Abstract

We study the $D = 5$ Emparan-Reall spinning black ring under an ultrarelativistic boost (in the sense of Aichelburg and Sexl) along an arbitrary direction. We analytically determine the resulting shock $pp$-wave geometry, in particular for two complementary boosts along axes orthogonal and parallel to the plane of rotation. The solution becomes physically more interesting and considerably simpler if one enforces equilibrium between the forces acting on the ring. We also briefly comment on the ultrarelativistic limit of recently found supersymmetric black rings with two independent angular momenta.

1 Introduction

Shock $pp$-wave geometries describe the spacetime surrounding very fast moving objects, and are thus relevant to the study of Planckian scattering [1]. They are also of interest in string theory, since strings may be exactly solved in such backgrounds [2]. The prototype of shock wave solutions is the Aichelburg-Sexl spacetime, which represents the gravitational field of a massless point particle. It was originally obtained by boosting the Schwarzschild black hole to the speed of light, while rescaling the mass to zero in an appropriate way [3]. According to recent extra-dimension scenarios [4], the fundamental Planck scale of (higher dimensional) gravity could be as low as a few TeV. This has stimulated renewed interest in the study of gravitational effects in high energy collisions, especially in view of the possible observation of microscopic black holes at near future colliders [5] (see, e.g., [6] for a recent review and for further references). It has been shown that closed trapped surfaces do indeed form in the ultrarelativistic collision of Aichelburg-Sexl point particles [7] and of finite-size beams [8], which can more accurately model string-size effects. Nevertheless, it is desirable to understand how other effects could influence high energy scattering. A first step in this direction is to investigate more general shock wave solutions of higher dimensional gravity, which can naturally be obtained by applying the boosting technique of [3] to black hole spacetimes. This has been done in any $D \geq 4$ for static black holes with electric charge [9] or immersed in an external magnetic field [10]. The ultrarelativistic limit of the Myers-Perry rotating black holes [11] has been studied in [12] (for the case of one non-vanishing spin). However, a striking feature of General Relativity in $D > 4$ is the non-uniqueness of the spherical black holes of [11]. In five-dimensional vacuum gravity, there exist also asymptotically flat rotating black rings with an event horizon of topology $S^1 \times S^2$ [13]. In the present contribution, we aim at studying the gravitational field generated by such rings in the Aichelburg-Sexl limit. As we will see in detail, this results in shock waves generated by extended lightlike sources (with a characteristic length-scale) which are remnants of the ring singularity of the original spacetime [13]. Our recent results on boosted non-rotating black rings [14] will be recovered as a special subcase. In general, the presence of spin is important because it allows black rings to be in equilibrium [13] without introducing “unphysical” membranes via conical singularities [15]. This will be reflected also in the

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shock geometry resulting from the boost. From a supergravity and string theory point of view, it is remarkable that supersymmetric black rings have been also constructed [16–19]. We will conclude this article with a brief comment on the boost of such solutions.

2 The black ring solution

In this section we briefly summarize the basic properties of the black ring, referring to [13,20] for details. In the coordinates of [20],\(^1\) the line element reads

\[
ds^2 = -\frac{F(y)}{F(x)} \left( dt + C(\nu, \lambda)L \frac{1 + y}{F(y)} d\psi \right)^2 + \frac{L^2}{(x - y)^2} F(x) \left[ \frac{-G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right],
\]

where

\[
F(\zeta) = \frac{1 + \lambda \zeta}{1 - \lambda}, \quad G(\zeta) = (1 - \zeta^2)^{\frac{1 + \nu \zeta}{1 - \nu}}, \quad C(\nu, \lambda) = \sqrt{\frac{\lambda(\lambda - \nu)(1 + \lambda)}{(1 - \nu)(1 - \lambda)^3}}.
\]

