Entanglement of Formation of Rotationally Symmetric States

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Computing the entanglement of formation of a bipartite state is generally difficult, but special symmetries of a state can simplify the problem. For instance, this allows one to determine the entanglement of formation of Werner states and isotropic states. We consider a slightly more general class of states, rotationally symmetric states, also known as SU(2)-invariant states. These states are invariant under global rotations of both subsystems, and one can examine entanglement in cases where the subsystems have different dimensions. We derive an analytic expression for the entanglement of formation of rotationally symmetric states of a spin-$j$ particle and a spin-$\frac{1}{2}$ particle. We also give expressions for the I-concurrence, I-tangle, and convex-roof-extended negativity.

I. INTRODUCTION

An entangled pure state is one in which complete knowledge about the overall state is incomplete with regard to the subsystems [1]. Such states are strongly correlated in the sense that the correlations cannot be reproduced by a local hidden variable theory [2, 3]. Besides being integral ingredients in no-go theorems regarding the interpretation of quantum mechanics, entangled states open new avenues in quantum engineering and information processing [1, 3]. Using entangled states, we can teleport quantum information, we can in principle build quantum computers that provide exponential speedup over classical computers, and we can also prove the security of assorted cryptographic protocols [6]. We can do all this and more without a complete understanding of entanglement, because the tasks rely only on pure states. Yet one often must deal with mixed states that arise from physical processes such as noise. This makes it important to have a measure of entanglement for mixed states so that potential applications can be explored and evaluated.

A commonly used entanglement measure for a pure state $|\psi\rangle$ of two systems, $A$ and $B$, is the entropy of the marginal density operator $\rho_A$ (or $\rho_B$),

$$E(\psi) = S(\rho_A) = -\text{tr}(\rho_A \log \rho_A) = -\text{tr}(\rho_B \log \rho_B). \quad (1.1)$$

The importance of this measure comes chiefly from the fact that it gives the rate at which copies of a pure state can be converted, by using only local operations and classical communication (LOCCs), into copies of maximally entangled states and vice versa [7, 8]. The entanglement of formation [7, 8],

$$E_F(\rho) \equiv \min_{\{p_j,|\psi_j\rangle\}} \left( \sum_j p_j E(\psi_j) \right), \quad (1.2)$$

is the so-called convex-roof extension of the pure-state measure (1.1) to bipartite mixed states $\rho$. The need to search over all ensemble decompositions of $\rho$ generally makes it impossible
to calculate the entanglement of formation, unless some efficient method to short-circuit a complete search can be found. The entanglement of formation provides an upper bound on the rate at which the maximally entangled states must be supplied to create copies of $\rho$ \cite{10,11} and the rate at which one can distill maximally entangled states from $\rho$. It is also a measure for which analytic results are known. The entanglement of formation is known for arbitrary bipartite qubit states \cite{12}, isotropic states \cite{13} and Werner states \cite{14} in arbitrary dimensions, and symmetric gaussian states in infinite dimensions \cite{15}.

Many other mixed-state entanglement measures have been defined: entanglement of distillation \cite{7,9}, negativity \cite{16}, relative entropy \cite{17}, robustness of entanglement \cite{18}, I-concurrence \cite{19}, I-tangle \cite{20}, the geometric measure \cite{21}, and others. Each is useful in particular physical contexts, but the different measures produce different orderings on mixed states even when they agree on pure states \cite{22}. Except for negativity, these measures are also difficult to calculate for an arbitrary mixed state. Here we focus on the entanglement of formation, but provide results for other measures since our results on entanglement of formation can easily be extended to these other measures.

Terhal and Vollbrecht \cite{13} showed how to use the symmetry of the isotropic states to find the entanglement of those states. Vollbrecht and Werner \cite{14} elucidated and extended the results of that paper; in particular, they formulated a general technique that leverages symmetries to simplify the calculation of convex-roof measures for symmetric states. We apply their method to a bipartite system consisting of a spin-$j$ particle and a spin-$\frac{1}{2}$ particle, i.e., a $[(2j+1) \times 2]$-dimensional Hilbert space, where the states we consider are those that are invariant under global rotations of the two particles. Such states are also known as SU(2)-invariant states \cite{23}. They are functions of a single parameter $p$,$$
abla\nabla\nabla\nabla\rho(p) = \frac{1-p}{2j+2} \Pi_{j+1/2} + \frac{p}{2j} \Pi_{j-1/2}, \quad (1.3)$$where the operators$$\Pi_{j\pm1/2} = \sum_{m=-(j+1/2)}^{j\pm1/2} |j \pm \frac{1}{2}, m\rangle \langle j \pm \frac{1}{2}, m|, \quad (1.4)$$are the projectors onto the subspaces of total angular momentum $j \pm \frac{1}{2}$. These states can appear when one loses information regarding the Cartesian reference frame of the two systems \cite{24}. They can also arise from bipartite splits of a rotationally symmetric chain of spin-$\frac{1}{2}$ particles into a single qubit and the rest. Such states can also arise as the multi-photon states that are generated by parametric down-conversion and then undergo photon losses \cite{25}.

The rotationally symmetric state \cite{13} is known to be separable if and only if \cite{23,26}$$p \leq \frac{2j}{2j+1}. \quad (1.5)$$In this paper we show that the entanglement of formation of the SU(2)-invariant state \cite{13} is given by$$E_F(\rho(p)) = \begin{cases} 0, & p \in [0, 2j/(2j+1)], \\ H\left(\frac{1}{2j+1}\left(\sqrt{p} - \sqrt{2j(1-p)}\right)^2\right), & p \in [2j/(2j+1), 1]. \end{cases} \quad (1.6)$$
Here $H(x) = -x \log x - (1 - x) \log(1 - x)$ is the binary entropy; we use the natural logarithm in calculating actual values of the entanglement of formation and when taking derivatives of entropic expressions. The formula (1.6) is noteworthy in that it provides the first example of an entanglement of formation for subsystems having different dimensions.

