Gravitating $SU(N)$ Monopoles from Harmonic maps

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Abstract

Spherically symmetric solutions of the $SU(N)$ Einstein-Yang-Mills-Higgs system are constructed using the harmonic map ansatz. This way the problem reduces to solving a set of ordinary differential equations for the appropriate profile functions. In the $SU(2)$ case, we recover the equations studied in great detail previously, while in the $SU(N)$ ($N > 2$) one we find new solutions which correspond to monopole-antimonopole configurations.

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I. INTRODUCTION

Topological defects are thought to have formed during the phase transitions that took place in the early universe. If a symmetry is spontaneously broken \((G \rightarrow H)\), the topology of the vacuum manifold \(\mathcal{M} = G/H\) determines which type of the defect arises during the symmetry breakdown. This way we could get domain walls, cosmic strings, monopoles and textures if \(\mathcal{M}\) is (respectively) disconnected, has contractible loops, non-contractible 2-spheres and non-contractible 3-spheres. In 1974, 't Hooft and Polyakov realised that the bosonic part of the Georgi-Glashow model, which is essentially an \(SU(2)\) Yang-Mills-Higgs system, possesses soliton solutions which, due to their topological properties, carry a magnetic charge. Since the unbroken \(U(1)\) group is associated with the electromagnetic field, these solutions are said to be describing “magnetic monopoles”. Soon afterwards people started looking at various embeddings of \(SU(2)\) into higher gauge groups. The embedding into \(SU(3)\) was first studied and the two possible embeddings which correspond to a \(SU(2)\), respectively \(SO(3)\) subgroup, were investigated. For the latter embedding, solutions with magnetic charge \(\pm \sqrt{3}\) as well as solutions with zero topological charge were constructed. A systematic analysis of the solutions in an \(SU(3)\) model with a non-vanishing potential has been done. Recently, static monopole solutions of the second order \(SU(N)\) BPS Yang-Mills-Higgs equations, which are not solutions of the first order Bogomolnyi equations, have also been constructed. These spherically symmetric solutions may be interpreted as monopole-antimonopole configurations and their construction involves the use of harmonic maps into complex projective spaces.

When an \(SU(2)\) Yang-Mills-Higgs system is minimally coupled to gravity, three different types of solutions are possible; namely embedded Reissner-Nordström solutions, gravitating monopoles and non-abelian black holes. Gravitating monopoles exist only up to a maximal value of the gravitational coupling beyond which their Schwarzschild radius becomes larger than the radius of the monopole core. At this maximal value of the gravitational coupling the solutions bifurcate producing a branch of extremal Reissner-Nordström solutions. Non-abelian black holes can be thought of as black holes situated inside the core of a magnetic monopole. Consequently, they exist only in a limited domain of the gravitational coupling-horizon plane. Note that gravitating monopoles in \(SU(3)\) corresponding to a \(SU(2)\) subgroup have been studied, while the gravitating monopoles and non-
abelian black holes in $SU(5)$, corresponding to a $SU(2)$ subgroup, were studied in [13]. Here we will construct the spherical symmetric gravitating solutions of the second order $SU(N)$ Einstein-Yang-Mills-Higgs equations which are neither solutions of the first order Bogomolnyi equations nor simple embedding of the $SU(2)$ ones.

When one considers fields in three dimensional space, it is sometimes convenient to introduce spherical polar coordinates to describe all points in space with the origin of coordinate system located at a specific point - like the centre of the soliton (e.g. monopole). Then one introduces a radial variable $r$ and two angular variables describing points on the sphere of radius $r$ - the complex variables $z$ and $\bar{z}$ (discussed later in more detail). Therefore, for a given $r$, we have maps from $S^2$ of radius $r$, which we can interpret as compactified $R^2$. The harmonic map ansatz [14] exploits this property; it uses maps of $S^2 \rightarrow G$ to construct maps of $R^3 \rightarrow G$. This is done by considering maps of $R^2 \rightarrow G$ and assuming that the parameters of these maps are functions of $r$ only. This cannot be done in an arbitrary way; the fact that the resultant maps are those of $R^3 \rightarrow G$ imposes some constraints of continuity etc (discussed in the next section).