The dimensionless parameters \(\lambda\) and \(\nu\) satisfy \(0 \leq \nu \leq \lambda < 1\), and for \(\lambda = 0 = \nu\) the spacetime is flat. The constant \(L > 0\) represents a length related to the radius of the “central circle” of the ring. For a physical interpretation of the spacetime we take \(y \in (-\infty, -1], x \in [-1, +1]\) (see a discussion in [21] for other possible choices) and \(\psi\) and \(\phi\) as periodic angular coordinates (see below). Surfaces of constant \(y\) have topology \(S^1 \times S^2\). The coordinate \(\psi\) runs along the \(S^1\) factor, whereas \((x, \phi)\) parametrize \(S^2\) (see [15,17,20] for illustrative pictures). Within the above range, \(y\) parametrizes “distances” from the ring circle. At \(y \to -\infty\) the spacetime has a inner spacelike curvature singularity, \(y = -1/\nu\) is a horizon and \(y = -1/\lambda\) an ergosurface, both with topology \(S^1 \times S^2\). The black ring solution is asymptotically flat near spatial infinity \(x, y \to -1\), where it tends to Minkowski spacetime in the form

\[
ds_0^2 = -dt^2 + \frac{L^2}{(x - y)^2} \left[ (y^2 - 1) d\psi^2 + \frac{dy^2}{y^2 - 1} + \frac{dx^2}{1 - x^2} + (1 - x^2) d\phi^2 \right].
\]

To avoid conical singularities at the axes \(x = -1\) and \(y = -1\), the angular coordinates must have the standard periodicity\(^2\)

\[
\Delta \phi = 2\pi = \Delta \psi.
\]

Centrifugal repulsion and gravitational self-attraction of the ring are in balance if conical singularities are absent also at \(x = +1\), which requires

\[
\lambda = \frac{2\nu}{1 + \nu^2}.
\]

When this equilibrium condition holds, the metric is a vacuum solution (of \(D = 5\) General Relativity) everywhere. With different choices (e.g., in the static limit \(\nu = \lambda\) [15]), the conical singularity at \(x = +1\) describes a disk-shaped membrane inside the ring.

The mass, angular momentum and angular velocity (at the horizon) of the black ring are

\[
M = \frac{3\pi L^2 \lambda}{4(1 - \lambda)}, \quad J = \frac{\pi L^3}{2} \sqrt{\frac{\lambda(\lambda - \nu)(1 + \lambda)}{(1 - \nu)(1 - \lambda)^3}}, \quad \Omega = \frac{1}{L} \sqrt{\frac{\lambda(\lambda - \nu)(1 - \lambda)}{\lambda(1 + \lambda)(1 - \nu)}}.
\]

The algebraic type of the Weyl tensor of the ring spacetime is \(I\) [21].

3 General boost

For our purposes, it is convenient to decompose the line element as

\[
ds^2 = ds_0^2 + \Delta,
\]

\(^1\)Up to simple constant rescalings of \(F(\zeta), G(\zeta), C(\lambda, \nu), \psi\) and \(\phi\), cf. Eqs. \(2\) and \(11\) with the corresponding ones in [20]. In addition, multiply our \(L^2\) by \((1 - \nu)/(1 - \lambda)\) to obtain the parameter used in [20].
in which $ds_0^2$ is Minkowski spacetime and

$$\Delta = \lambda \frac{x-y}{1+\lambda x} dt^2 - 2(1-\lambda)C(\lambda, \nu)L \frac{1+y}{1+\lambda y} dt d\psi$$

$$+ \frac{\lambda - \nu}{1-\nu} \frac{L^2}{1+\lambda y} \left[ -\frac{1}{1-\lambda} \left( \frac{1}{1+y} + \frac{y^2-1}{x-y} \right) \right] d\psi^2$$

$$+ \frac{L^2}{(x-y)^2} \left[ \frac{\nu}{1-\nu} (y^2-1) d\psi^2 + \frac{\lambda(1-\nu)(x-y) + (\lambda-\nu)(1+y)}{(1-\lambda)(1+\nu y)} \frac{y^2-1}{y^2-1} \right]$$

$$+ \frac{\lambda - \nu}{1-\nu} \frac{L^2}{1+\lambda x} \left[ \frac{\nu}{1-\nu} \frac{1}{1-x} \left( 1-x^2 \right) d\phi^2 \right].$$

The above splitting is such that near infinity $(x, y \to -1)$ one has $ds^2 \to ds_0^2$, while $\Delta$ becomes “negligible” (in the sense of the “background” metric $ds_0^2$). This enables us to define a notion of Lorentz boost using the symmetries of the asymptotic Minkowskian background $ds_0^2$. Cartesian coordinates will visualize it most naturally. These can be introduced in two steps. First, we replace the coordinates $(y, x)$ with new coordinates $(\xi, \eta)$ via the substitution

$$y = -\frac{\xi^2 + \eta^2 + L^2}{\Sigma}, \quad x = -\frac{\xi^2 + \eta^2 - L^2}{\Sigma},$$

where

$$\Sigma = \sqrt{(\eta^2 + \xi^2 - L^2)^2 + 4L^2\eta^2}.$$ (10)