In Sec. II we briefly review the Terhal-Vollbrecht-Werner program for determining the entanglement of symmetric states. Section III considers rotationally invariant states and their properties. In Sec. IV we calculate the entanglement of formation of rotationally invariant states of a spin-$j$ particle and a spin-$\frac{1}{2}$ particle. Section V contains analytic expressions for other mixed-state entanglement measures. Finally, in Sec. VI we address possible extensions of our results.

II. CONVEX-ROOF ENTANGLEMENT MEASURES UNDER SYMMETRY

In this section we summarize the Terhal-Vollbrecht-Werner procedure [14] for determining the entanglement of symmetric states.

We begin with a few definitions. Let $K$ be a compact convex set (e.g., density matrices), which is itself a subset of a finite-dimensional vector space $V$, and let $M \subset K$ be an arbitrary subset of $K$ (e.g., pure states). Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a real-valued function on $M$ (e.g., an entanglement measure on pure states). The convex roof of $f$, $\co f : K \to \mathbb{R} \cup \{+\infty\}$, is a function on the entire set $K$, defined by

$$
\co f(x) \equiv \min_{\{s_j \in M\}} \left( \sum_j \lambda_j f(s_j) \right) \text{ s.t. } \sum_j \lambda_j s_j = x, \ \lambda_j \geq 0, \ \sum_j \lambda_j = 1.
$$

(2.1)

The convex roof is computed by minimizing the average value of $f$ over all possible ways of writing an element $x \in K$ as a convex combination of elements in $M$. In the case of density operators and pure states, such a convex combination is called a pure-state ensemble decomposition, so we refer generally to a decomposition of $x$. If $x$ has no decomposition in terms of elements of $M$, $\co f(x)$ is infinite. The convex roof $\co f(x)$ is the largest convex function $g$ on $K$ such that $g(s) \leq f(s)$ for $s \in M$; if $M$ is a set of extreme points of $K$, as in the case of pure states and density operators, $\co f(s) = f(s)$ for $s \in M$. In this notation, the entanglement of formation (1.2) is

$$
E_F(\rho) = \co E(\rho).
$$

(2.2)

Now suppose there exists a symmetry group $G$ that acts on $V$ through its matrix representations $\alpha_G$. Assume that $\alpha_G K \subset K$ and also that $\alpha_G M \subset M$, with $f(\alpha_G(s)) = f(s)$ for all $s \in M$. In the case of entanglement, the linear transformations $\alpha_G$ are tensor products of local unitary representations of a symmetry group; as unitary transformations, they take pure states to pure states, and as tensor products of local unitary transformations on the two subsystems, they preserve the entanglement measure. Define a projection $P : K \to K$ by averaging uniformly over the group,

$$
P(x) \equiv \int dG \alpha_G(x).
$$

(2.3)

The projection $P$ is often called twirling. We denote the range of $P$ by $PK$. Elements of $PK$ are precisely those elements of $K$ that are invariant under the projection. More generally,
\( PK \) consists of all elements of \( K \) that are invariant under the group operations and thus are known as group-invariant elements. It is clear that \( PK \) is a convex set.

The problem of finding \( \text{co } f(x) \) for \( x \in PK \) is simplified, because the problem can be divided into two parts. One first determines the function \( \epsilon : PK \rightarrow \mathbb{R} \cup \{+\infty\} \) defined by

\[
\epsilon(x) \equiv \min_{\{s \in M\}} \left( f(s) \mid P(s) = x \right) .
\]

(2.4)

Thus instead of minimizing over all possible decompositions, one first minimizes over a subset of the pure states. Then the convex roof of \( f \) on \( PK \) is the convex hull of \( \epsilon \):

\[
\text{co } f(x) = \text{co } \epsilon(x) , \quad x \in PK .
\]

(2.5)

The convex hull of \( \epsilon \) is the largest convex function on \( PK \) that nowhere exceeds \( \epsilon \). We use the same notation for convex hull and convex roof, because the convex hull of a function \( f \) is the convex roof for the special case \( M = K \), i.e., in this case,

\[
\text{co } \epsilon(x) = \min_{\{x_j \in PK\}} \left( \sum_j \lambda_j \epsilon(x_j) \mid \sum_j \lambda_j x_j = x, \ \lambda_j \geq 0, \ \sum_j \lambda_j = 1 \right) .
\]

(2.6)

For highly symmetric states, the more difficult of the two steps is determining \( \epsilon(x) \), since finding \( \text{co } \epsilon(x) \) is usually straightforward or even unnecessary because \( \epsilon(x) \) is already convex.

