. Because one can measure the particles in the ensembles one by one, the absolute value of $N\delta$ increases with $N$. Thus as $N$ goes large, the fluctuation becomes large too. This makes the distinction of the ensembles more easily. Even if Bob repeats the same kind of measurement, say $\sigma_x$, he would not be able to produce exactly two identical ensembles if the particle number is finite. Thus strictly speaking, the density matrices of the two ensembles produced by $\sigma_x$ and $\sigma_z$ measurement are not identical. Though the two ensembles have different density matrices, one could not use them for communication as the fluctuation is uncontrollable. Hence there is no question of distinction of ensembles having the same density matrix at all in this problem.

**Now we show that even one can distinguish ensembles with the same density matrix, it is still impossible to perform faster-than-light communication.** We make the following modifications to Preskill’s scheme. While preparing the ensemble by measuring each qubit with $\sigma_z(\sigma_x)$, Bob can make the number of particles in $|\uparrow_z\rangle$ ($|\uparrow_x\rangle$) and $|\downarrow_z\rangle$ ($|\downarrow_x\rangle$) exactly the same by dropping some qubits. He then tells Alice which qubits should be excluded from her ensemble, so that Alice’s ensemble is prepared with exactly equal numbers of qubits in opposite polarization. Of course, Bob and Alice can produce several such copies with equal or near equal total number of particles. But they all have equal number of particles in opposite directions. Of course, the communication in the preparation is classical, hence it is not faster-than-light communication. Now Alice’s ensemble has exactly the density matrix $\rho_A = \frac{1}{2}I_2$ and with two possible form of constituents: either polarized or anti-polarized along $z$-direction, or polarized or anti-polarized along $x$-direction. The question is that can Alice distinguish the two cases with whatever methods that is available. Alice can determine which one of the two constructions the ensemble is by making a $\sigma_z$ measurement on each of the particles in the ensemble. If the ensemble was prepared by the $\sigma_z$ measurement with equal numbers of particles in opposite directions with the help of classical communication, then the sum of the all the measured results is zero, namely

\[ \Sigma_z = \sum_{i=1}^{N} \sigma_z(i) = 0. \quad (4) \]

However if the ensemble was prepared by the $\sigma_x$ measurement instead, then the sum of all the measurement will be,

\[ \Sigma'_z = \sum_{i=1}^{N} \sigma_z(i) \approx \pm \sqrt{N}. \quad (5) \]
Because in the first ensemble, the state of the individual particle is the eigenstate of $\sigma_z$, and it will give a definite result when $\sigma_z$ is measured. While in the second case, the state of an individual particle is in the eigenstate of $\sigma_x$, there is fluctuations in the measured results though the average total sum is zero. Because Alice and Bob have several copies, these fluctuations can be easily found when Alice repeats the measurement on several such copies. By observing the fluctuation, Alice can easily determine what measurement Bob has performed. This has been suggested by d'Espagnat [7] and has been generalized into general ensembles in Ref. [8].

To summarize, it has been shown that ensembles with the same density matrix can be distinguished physically does not contradict the claim of impossibility of faster-than-light communication for two reasons. First, it is impossible to produce ensembles having the same density by measurements even if a single atom precision is available. This is because the collapse of state under measurement is random and the measured results have fluctuations. This makes the precise density matrix to fluctuate around the average value. Second, even if two ensembles are produced with exactly the same density matrix and they are distinguishable by observing the fluctuations of observables related to the whole ensemble, this could not be used for information transmission because the classical communication is necessary to prepare the ensembles with the same density matrix.

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