The flat term $ds_0^2$ in Eq. \[11\] now takes the form $ds_0^2 = -dt^2 + dy^2 + \eta^2 d\phi^2 + d\xi^2 + \xi^2 d\psi^2$. Then, cartesian coordinates adapted to the Killing vectors $\partial_\xi$ and $\partial_\eta$ are given by

$$x_1 = \eta \cos \phi, \quad x_2 = \eta \sin \phi, \quad y_1 = \xi \cos \psi, \quad y_2 = \xi \sin \psi,$$ (11)

so that $\eta = \sqrt{x_1^2 + x_2^2}$, $\xi = \sqrt{y_1^2 + y_2^2}$, and $ds_0^2 = -dt^2 + dx_1^2 + dx_2^2 + dy_1^2 + dy_2^2$. This enables us to study a boost along a general direction. Since the original spacetime \[12\] is symmetric under (separate) rotations in the $(x_1, x_2)$ and $(y_1, y_2)$ planes, such a direction can be specified by a single parameter $\alpha$, namely introducing rotated axes $z_1$ and $z_2$

$$x_1 = z_1 \cos \alpha - z_2 \sin \alpha, \quad y_1 = z_1 \sin \alpha + z_2 \cos \alpha.$$ (12)

Defining now suitable double null coordinates $(u', v')$ by

$$t = -\frac{u' + v'}{\sqrt{2}}, \quad z_1 = \frac{u' + v'}{\sqrt{2}}.$$ (13)

a Lorentz boost along $z_1$ takes the simple form

$$u' = \epsilon^{-1} u, \quad v' = \epsilon v.$$ (14)

The parameter $\epsilon > 0$ is related to the standard Lorentz factor via $\gamma = (\epsilon + \epsilon^{-1})/2$. We are interested in “ultrarelativistic” boosts to the speed of light, i.e. in taking the limit $\epsilon \to 0$ in the transformation \[14\]. While $\epsilon \to 0$, we will rescale the mass as $M = \gamma^{-1}p_M \approx 2\epsilon p_M$ \[3\], which physically means that the total energy remains finite in the limit ($p_M > 0$ is a constant). Moreover, during the ultrarelativistic limit we wish to keep the angular velocity $\Omega$ finite (a similar condition was imposed in \[12\]), and to allow for the possibility of black rings in equilibrium \[when the condition \[6\] holds\]. From Eq. \[6\], these requirements imply the rescalings\footnote{This appears to be physically the most interesting and simple choice. See Footnote \[3\] for a subtler, slightly more general comment.}

$$\lambda = \epsilon p_\lambda, \quad \nu = \epsilon p_\nu,$$ (15)

where $p_\lambda = 8p_M/(3\pi L^2)$ and $p_\nu$ is another positive constant such that $p_\lambda \geq p_\nu$. In terms of these parameters, for $\epsilon \to 0$ the equilibrium condition \[14\] becomes

$$p_\lambda = 2p_\nu.$$ (16)

Values $p_\nu \leq p_\lambda < 2p_\nu$ correspond to black rings \[11\] which are “underspinning” before the boost (and therefore balanced by a membrane of negative energy density), values $p_\lambda > 2p_\nu$
to “overspinning” black rings (with a membrane of positive energy density). Notice, however, that under the limit $\epsilon \to 0$ the angular momentum $J$ will tend to zero ($\sim \epsilon$).

We can now evaluate how the black ring metric $[11]$ [that is, Eq. (7) with Eqs. (8) and (9)] transforms under the boost $[11]$. We have first to substitute Eq. (9) into Eqs. (8) and (9). Then, we apply the sequence of substitutions $[11]$, $[12]$, $[13]$ into the thus obtained expressions for $ds_0^2$ and for $\Delta$. Finally, we perform the boost $[14]$ with the rescalings $[15]$, which make $\Delta = \Delta_\epsilon$ dependent on $\epsilon$. The $ds_0^2$ is invariant under the boost and at the end it reads