Demonstrating the reduction of the minimization (2.1) to Eqs. (2.4) and (2.5) is sufficiently simple that we include it here for completeness. First, let \( x \in PK \) have the optimal decomposition \( x = \sum_j \lambda_j s_j \) relative to Eq. (2.1), i.e., \( \text{co } f(x) = \sum_j \lambda_j f(s_j) \). Defining \( x_j = P(s_j) \), we have \( \epsilon(x_j) \leq f(s_j) \), \( x = P(x) = \sum_j \lambda_j x_j \), and

\[
\text{co } f(x) = \sum_j \lambda_j f(s_j) \geq \sum_j \lambda_j \epsilon(x_j) \geq \text{co } \epsilon(x) .
\]

(2.7)

Second, let \( x \in PK \) have the optimal decomposition \( x = \sum_j \lambda_j x_j \) relative to Eq. (2.6), i.e., \( \text{co } \epsilon(x) = \sum_j \lambda_j \epsilon(x_j) \), and let \( s_j \) achieve the minimum in Eq. (2.4) for \( x_j \), i.e., \( x_j = P(s_j) \) and \( \epsilon(x_j) = f(s_j) \). Then

\[
x = \sum_j \lambda_j \int dG \alpha_G(s_j)
\]

(2.8)

is a decomposition of \( x \), and

\[
\text{co } f(x) \leq \sum_j \lambda_j \int dG f(\alpha_G(s_j)) = \sum_j \lambda_j f(s_j) = \sum_j \lambda_j \epsilon(x_j) = \text{co } \epsilon(x) .
\]

(2.9)

Together, Eqs. (2.7) and (2.9) give Eq. (2.5).

This method has been used to compute the entanglement of the Werner states [14], which are invariant under \( U \otimes U \), and the isotropic states [13], which are invariant under \( U \otimes U^* \). One can generalize to a larger class of states by considering a subgroup of either \( U \otimes U \) or \( U \otimes U^* \). In the next section we examine groups of the form \( R \otimes R \), where \( R \) is a rotation.
III. SU(2)-IN Variant States

Consider two particles, one of spin $j_1$ and the other of spin $j_2$. We consider states that are symmetric under global rotations. The symmetry group is $\mathcal{R} = \{D^{(j_1)}(R) \otimes D^{(j_2)}(R)\}$, where $D^{(j)}(R) = \exp(-i\theta J \cdot n)$ is the spin-$j$ representation of the rotation $R \in SO(3)$ [or equivalently in SU(2)]. Here $J$ is the angular-momentum vector, with components $J_x$, $J_y$, and $J_z$. The twirling operator is then

$$P_{\mathcal{R}}(\rho) = \int d\mu(R) D^{(j_1)}(R) \otimes D^{(j_2)}(R)\rho D^{(j_1)}(R)^\dagger \otimes D^{(j_2)}(R)^\dagger,$$ (3.1)

where $\mu(R)$ is the group-invariant measure for the rotation group. The twirling operation describes a process where two parties use classical communication to select a random rotation $R$ that each party implements locally. This makes twirling a LOCC operation, which implies that entanglement does not increase under twirling.

The states that are invariant under the twirling operation are those that are convex combinations of the states associated with the projectors onto the irreducible subspaces of total angular momentum $J = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$. These projectors, $\Pi_J = \sum_{m=-J}^{J} |J, m\rangle \langle J, m|$, (3.2)

have associated normalized states $\Pi_J/(2J + 1)$. Thus the $\mathcal{R}$-invariant states are

$$\rho(p) = \sum_{J = |j_1 - j_2|}^{j_1 + j_2} \frac{p_J}{2J + 1} \Pi_J,$$ (3.3)

where $p_J = \text{tr}(\rho(p)\Pi_J) \geq 0$ and $\sum p_J = 1$. Any state $\sigma$ twirls to a $\mathcal{R}$-invariant state, i.e., $P_{\mathcal{R}}(\sigma) = \rho(p)$, where $p_J = \text{tr}(P_{\mathcal{R}}(\sigma)\Pi_J) = \text{tr}(\sigma P_{\mathcal{R}}(\Pi_J))$. Since $\Pi_J$ is $\mathcal{R}$ invariant, we find that $p_J = \text{tr}(\sigma \Pi_J)$ is the overlap of $\sigma$ with the subspace of total angular momentum $J$. Notice also that

$$P_{\mathcal{R}}(|J, m\rangle \langle J, m|) = \frac{\Pi_J}{2J + 1}$$ (3.4)

for any value of $m$.

We note that, for the $\mathcal{R}$-invariant state (3.3), the positive partial transpose condition is necessary and sufficient for separability when $j_1 = \frac{1}{2}$ and $j_2$ is arbitrary and when $j_1 = 1$ and $j_2$ is an integer [22, 26, 27, 28, 29].

The general problem of finding the entanglement of formation of rotationally invariant states is first to determine the function

$$\epsilon(p) = \min_{\{|\psi\rangle\}} \left( E(\psi) \mid P_{\mathcal{R}}(|\psi\rangle \langle \psi|) = \rho(p) \right) = \min_{\{|\psi\rangle\}} \left( E(\psi) \mid \langle \psi| \Pi_J |\psi\rangle = p_J \right)$$ (3.5)

and then to find its convex hull on the convex set of probabilities $p$.

It is interesting to note that although states invariant under $U \otimes U$ and $U \otimes U^*$ are quite different, this is not the case if one compares states invariant under $R \otimes R$ and $R \otimes R^*$. The reason is that conjugation of a representation is equivalent to rotating by $\pi$ about the $y$ axis, i.e., $D^{(j)}(R)^* = e^{i\pi J_y} D^{(j)}(R) e^{-i\pi J_y}$. Thus the states that are invariant under $\mathcal{R} = \{D^{(j_1)}(R) \otimes D^{(j_2)}(R)^*\}$ are obtained from the rotationally invariant states by rotating one of the two systems by $\pi$ about the $y$ axis.
IV. ENTANGLEMENT OF FORMATION FOR SU(2)-INVARIANT STATES OF SPIN-$j$ AND SPIN-$1/2$ PARTICLES