$$ds_0^2 = 2dudv + dx_2^2 + dy_2^2 + dz_2^2. \quad (17)$$

The next step is to take the ultrarelativistic limit $ds^2 = ds_0^2 + \lim_{\epsilon \to 0} \Delta_\epsilon$. This is delicate because the expansion of $\Delta_\epsilon$ in $\epsilon$ has a different structure in different regions of the spacetime (even away from the singularity $y = -\infty$). In particular, a peculiar behaviour is obtained for $u = 0$, because $\Delta_\epsilon$ depends on $u$ through the combination

$$z_\epsilon = \frac{1}{\sqrt{2}}(\epsilon^{-1}u + \epsilon v). \quad (18)$$

In order to have control over the exact distributional structure of the limit, it is convenient to isolate such dependence on $\epsilon^{-1}u$ by performing first an expansion of $\Delta_\epsilon$ with $z_\epsilon$ unexpanded. This leads to an expression

$$\Delta_\epsilon = \frac{1}{\epsilon} h(z_\epsilon)du^2 + [k_1(z_\epsilon)dx_2 + k_2(z_\epsilon)dy_2 + k_3(z_\epsilon)dz_2 + k_4(z_\epsilon)du] \, du + \ldots, \quad (19)$$

where the dots denote terms proportional to higher powers of $\epsilon$, which are negligible in the limit. We have emphasized here the dependence of the functions $h$ and $k_i$ $(i = 1, \ldots, 4)$ on $z_\epsilon$ (and thus on $\epsilon$), because this is essential in our limit, but they depend also on $x_2$, $y_2$ and $z_2$. The quantities $k_i$ are rather involved, but it suffices to observe here that $\lim_{\epsilon \to 0} k_i(z_\epsilon) = 0$. We can thus also drop all the terms of order $\epsilon^0$ in $[19]$. For $h$, after all the steps described above, we obtain explicitly

$$h(z_\epsilon) = \frac{L^2}{\Sigma} p_\lambda + \frac{L^2}{\Sigma_\alpha} \left[ (\xi^2 - \eta^2 - L^2) \frac{y_1}{\xi} \sin \alpha + 2 \xi x_1 \cos \alpha \right]^2 + \frac{1}{2} \left( 2p_\nu - p_\lambda \right) \left( 1 - \frac{\xi^2 + \eta^2 - L^2}{\Sigma} \right) \left( \frac{y_2^2}{\xi^2} \sin^2 \alpha + \frac{x^2}{\eta^2} \cos \alpha \right) + \sqrt{p_\lambda (p_\lambda - p_\nu)} \frac{L y_2 \sin \alpha}{\xi^2} \left( -1 + \frac{\xi^2 + \eta^2 + L^2}{\Sigma} \right) + (p_\lambda - p_\nu) \frac{L^2 y_2^2}{\xi^2 \Sigma} \sin^2 \alpha + \frac{1}{2} \left( p_\lambda - p_\nu \right) \left( 1 - \frac{\xi^2 + \eta^2 - L^2}{\Sigma} \right), \quad (20)$$

Recall that the dependence of $h$ on $\epsilon$ is contained in $x_1$ and $y_1$ via Eqs. $[12]-[14]$, in $\eta$ and $\xi$ via Eq. $[11]$ and in $\Sigma$ via Eq. $[10]$. In taking the limit $\epsilon \to 0$ of Eq. $[19]$, we apply the distributional identity

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} f(z_\epsilon) = \sqrt{2} \delta(u) \int_{-\infty}^{+\infty} f(z) \, dz. \quad (21)$$

The final metric is thus [cf. Eqs. $[17]$ and $[19]$]

$$ds^2 = 2dudv + dx_2^2 + dy_2^2 + dz_2^2 + H(x_2, y_2, z_2) \delta(u) \, du^2, \quad (22)$$