A. General considerations

To obtain insight into the entanglement of rotationally symmetric states, we examine the simplest case, a particle of arbitrary spin $j$ and a particle of spin-$1/2$. The eigenstates of total angular momentum are well known, given by the Clebsch-Gordan coefficients [30], and can be written as

$$|j \pm 1/2, m\rangle = \pm \sqrt{\frac{j + 1/2 \pm m}{2j + 1}} |j, m - 1/2 \rangle \otimes |1/2, \frac{1}{2} \rangle + \sqrt{\frac{j + 1/2 \mp m}{2j + 1}} |j, m + 1/2 \rangle \otimes |1/2, -1/2 \rangle . \quad (4.1)$$

It is straightforward to calculate the entanglement (1.1) of these states:

$$E(|j \pm 1/2, m\rangle) = H\left(\frac{j + 1/2 \pm m}{2j + 1}\right) = H\left(\frac{1}{2} - \frac{|m|}{2j + 1}\right). \quad (4.2)$$

Eigenstates with identical values of $|m|$ have the same entanglement. In the $j + 1/2$ subspace, the minimum entanglement is 0, achieved when $|m| = j + 1/2$ and in the $j - 1/2$ subspace, the minimum entanglement is $H(1/(2j + 1))$, achieved when $|m| = j - 1/2$.

The $\mathcal{R}$-invariant states are convex combinations of the states associated with the projectors (1.4) onto the subspaces of total angular momentum $j + 1/2$ and $j - 1/2$,

$$\rho(p) = \frac{1 - p}{2(j + 1)} \Pi_{j+1/2} + \frac{p}{2j} \Pi_{j-1/2} , \quad (4.3)$$

where $p = \text{tr}(\rho(p)\Pi_{j-1/2})$. Any state $\sigma$ twirls to $\mathcal{P}_\mathcal{R}(\sigma) = \rho(p)$, where $p = \text{tr}(\sigma\Pi_{j-1/2})$ is the overlap of $\sigma$ with the $j - 1/2$ subspace.

The first reason this problem is simpler than the general case of two arbitrary spins is that the rotationally invariant states are specified by the single parameter $p$. The problem of finding the entanglement of formation reduces to determining a function of this one parameter,

$$\epsilon(p) = \min_{\{\psi\}} \{ E(\psi) \mid \langle \psi | \Pi_{j-1/2} | \psi \rangle = p \} , \quad (4.4)$$

and then finding its convex hull, $\text{co} \epsilon(p)$.

B. Determining $\epsilon(p)$

To find $\epsilon(p)$, we begin by looking at a couple of example states. The first,

$$|\phi\rangle = |j, j\rangle \otimes (\sqrt{1 - \nu}|1/2, 1/2\rangle + \sqrt{\nu}|1/2, -1/2\rangle)$$

$$= \sqrt{1 - \nu}|j + 1/2, j + 1/2\rangle + \sqrt{\nu|2j + 1|j + 1/2, j - 1/2\rangle + \sqrt{2j}\nu|2j + 1|j - 1/2, j - 1/2\rangle , \quad (4.5)$$

is a product state, thus having no entanglement, i.e., $E(\phi) = 0$. This state has overlap $p = \langle \phi | \Pi_{j-1/2} | \phi \rangle = 2j\nu/(2j + 1)$. As $\nu$ ranges from 0 to 1, $p$ varies from 0 to $2j/(2j + 1)$,
which shows that $\epsilon(p) = 0$ for $p \in [0, 2j/(2j + 1)]$. Thus in determining $\epsilon(p)$, we can restrict our attention to $p \in [2j/(2j + 1), 1]$.

The second state,

\[
|\chi\rangle = -\sqrt{\mu} |j, j - 1\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{1 - \mu} |j, j\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle
\]

\[
= \frac{1}{\sqrt{2j + 1}} \left[ (-\sqrt{2j\mu} + \sqrt{1 - \mu}) |j + \frac{1}{2}, j - \frac{1}{2}\rangle + (\sqrt{\mu} + \sqrt{2j(1 - \mu)}) |j - \frac{1}{2}, j - \frac{1}{2}\rangle \right],
\]

(4.6)
is entangled, with $E(\chi) = H(\mu)$, and has overlap

\[
p = \langle \chi | \Pi_{j-1/2} |\chi\rangle = \frac{1}{2j + 1} \left( \sqrt{\mu} + \sqrt{2j(1 - \mu)} \right)^2 \equiv p_\mu.
\]

(4.7)

As $\mu$ increases from 0, $p$ increases monotonically from a value of $2j/(2j + 1)$ at $\mu = 0$ to a maximum value of 1 at $\mu = 1/(2j + 1)$; for larger values of $\mu$, $p$ decreases monotonically to a value of $1/(2j + 1)$ at $\mu = 1$.

Inverting to find $\mu$ as a function of $p$ and using the branch that gives the smaller values of $\mu$, we find that for $p \in [2j/(2j + 1), 1]$,

\[
\mu = \frac{1}{2j + 1} \left( \sqrt{p} - \sqrt{2j(1 - p)} \right)^2 \equiv \mu_{\min}(p).
\]

(4.8)

The reason for the functional notation $\mu_{\min}(p)$ becomes clear below. The function $\mu_{\min}(p)$ increases monotonically from a value of $\mu = 0$ at $p = 2j/(2j + 1)$ to $\mu = 1/(2j + 1)$ at $p = 1$. The upshot is that the state (4.6) tells us that for $p \in [2j/(2j + 1), 1]$,

\[
\epsilon(p) \leq H(\mu_{\min}(p)) = H \left( \frac{1}{2j + 1} \left( \sqrt{p} - \sqrt{2j(1 - p)} \right)^2 \right) \leq H \left( \frac{1}{2j + 1} \right).
\]

(4.9)

It turns out that $|\chi\rangle$ achieves the minimum value of $E(\psi)$ in Eq. (4.4) and thus that $\epsilon(p)$ is actually given by the expression in the middle of Eq. (4.9). To proceed with the proof of this, however, the only information we need from $|\chi\rangle$ is the final inequality in Eq. (4.9), i.e., $\epsilon(p) \leq H(1/(2j + 1))$.