with a profile function given by

$$H(x_2, y_2, z_2) = \sqrt{2} \int_{-\infty}^{+\infty} h(z) \, dz. \quad (23)$$

$^3$A remark on the “triviality” of the $\epsilon^0$ terms is in order, since they could be non-vanishing for certain more general scalings of the original metric parameters. While with higher order (in $\epsilon$) corrections in Eq. $[15]$ $\lim_{\epsilon \to 0} k_i(z_\epsilon) = 0$ would still hold, we could introduce a non-vanishing contribution by allowing an $\epsilon$-dependence in the ring “radius” via $L_0 = L + c_1 \epsilon + c_2 \epsilon^2 + \ldots$. The convergence of the integral $[23]$ would then require $c_1 = 0$, but the quantity $c_2 \epsilon^2$ would affect the limit of $[19]$ via $\lim_{\epsilon \to 0} k_4(z_\epsilon) = c_2$. The resulting term $c_2 \epsilon^2$ is, however, obviously removable with a coordinate transformation.
A black ring boosted to the speed of light in a general direction $z_1$ is thus described by the metric \( \text{[22]} \) with Eq. \( \text{[23]} \). This is evidently a $D = 5$ impulsive $pp$-wave with wave vector $\partial_v$. Such a spacetime is flat everywhere except on the null hyperplane $u = 0$, which represents the impulsive wave front. Note that the equilibrium condition \( \text{[16]} \) has not yet been enforced in the above expression for $h$ (in particular, in the static limit $p_v = p_\lambda$ we recover the result of \[14\]). In order to write the solutions in a completely explicit form, it only remains to perform the integration in Eq. \( \text{[26]} \), with $h$ given by Eq. \( \text{[20]} \) with Eqs. \( \text{[10]} - \text{[14]} \) and \( \text{[15]} \). For any $\alpha$, this integral is always convergent and can in principle be expressed using elliptic integrals (because $\Sigma$ is a square root of a fourth order polynomial in $z$, see \[14\] for related comments). Therefore, no singular coordinate transformation of the type of \[3\] has to be performed. In the following, we will explicitly calculate the integral, and study the corresponding solution in the case of two different boosts of the black ring along the privileged axes $x_1$ ($\alpha = 0$) and $y_1$ ($\alpha = \pi/2$), which are respectively “orthogonal” and “parallel” to the 2-plane $(y_1, y_2)$ [i.e., $(\xi, \psi)$] in which the ring rotates.

### 4 Orthogonal boost: $\alpha = 0$

For the orthogonal boost $\alpha = 0$, from Eq. \( \text{[12]} \) one has $z_1 = x_1$ and $z_2 = y_1$, so that the general $pp$-wave \( \text{[22]} \) reduces to

$$
\text{ds}^2 = 2dudv + dx_1^2 + dy_1^2 + H_p(x_1, y_1, y_2)\delta(u)dv^2.
$$

Also, it is now convenient to rewrite $h$ in Eq. \( \text{[20]} \) as

$$
\begin{align*}
h_p(z_2) & = \left[3p_\lambda L^2 - (p_\lambda - p_v)\xi^2 - p_v(x_2^2 + L^2)\right] \frac{1}{2\Sigma} + p_v \frac{4L^2\xi^2z_2^2}{\Sigma^3} + \frac{1}{2}(p_\lambda - p_v) \left(1 - \frac{z_2^2}{\Sigma}\right),
\end{align*}
$$

and $\Sigma$ [from Eq. \( \text{[10]} \)] as

$$
\Sigma = \sqrt{[z_2^2 + x_2^2 + (\xi + L)^2][z_2^2 + x_2^2 + (\xi - L)^2]}.
$$

Hereafter, it is understood that $\xi = \sqrt{y_1^2 + y_2^2}$. In the orthogonal boost there is no contribution to $h_p$ from the off-diagonal term $\partial_\psi$ in the metric \( \text{[1]} \). Performing the integration \( \text{[26]} \) with $h$ given by Eqs. \( \text{[20]} \) and \( \text{[20]} \), we find

$$
H_p(x_1, y_1, y_2) = \sqrt{2} \frac{3p_\lambda L^2 + (2p_\lambda - p_v)\xi^2}{\sqrt{(\xi + L)^2 + x_2^2}} K(k) + \sqrt{2}(2p_\lambda - p_v)
\times \left[-\sqrt{(\xi + L)^2 + x_2^2} E(k) + \frac{\xi - L}{\xi + L} \sqrt{(\xi + L)^2 + x_2^2} \right] \Pi(\rho, k) + \pi|x_2|\Theta(L - \xi).
$$

where

$$
k = \sqrt{\frac{4\xi L}{(\xi + L)^2 + x_2^2}}, \quad \rho = \frac{4\xi L}{(\xi + L)^2},
$$

and $\Theta(L - \xi)$ denotes the step function. In the above calculation, we have used the standard elliptic integrals and their properties summarized in the Appendix of \[14\], and the additional integral [\( \Sigma \) given by Eq. \( \text{[20]} \) with $z$, replaced by $z$]

$$
\int_0^\infty \left(1 - \frac{z^2}{\Sigma}\right) dz = \sqrt{(\xi + L)^2 + x_2^2} E(k).
$$

It is remarkable that for the physically more interesting case of black rings in equilibrium, i.e. those satisfying $p_\lambda = 2p_v$ [see Eq. \( \text{[10]} \)], the profile function simplifies significantly to

$$
H_p^{\text{eq}}(x_1, y_1, y_2) = \frac{3\sqrt{2}p_\lambda L^2}{\sqrt{(\xi + L)^2 + x_2^2}} K(k).
$$

Interestingly, this is just the Newtonian potential generated by a uniform ring of radius $L$ and linear density $\mu = 3\sqrt{2}p_\lambda L/4$ located at $x_2 = 0$ in the flat three-dimensional space.
Figure 1: The profile function $H_\perp$, given by Eq. (27), in the case of underspinning ($p_\lambda < 2 p_\nu$, left), overspinning ($p_\lambda > 2 p_\nu$, right) and balanced ($p_\lambda = 2 p_\nu$, bottom) black rings [cf. Eq. (16)] boosted along an orthogonal direction $x_1$. It is represented over the plane $(x_2, \xi)$ [cf. Eq. (11)], and the Killing coordinate $\psi$ is suppressed. In the equilibrium case, $H_\perp$ reduces to $H_\perp^e$ of Eq. (30) and the disk membrane at $x_2 = 0, \xi < L$ disappears (no jump of $\partial H_\perp^e / \partial x_2$ occurs at $x_2 = 0$). In all cases, there is a ring singularity at $x_2 = 0, \xi = L$, as indicated by the thick points in the pictures.

$(x_2, y_1, y_2)$. Since for a general $pp$-wave (24) the only component of the Ricci tensor is $R_{uu} = -\frac{1}{2} \delta(u) \Delta H_\perp$, $\Delta$ denoting the Laplace operator over the transverse space $(x_2, y_1, y_2)$, it follows that the profile function (30) represents a spacetime which is vacuum everywhere except on the circle $u = 0 = x_2, \xi = L$ [so that $k = 1$ in Eq. (28)]. This lies on the wave front and corresponds to a singular ring-shaped source moving with the speed of light. It is obviously a remnant of the curvature singularity ($y = -\infty$) of the original static black ring (1). For the non-equilibrium solution (27), the discontinuous term proportional to $\Theta(L - \xi)$ is responsible for a disk membrane supporting the ring [14]. We have plotted typical profile functions $H_\perp$ and $H_\perp^e$ in Fig. 1.

5 Parallel boost: $\alpha = \pi/2$

For the parallel boost $\alpha = \pi/2$, from Eq. (12) one has $z_1 = y_1$ and $z_2 = -x_1$, and the general $pp$-wave (22) reduces to

$$ds^2 = 2dudv + dx_1^2 + dx_2^2 + dy_2^2 + H_\perp(x_1, x_2, y_2)\delta(u)du^2. \quad (31)$$

The function $h$ can be reexpressed as

$$h_\perp(z_\perp) = \left[ (3p_\lambda + p_\nu)L^2 - p_\nu y_2^2 + 2 \sqrt{p_\lambda(p_\lambda - p_\nu)}Ly_2 - (p_\lambda - p_\nu)\eta^2 \right] \frac{1}{2\Sigma} - p_\nu \frac{4L^2\eta^2 z_\perp^2}{\Sigma^3}$$

$$+ \frac{1}{2} \left[ (2p_\nu - p_\lambda)y_2^2 - 2 \sqrt{p_\lambda(p_\lambda - p_\nu)}Ly_2 \right] \left[ - \frac{L^2 + \eta^2}{(z_\perp^2 + y_2^2)\Sigma} + \frac{1}{z_\perp^2 + y_2^2} \right]$$

$$+ \frac{1}{2}(p_\lambda - p_\nu) \left( 1 - \frac{z_\perp^2}{\Sigma} \right), \quad (32)$$
Figure 2: Plot of the profile function $H_{\parallel}^{\mu}(y_2, \eta)$, given by Eq. (38), for balanced black rings ($p_\lambda = 2p_\nu$) boosted along the direction $y_1$ in the plane of rotation. It is depicted over the plane $(y_2, \xi)$ [cf. Eq. (11)], and the Killing coordinate $\phi$ is suppressed. The profile function $H_{\parallel}^{\mu}$ diverges at the rod singularity $\eta = 0$, $|y_2| \leq L$, as indicated by the thick line. The two smaller pictures represent the symmetric and antisymmetric part (with respect to the origin of the $y_2$-axis) of $H_{\parallel}^{\mu}$, respectively. The case of unbalanced black rings, Eq. (35), does not produce significant changes, since the disk membrane Lorentz-contracts to the singular rod region.