The next step is to characterize the set of pure states over which we must minimize in Eq. (4.4). Any pure state of a $(2j + 1) \times 2$ system can be written in terms of a two-term Schmidt decomposition,

\[
|\psi\rangle = \sqrt{\mu} |e_1\rangle \otimes |f_1\rangle + \sqrt{1 - \mu} |e_2\rangle \otimes |f_2\rangle.
\]

(4.10)

which has entanglement $E(\psi) = H(\mu)$. The second reason this problem is simpler than the general case is that the pure states that twirl to invariant states are specified by the single parameter $\mu$. The final inequality in Eq. (4.9) implies that the only values of $\mu$ we need to consider are $\mu \in (0, 1/(2j + 1)]$, for which $H(\mu)$ is monotonically increasing.

We can introduce two unitary operators, $V$ on the spin-$j$ particle and $W$ on the spin-$\frac{1}{2}$ particle, which transform bases of our choice to the Schmidt bases. In particular, we can write $|\psi\rangle$ as

\[
|\psi\rangle = V \otimes W |\chi\rangle = V \otimes W \left( -\sqrt{\mu} |j, j - 1\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{1 - \mu} |j, j\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle \right).
\]

(4.11)
Any unitary on the spin-$\frac{1}{2}$ system is equivalent (up to an irrelevant phase) to a rotation $R$, i.e., $W = D^{(1/2)}(R)$. Rotating both particles by the inverse of $R$, we obtain the state

$$|\psi_\mu(U)\rangle = U \otimes I |\chi\rangle = U \otimes I \left(-\sqrt{\mu} |j, j - 1\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{1-\mu} |j, j\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle\right),$$

(4.12)

where $U = D^{(j)}(R)^\dagger V$.

The significance of this move is that $|\psi_\mu(U)\rangle$ has the same entanglement and the same overlap with the $j - \frac{1}{2}$ subspace as does $|\psi\rangle$. Thus in doing the minimization (4.14), we only need to consider the states $|\psi_\mu(U)\rangle$, i.e.,

$$\epsilon(p) = \min_{\{\mu, U\}} \left(H(\mu) \mid p_\mu(U) = p\right),$$

(4.13)

where

$$p_\mu(U) = \langle \psi_\mu(U) | \Pi_{j-1/2} | \psi_\mu(U) \rangle = \langle \chi | U^\dagger \otimes I \Pi_{j-1/2} U \otimes I | \chi \rangle.$$

(4.14)

Moreover, since $H(\mu)$ is a monotonically increasing function of $\mu$ in the interval of interest, we can simply minimize $\mu$ over all unitaries $U$,

$$\tilde{\mu}(p) \equiv \min_{\{U\}} \left(\mu \mid p_\mu(U) = p\right),$$

(4.15)

and plug the result into the binary entropy to give

$$\epsilon(p) = H(\tilde{\mu}).$$

(4.16)

The final reason that the problem of finding the entanglement of formation in this case is doable is that the problem can be reduced to the minimization (4.15) over a single unitary, that being a unitary acting on the spin-$j$ particle.

Our strategy now is to show that $p_\mu(U) \leq p_\mu(I) = p_\mu$, where $p_\mu$ is the function (4.17). This result, which we will prove below, allows us to immediately determine $\tilde{\mu}(p)$. Since $p_\mu$ is monotonically increasing on the relevant interval $[0, 1/(2j+1)]$, we can find a $\mu' \leq \mu$ such that $p_\mu(U) = p_{\mu'}(I)$ for any $\mu$ and $U$, implying that the minimum is always achieved by $U = I$. This means that $\tilde{\mu}$ is obtained by inverting $p_\mu = p$, and thus

$$\tilde{\mu}(p) = \mu_{\min}(p),$$

(4.17)

as given by Eq. (4.18). As promised, $\epsilon(p)$ on the interval $[2j/(2j+1), 1]$ is given by the middle expression in Eq. (4.19). The result is that

$$\epsilon(p) = \begin{cases} 0, & p \in [0, 2j/(2j+1)], \\ H \left(\frac{1}{2j+1} \left(\sqrt{p} - \sqrt{2j(1-p)}\right)^2\right), & p \in [2j/(2j+1), 1]. \end{cases}$$

(4.18)

We now must show that $p_\mu(U) \leq p_\mu$. We begin by noting that

$$p_\mu(U) = \sum_{m=-j+1/2}^{j-1/2} \left|\langle \chi | U^\dagger \otimes I | j - \frac{1}{2}, m \rangle\right|^2 = \frac{1}{2j+1} \sum_{m=-j+1/2}^{j-1/2} \left|\sqrt{1-\mu} r_m x_m e^{i\alpha_m} + \sqrt{\mu r_m} y_m e^{i\beta_m}\right|^2,$$

(4.19)
where we define two rows of matrix elements of \( U^\dagger \),

\[
\langle j, j|U^\dagger|j, m + \frac{1}{2}\rangle = x_m e^{i\alpha_m}, \quad x_m \geq 0, \quad m = -j - \frac{1}{2}, \ldots, j - \frac{1}{2}, \quad (4.20)
\]

\[
\langle j, j - 1|U^\dagger|j, m - \frac{1}{2}\rangle = y_m e^{i\beta_m}, \quad y_m \geq 0, \quad m = -j + \frac{1}{2}, \ldots, j + \frac{1}{2}. \quad (4.21)
\]

and where

\[
r_m \equiv j + \frac{1}{2} + m \geq 0. \quad (4.22)
\]