and

$$\Sigma = \sqrt{z^4 + 2(y_2^2 + \eta^2 - L^2)z^2 + a^4}, \quad (33)$$

with

$$a = \left[ (\eta^2 + y_2^2 - L^2)^2 + 4\eta^2 L^2 \right]^{1/4}. \quad (34)$$

It is understood that $\eta = \sqrt{x_1^2 + x_2^2}$. Performing the integration (36) with $h$ given by Eqs. (32)–(34), one obtains

$$H_{\parallel}(x_1, x_2, y_2) = \left[ 2(2p_\lambda - p_\nu)L^2 + (2p_\nu - p_\lambda)a^2 \left( 1 + \frac{L^2 + \eta^2}{a^2 - y_2^2} \right) \right]$$

$$+ 2\sqrt{p_\lambda(p_\lambda - p_\nu)}Ly_2 \left( 1 - \frac{L^2 + \eta^2}{a^2 - y_2^2} \right) \frac{\sqrt{2}}{a} K(k) - 2\sqrt{2}(2p_\nu - p_\lambda)aE(k)$$

$$+ \frac{\sqrt{2}}{2} \left[ (2p_\nu - p_\lambda)y_2 - 2\sqrt{p_\lambda(p_\lambda - p_\nu)}L \right] \left[ \frac{\eta^2 + L^2 a^2 + y_2^2}{a^2 - y_2^2} \right] \Pi(\rho, k) + \pi \text{sgn}(y_2) \right], \quad (35)$$

where

$$k = \frac{(a^2 - \eta^2 - y_2^2 + L^2)^{1/2}}{\sqrt{2a}}, \quad \rho = \frac{(a^2 - \eta^2 - y_2^2)^{1/2}}{4a^2y_2^2}. \quad (36)$$

Again, we refer to the Appendix of [14], the only additional integral used here being $|\Sigma$ given by Eq. (33) with $z \rightarrow$ replaced by $z$]

$$\int_0^\infty \left( 1 - \frac{z^2}{\Sigma} \right) dz = 2aE(k) - aK(k). \quad (37)$$

We are especially interested in black rings in equilibrium (16), for which one is left with

$$H_{\parallel}^{\mu}(x_1, x_2, y_2) = p_\lambda L \left[ \frac{3\sqrt{2}L}{a} + \frac{2y_2}{a} \left( 1 - \frac{L^2 + \eta^2}{a^2 - y_2^2} \right) \right] K(k)$$

$$+ p_\lambda L \left[ \eta^2 + L^2 a^2 + y_2^2 \frac{\eta^2 + L^2 a^2 + y_2^2}{a^2 - y_2^2} \Pi(\rho, k) - \pi \text{sgn}(y_2) \right]. \quad (38)$$

This function is singular at the points satisfying $u = 0 = \eta$ and $|y_2| \leq L |k| = 1$ in Eq. (36), i.e. on a rod of length $2L$ contained within the wave front. This is a remnant of the curvature singularity of the original static black ring (11), which has (infinitely) Lorentz-contraction, because of the ultrarelativistic boost in the plane of the ring. For the same reason, and because
the original ring was rotating, the rod-source corresponding to Eq. (38) is not uniform. The profile (38) corresponds to a vacuum spacetime everywhere except on the rod. Notice also that the apparent divergences of $H^c$ at $y_2^2 = a^2$ and $y_2 = 0$ is only a fictitious effect: the singular behaviour of the coefficient of $\Pi$ in Eq. (38) is exactly compensated from that of $K$ in the first case and from the $\text{sgn}(y_2)$ function in the second case [recall also the form of $\rho$ in Eq. (36)]. Finally, it is interesting to observe that the antisymmetric part (in the coordinate $y_2$) of $H^c$ and $H^e$ comes entirely from the off-diagonal term $y_2\phi$ in the metric (11), which was responsible for rotation before the boost. The profile function $H^e$ is plotted in Fig. 2.