Only two rows of the unitary matrix are involved in the overlap (4.19) because one particle is a qubit. Now we can write

\[
p_\mu(U) \leq \frac{1}{2j+1} \sum_{m=-j+1/2}^{j-1/2} \left[ \sqrt{(1-\mu) r_m x_m + \sqrt{\mu r_m y_m}} \right]^2 = \frac{1}{2j+1} F(x, y), \quad (4.23)
\]

with equality when all the phase factors are equal to 1. For the remainder of the proof, we omit the range of the sum since it is always that of Eq. (4.23), i.e., \( m = -j + \frac{1}{2}, \ldots, j - \frac{1}{2} \). Notice that on this range, \( r_m \) and \( r_{-m} \) are strictly positive.

The problem now is one of maximizing \( F(x, y) \) subject to the constraints on the two relevant rows of \( U^\dagger \). Normalization of these rows gives the constraints

\[
\sum x_m^2 = 1 - x_{-j+1/2}^2 \quad \text{and} \quad \sum y_m^2 = 1 - y_{j+1/2}^2. \quad (4.24)
\]

The two rows must also be orthogonal, but we ignore this constraint on the grounds that doing so can only lead to a bigger maximum, which is still an upper bound for \( p_\mu(U) \). With this constraint ignored, it is clear from the form of Eq. (4.23) that the maximum is achieved when \( x_{-j+1/2} = 0 = y_{j+1/2} \). Thus we introduce two Lagrange multipliers and maximize the function

\[
G(x, y) = F(x, y) + \lambda_1 \left( \sum x_m^2 - 1 \right) + \lambda_2 \left( \sum y_m^2 - 1 \right). \quad (4.25)
\]

It turns out that solution to this maximization satisfies the orthogonality constraint.

Setting the first derivatives equal to zero yields the following equations,

\[
[(1-\mu)r_m + \lambda_1]x_m + \sqrt{(1-\mu)\mu r_m r_{-m}} y_m = 0, \quad (4.26)
\]

\[
\sqrt{(1-\mu)\mu r_m r_{-m}} x_m + (\mu r_{-m} + \lambda_2) y_m = 0,
\]

which hold for all \( m \). Since \( x_m \) and \( y_m \) are both nonnegative, we must have \( \lambda_1 \leq -(1-\mu)r_m < 0 \) and \( \lambda_2 \leq -\mu r_{-m} < 0 \). Moreover, if the solution is not to be \( x_m = y_m = 0 \), the determinant of the matrix of coefficients must vanish, which gives

\[
\lambda_1 \lambda_2 + (1-\mu)r_m \lambda_2 + \mu r_{-m} \lambda_1 = 0. \quad (4.27)
\]

Now suppose Eq. (4.27) holds for two or more values of \( m \). By considering any pair of \( m \) values for which Eq. (4.27) holds, it is easy to show that the Lagrange multipliers are given by \( \lambda_1 = \lambda_2 = 0 \) or by

\[
\lambda_1 = -(1-\mu)(r_m + r_{-m}) = -(1-\mu)(2j + 1), \quad (4.28)
\]

\[
\lambda_2 = -\mu(r_m + r_{-m}) = -\mu(2j + 1). \quad (4.29)
\]
The former case is ruled out by the requirement that \( \lambda_1 \) and \( \lambda_2 \) be negative, so we need only consider Eqs. (4.28) and (4.29), which imply

\[ x_m = \sqrt{\frac{\mu}{1 - \mu}} \sqrt{r_m} y_m . \]  

(4.30)

Now we find that

\[ \sum x_m^2 = \frac{\mu}{1 - \mu} \sum r_m y_m^2 \leq \frac{2j \mu}{1 - \mu} \sum y_m^2 \leq 1 , \]  

(4.31)

with equality if and only if the only nonzero \( y_m \) is \( m = j - \frac{1}{2} \) and \( \mu = 1/(2j + 1) \). We conclude that it is impossible to have nonzero solutions for \( x_m \) and \( y_m \) for more than one value of \( m \).

The result is that the only solutions of the derivative equations (4.26) have just one value of \( m \) for which \( x_m \) and \( y_m \) are nonzero, and for that value, the constraints imply that \( x_m = y_m = 1 \), giving \( 2j \) extrema of \( F(x,y) \). For all these extrema, the two rows of \( U^\dagger \) are orthogonal, as required by the constraint we neglected. The value of \( F \) at the extreme points is a function of \( m \),

\[ F(x,y) = F_m = \left( \sqrt{(1 - \mu)r_m + \sqrt{\mu r_m}} \right)^2 . \]  

(4.32)

For \( 0 \leq \mu \leq 1/(2j + 1) \), \( F_m \) is monotonically increasing for \( m \in [-j + \frac{1}{2}, j - \frac{1}{2}] \). Hence the maximum occurs at \( m = j - \frac{1}{2} \). Pulling all this together, we have

\[ p_\mu(U) \leq \frac{1}{2j + 1} F_{j-1/2} = \frac{1}{2j + 1} \left( \sqrt{\mu + \sqrt{2j(1 - \mu)}} \right)^2 = p_\mu . \]  

(4.33)

This establishes the result we needed above to complete our determination of \( \epsilon(p) \).

C. Determining \( \text{co} \epsilon(p) \)

We now have to find \( \text{co} \epsilon(p) \). What we show is that \( \epsilon(p) \) is convex, so \( \text{co} \epsilon(p) = \epsilon(p) \).

The second derivative of \( \epsilon(p) \) is given by

\[ (p(1 - p))^{3/2} \epsilon''(p) = \frac{\sqrt{2j}}{2j + 1} \log \left( \frac{\sqrt{2jp + \sqrt{1 - p}}}{\sqrt{p + \sqrt{2j(1 - p)}}} \right) - \sqrt{p(1 - p)}. \]  

(4.34)

We plot \( \epsilon''(p) \) for \( j = \frac{1}{2}, 1, \) and 3 in Fig. 11 and we see that the second derivative is positive for these values of \( j \). This turns out to be true for all \( j \), as we see in the following way. It is straightforward to show that the right-hand side of Eq. (4.34) is a nonincreasing function in the region of interest by showing that the derivative is bounded above by zero, and therefore its minimum value occurs at \( p = 1 \). We thus get the result

\[ (p(1 - p))^{3/2} \epsilon''(p) \geq \frac{\sqrt{2j}}{2(2j + 1)} \log 2j , \]  

(4.35)

and so the second derivative of \( \epsilon(p) \) is always positive for \( j \geq 1/2 \), which establishes that \( \epsilon(p) \) is convex.

We have established that the entanglement of formation is given by \( E_F(\rho(p)) = \text{co} \epsilon(p) = \epsilon(p) \), thus verifying Eq. (1.6). We plot the \( E_F(\rho(p)) \) for \( j = \frac{1}{2}, 1, \) and 3 in Fig. 2.
FIG. 1: $\log \epsilon''(p)$ for $j = \frac{1}{2}$ (solid), $j = 1$ (long-dashed), and $j = 3$ (short-dashed). We plot the log because $\epsilon''$ diverges rapidly at $p = 1$ for all $j \geq 1$. Since the log is positive, the second derivative is bigger than 1, and therefore $\epsilon(p)$ is convex. Equation (4.35) shows that this holds for all $j \geq 1/2$.

FIG. 2: The entanglement of formation in units of nats (we use the natural log in the binary entropy) for $j = \frac{1}{2}$ (solid), $j = 1$ (long-dashed), and $j = 3$ (short-dashed). The rotationally invariant states become less entangled as $j$ increases.

V. CONVEX-ROOF MEASURES

Many entanglement measures can be constructed using the the convex-roof extension. Given any pure-state entanglement measure $X(\psi) = f(\rho_A)$ that is (i) invariant under local unitaries and (ii) a concave function of the marginal density matrix $\rho_A$, co $X(\rho)$ is an entanglement monotone. The procedure used here to find the entanglement of formation of rotationally invariant states applies to any other convex-roof extension: first determine

$$\xi(p) \equiv \min_{\{\psi\}} \left( X(\psi) \mid \langle \psi | \Pi_{j-1/2} | \psi \rangle = p \right),$$

(5.1)

the minimum value of $X$ over all pure states that project to $\rho(p)$, and then compute the convex hull of $\xi(p)$. The state $|\chi\rangle$ of Eq. (4.6), which minimizes $\epsilon(p)$, also minimizes $\xi(p)$, because different pure-state entanglement measures give the same ordering of pure states. In terms of the quantities introduced in our consideration of the entanglement of formation,
this is the statement that for the state (4.10), \( f(\mu) \equiv X(\psi) \) is an increasing function of \( \mu \) for \( 0 \leq \mu \leq 1/(2j + 1) \), thus giving \( \xi(p) = f(\mu_{\min}(p)) \), in analogy to Eq. (4.10).

In this section we apply this technique to find the concurrence, the tangle, and the negativity of rotationally invariant states of a spin-\( j \) particle and a spin-\( \frac{1}{2} \) particle.

A. I-Concurrence

The generalized concurrence \[19\] of a joint pure state \( |\psi\rangle \) of systems A and B measures the purity of the marginal states.

\[
C(\psi) = \sqrt{2 [1 - \text{tr}(\rho_A^2)]} = \sqrt{2 [1 - \text{tr}(\rho_B^2)]}. \tag{5.2}
\]

The concurrence of the state (4.10) is \( C(\psi) = 2\sqrt{\mu(1 - \mu)} \). The convex-roof extension of the concurrence, \( C(\rho) \equiv \text{co} C(\rho) \), is called the I-concurrence.

As discussed above, our results for the entanglement of formation determine the function

\[
c(p) \equiv \min_{\{\psi\}} \left( C(\psi) \mid \langle \psi | \Pi_{j-1/2} | \psi \rangle = p \right) = 2\sqrt{\mu_{\min}(1 - \mu_{\min})}
= \frac{2}{2j + 1} \left( \sqrt{2j(2p - 1)} - (2j - 1)\sqrt{p(1 - p)} \right). \tag{5.3}
\]

The function \( c(p) \) is clearly convex because it is the sum of convex functions. When both particles have spin-\( \frac{1}{2} \), \( c(p) \) is linear, \( c(p) = 2p - 1 \) for \( 1/2 \leq p \leq 1 \), and it becomes more convex as the spin \( j \) of the first particle increases. Thus the I-concurrence of a rotationally symmetric state is

\[
C(\rho(p)) = \begin{cases} 0, & p \in [0, 2j/(2j + 1)], \\
c(p), & p \in [2j/(2j + 1), 1]. \end{cases} \tag{5.4}
\]

The I-concurrence increases from 0 at \( p = 2j/(2j + 1) \) to \( 2\sqrt{2j}/(2j + 1) \) at \( p = 1 \).