6 Boost of the supersymmetric black ring

To conclude, we demonstrate that the above method can also be employed to calculate the gravitational field generated by other black rings in the ultrarelativistic limit. The first supersymmetric black ring (solution of $D = 5$ minimal supergravity) was presented in [16] (and subsequently generalized in [17–19]). The line element reads

$$ds^2 = -f^2(dt + \omega_\psi d\psi + \omega_\phi d\phi)^2 + f^{-1}(ds_0^2 + dt^2),$$

with $ds_0^2$ as in Eq. (33) and

$$f^{-1} = 1 + \frac{Q - q^2}{2L^2}(x - y) - \frac{q}{4L^2}(x^2 - y^2),$$

$$\omega_\psi = \frac{3}{2} q(1 + y) + \frac{q}{8L^2}(1 - y^2) \left[3Q - q^2(3 + x + y)\right],$$

$$\omega_\phi = -\frac{q}{8L^2}(1 - x^2) \left[3Q - q^2(3 + x + y)\right].$$

The $S^1 \times S^2$ horizon is localized at $y \to -\infty$, and asymptotic infinity at $x, y \to -1$. The Maxwell field $F = dA$ is determined by

$$A = \frac{\sqrt{3}}{2} f(dt + \omega_\psi d\psi + \omega_\phi d\phi) - \frac{\sqrt{3}}{4} q[(1 + x) d\phi + (1 + y) d\psi].$$

The net electric charge and the local dipole magnetic charge are proportional to the positive parameters $Q$ and $q$, respectively, which (for a physical interpretation) are assumed to satisfy $Q \geq q^2$ and $L < (Q - q^2)/(2q)$ [16]. The mass and angular momenta of the ring are

$$M = \frac{3\pi}{4Q}, \quad J_\psi = \frac{\pi}{8} q(6L^2 + 3Q - q^2), \quad J_\phi = \frac{\pi}{8} q(3Q - q^2).$$

In the limit $q = 0$ the black ring becomes a static charged naked singularity, solution of the pure Einstein-Maxwell theory. In order to boost the line element (33), we can follow a procedure almost identical to the one used for the vacuum ring. The standard mass rescaling of [3] together with the inequality $L < (Q - q^2)/(2q)$ suggests that during the boost we rescale the charges as

$$Q = \epsilon p_Q, \quad q = \epsilon p_q \quad (p_Q > 2Lp_q).$$

Omitting straightforward intermediate steps, in the case of a boost orthogonal to the plane $(\xi, \psi)$ we obtain a shock $pp$-wave (24) with

$$H^e(x_2, y_1, y_2) = \frac{3\sqrt{2} p_q}{\sqrt{(\xi + L)^2 + x_2^2}} K(k),$$

and $k$ given by Eq. (25). For a parallel boost, we obtain the metric (31) with

$$H^e(x_1, x_2, y_2) = 3\sqrt{2} \left[\frac{1}{a} + \frac{y_2}{a} \left(1 - \frac{L^2 + y_2^2}{a^2 - y_2^2}\right)\right] K(k) + \frac{3\sqrt{2} p_q}{2} \left[\frac{y_2^2 + L^2 a^2 + y_2^2}{a^2 - y_2^2} \Pi(\rho, k) - \pi \text{sgn}(y_2)\right],$$

where $k$ and $\rho$ as in Eq. (50). To obtain the field of a boosted naked singularity ($q = 0$) just set $p_q = 0$ in Eq. (17). Notice that the dipole charge $q$ has an effect only in the
case of a parallel boost, since \( p_4 \) does not appear in \( H_s^* \) [which is in fact equivalent to the expression (30) for balanced vacuum rings]. This is related to the “asymmetry” between the angular momenta \( J_\psi \) and \( J_\phi \) in Eq. (44). In both boosts, one also finds that \( F = dA \) tends to zero together with its associated energy-momentum tensor (so that the “peculiar configuration” of [9] does not arise here). In fact, both \( H_s^* \) and \( H_s^\parallel \) correspond to vacuum \( pp \)-waves. In principle, rescalings different from Eq. (45) can be considered if one drops the requirement \( L < (Q - q^2)/(2q) \). The detailed investigation of this and other possibilities is left for possible future work.

Acknowledgments

M.O. is supported by a post-doctoral fellowship from Istituto Nazionale di Fisica Nucleare (bando n.10068/03).

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