B. Tangle

The tangle is the convex-roof extension of the squared concurrence \[20\]:

\[
\tau(\rho) \equiv \text{co} C^2(\rho). \tag{5.5}
\]

Since

\[
\min_{\{\psi\}} \left( C^2(\psi) \mid \langle \psi | \Pi_{j-1/2} | \psi \rangle = p \right) = c^2(p) \tag{5.6}
\]

and the convexity of \( c(p) \) implies the convexity of \( c^2(p) \), the tangle of rotationally symmetric states is given by

\[
\tau(\rho(p)) = \begin{cases} 0, & p \in [0, 2j/(2j + 1)], \\
c^2(p), & p \in [2j/(2j + 1), 1]. \end{cases} \tag{5.7}
\]
C. Convex-Roof-Extended Negativity

The negativity of a joint density operator \( \rho \) is proportional to the sum of the negative eigenvalues of the partial transpose of \( \rho \):

\[
N(\rho) = -2 \sum_{\mu_j \leq 0} \mu_j = ||\rho^{TB}|| - 1. \tag{5.8}
\]

Here the \( \mu_j \)'s are the eigenvalues of the partial transpose \( \rho^{TB} \) of \( \rho \) with respect to system \( B \), and \( ||\rho^{TB}|| \) is the sum of the absolute values of these eigenvalues. The negativity is an entanglement monotone \[16\], and it is straightforward to evaluate. The negativity does not, however, identify all entangled states. There are states, the bound entangled states, that have a positive partial transpose and thus zero negativity, yet are still entangled. Notice that there are no such states in the one-parameter family of rotationally symmetric states. To get around the inability of the negativity to identify all entangled states, the convex-roof-extended negativity, \( coN \), has been proposed as a mixed-state entanglement measure. This comes with a considerably increased difficulty in computation, of the sort associated with any convex-roof entanglement measure, but the convex-roof-extended negativity has been evaluated for the isotropic states and Werner states \[32\].

For \( p \in [2j/(2j+1), 1] \), the partial transpose of the rotationally symmetric state \( \rho(p) \) of Eq. (1.3) has two eigenvalues

\[
\mu_+ = \frac{1}{2(j+1)} \left( \frac{1}{2j+1} + p \right),
\]
\[
\mu_- = \frac{1}{2j+1} - \frac{p}{2j}, \tag{5.9}
\]

the first of which is \( 2(j+1) \)-fold degenerate, and the second of which is \( 2j \)-fold degenerate and is negative for \( p \in (2j/(2j+1), 1] \). Thus the negativity of \( \rho(p) \) is

\[
N(\rho(p)) = \max \left[ 0, 2 \left( p - \frac{2j}{2j+1} \right) \right]. \tag{5.10}
\]

To find the convex-roof-extended negativity, we note that for the states \( \psi \), \( N(\psi) = 2\sqrt{\mu(1-\mu)} = C(\psi) \). This means that for rotationally symmetric states, the convex-roof-extended negativity is actually identical to the I-concurrence \[5.4\],

\[
coN(\rho(p)) = coC(\rho(p)) = C(\rho(p)). \tag{5.11}
\]

VI. SUMMARY

We have computed the entanglement of formation of rotationally symmetric states for a \([2j+1 \times 2]\)-dimensional system. Three features of the problem simplified the endeavor: the rotationally invariant states are specified by a single parameter, pure states that are twirled to an invariant state are determined by a single constraint, and only one local unitary is needed to relate the pure states to a fiducial state. The first makes finding the convex hull in Eq. \[2.5\] easier, and the other two allow one to efficiently characterize the subset of pure states over which one minimizes in Eq. \[2.4\]. These elements were also vital in
all other applications of the Terhal-Vollbrecht-Werner procedure to date, i.e., generalized Werner states and isotropic states.

The rotationally symmetric states of a spin-\(j\) particle and a spin-\(\frac{1}{2}\) particle are similar in some ways to the states of a two-qubit system: positive partial transpose is necessary and sufficient for separability, and the pure states \(R \otimes R|\psi\rangle\) in the optimal decomposition all have the same entanglement. The entanglement in these states vanishes as one spin becomes more classical (\(j \to \infty\)). We have also determined the I-concurrence, tangle, and the convex-roof negativity of the rotationally symmetric states. These all display the same features as the entanglement of formation. This is quite different from what occurs for the isotropic states in high dimensions, where the concurrence is linear [33], the tangle has the same behavior as the entanglement of formation [33], and the convex-roof negativity is the same as the negativity of the isotropic state [32].

Since we have determined an optimal decomposition, we can also find the entanglement of formation of states that are not rotationally symmetric. This procedure for extending optimal decompositions to other states was outlined by Vollbrecht and Werner [14]: arbitrary convex combinations of the pure states \(R \otimes R|\psi\rangle\) have the same entanglement as the rotationally symmetric states, which are uniform convex combinations of these states. This extension procedure does not cover the entire space; the states for which it does apply and their properties are currently being investigated.

We have examined the simplest case of rotationally invariant states, and the next step, which we are pursuing, is to investigate two spin-1 particles, where the rotationally symmetric states constitute a two-parameter family. For two spin-1 particles, it is known that the positive partial transpose condition is necessary and sufficient for separability, and the entanglement of a large fraction of the rotationally symmetric states can be obtained from extending the decompositions for generalized Werner states and isotropic states [14]. However the calculation for the rest of the states poses some difficulties, because the important features that facilitated the calculations in this paper are now absent. The states are functions of two variables, there are two constraints on the pure states, instead of one, and it becomes difficult to characterize efficiently the states that are twirled to an invariant state. In particular, the technique used to remove the unitary on one subsystem, thus obtaining Eq. (4.12), no longer applies. The entire problem gets progressively more difficult as the spins of the two particles increase.

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