Our paper is organized as follows. In Section II we present the model and introduce the harmonic map ansatz. In Section III we derive and then discuss the resulting equations for the $SU(2)$ case while in Section IV the equations for the $SU(3)$ model and also present our numerical results. Finally, in Section V we presents our conclusions.

II. THE MODEL

The $SU(N)$ Einstein-Yang-Mills-Higgs action is given by:

$$S = \int \left[ \frac{R}{16\pi G} - \frac{1}{2} \text{tr} (F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} \text{tr} (D_\mu \Phi D^\mu \Phi) + \frac{1}{8} \lambda (\text{tr} (\Phi^2 - \eta^2))^2 \right] \sqrt{-g} \, d^4x$$

(1)

where $g$ denotes the determinant of the metric while the field strength tensor is defined by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

(2)

and the covariant derivative of the Higgs field reads:

$$D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi].$$

(3)

The matrix $\eta$ represents a constant matrix of the form: $\eta = i\nu 1_N$, where $\nu \in \mathbb{R}$ and $1_N$ denotes the unit matrix in $N$ dimensions. The constants in the action represent Newton's
constant $G$, the Higgs self-coupling constant $\lambda$ and the vacuum expectation value of the Higgs field $v$. Note that for $N > 2$ the potential in (1) is not the most general one that could have been used.

Variation of the action (1) with respect to the metric $g^{\mu\nu}$ leads to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

with the stress-energy tensor $T_{\mu\nu} = g_{\mu\nu} \mathcal{L} - 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}$ given by

$$T_{\mu\nu} = \frac{1}{2} \text{tr} \left( D_\mu \Phi D_\nu \Phi - \frac{1}{2} g_{\mu\nu} D_\alpha \Phi D^\alpha \Phi \right) + 2 \text{tr} \left( g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) + \frac{1}{8} g_{\mu\nu} \lambda \left( \text{tr} (\Phi^2 - \eta^2) \right)^2. \tag{5}$$

Variation with respect to the gauge fields $A_\mu$ and the Higgs field $\Phi$ leads to the matter equations

$$\frac{1}{\sqrt{-g}} D_\mu \left( \sqrt{-g} F^{\mu\nu} \right) - \frac{1}{4} \left[ \Phi, D^{\nu} \Phi \right] = 0,$n

$$\frac{1}{\sqrt{-g}} D_\mu \left( \sqrt{-g} D^{\mu} \Phi \right) + \lambda \text{tr} (\Phi^2 - \eta^2) \Phi = 0. \tag{6}$$

In what follows we consider the static Einstein-Yang-Mills-Higgs equations in order to construct their spherically symmetric and purely magnetic (i.e. $A_0 = 0$) solutions based on the harmonic map ansatz first introduced in [9].

A. The Harmonic Map Ansatz

The starting point of our investigation is the introduction of the coordinates $r, z, \bar{z}$ on $\mathbb{R}^3$. In terms of the usual spherical coordinates $r, \theta, \phi$ the Riemann sphere variable $z$ is given by $z = e^{i\phi} \tan(\theta/2)$ and $\bar{z}$ is the complex conjugate of $z$. In this system of coordinates the Schwarzschild-like metric reads:

$$ds^2 = -A^2(r)B(r)dt^2 + \frac{1}{B(r)} dr^2 + \frac{4r^2}{(1 + |z|^2)^2} dz d\bar{z}, \quad B(r) = 1 - \frac{2m(r)}{r}, \tag{7}$$

where $A$ and $B$ are the metric functions which are real and depend only on the radial coordinate $r$, and $m(r)$ is the mass function. The (dimensionfull) mass of the solution is given by $m_\infty \equiv m(\infty)$. For this metric the square-root of the determinant takes the simple form:

$$\sqrt{-g} = iA(r) \frac{2r^2}{(1 + |z|^2)^2}. \tag{8}$$
Using (7) the matter equations (6) read:

\[ \frac{1}{4} [D_r \Phi, \Phi] - \frac{(1 + |z|^2)^2}{2r^2} (D_z F_{rz} + D_z F_{rz}) = 0, \]  

\[ \frac{1}{4} [D_z \Phi, \Phi] + \frac{1}{A} D_r (AB F_{rz}) - \frac{1}{2r^2} D_z ((1 + |z|^2)^2 F_{z\bar{z}}) = 0, \]  

\[ \frac{1}{A r^2} D_r (AB r^2 D_r \Phi) + \frac{(1 + |z|^2)^2}{2r^2} (D_z D_z \Phi + D_z D_z \Phi) + \lambda \text{tr} (\Phi^2 - \eta^2) \Phi = 0 \]  

In addition, the Einstein equations (4) take the form:

\[ \frac{2}{r^2} m' = 8\pi G T_0^0, \quad \frac{2 A'}{r} B = 8\pi G (T_0^0 - T_r^r) \]  

where the prime denotes the derivative with respect to \( r \) and

\[ T_0^0 = -\frac{B(1 + |z|^2)^2}{r^2} \text{tr} (|F_{rz}|^2) + \frac{(1 + |z|^2)^4}{4r^4} \text{tr} (|F_{z\bar{z}}|^2) - \frac{B}{4} \text{tr} ((D, \Phi)^2) - \frac{(1 + |z|^2)^2}{4r^2} \text{tr} (|D_z \Phi|^2) + \frac{\lambda}{8} (\text{tr} (\Phi^2 - \eta^2))^2, \]  

\[ T_0^0 - T_r^r = -\frac{2B(1 + |z|^2)^2}{r^2} \text{tr} (|F_{rz}|^2) - \frac{B}{2} \text{tr} ((D, \Phi)^2). \]  

Next we introduce the following ansatz (9) for the SU(\( N \)) gravitating monopoles:

\[ \Phi = i \sum_{j=0}^{N-2} h_j \left( P_j - \frac{1}{N} \right), \quad A_z = \sum_{j=0}^{N-2} g_j [P_j, \partial_z P_j], \quad A_r = 0 \]  

where \( h_j(r), g_j(r) \) are the radial dependeded matter profile functions and \( P(z, \bar{z}) \) are \( N \times N \) Hermitian projectors: \( P_j = P_j^\dagger = P_j^2 \), which are independent of the radius \( r \). Note that all \( N - 1 \) projectors \( P_i \) are orthogonal to each other since \( P_i P_j = 0 \) for \( i \neq j \) and that we are working in a real gauge, since \( A_z = -A_z^\dagger \).

As shown in [9], the projectors \( P_k \) defined as

\[ P_k = \frac{(\Delta^k f)^\dagger \Delta^k f}{|\Delta^k f|^2}, \quad k = 0, ..., N - 1 \]  

where

\[ \Delta f = \partial_z f - \frac{f \left( f^\dagger \partial_z f \right)}{|f|^2} \]  

give us our required set of orthogonal harmonic maps (for details see [15]). Moreover, the harmonic maps with spherical symmetry can obtained by applying the orthogonalization procedure to the initial holomorphic vector \( f \) given by

\[ f = (f_0, ..., f_j, ..., f_{N-1})^\dagger, \quad \text{where} \quad f_j = z^j \sqrt{\binom{N-1}{j}} \]
and \( \binom{N}{j} \) denote the binomial coefficients. Then equation (9) is automatically satisfied.

In dealing with the equations which arise from the harmonic map ansatz (15) it is convenient to replace the profile functions \( h_j(r), g_j(r) \) by the functions \( b_j(r), c_j(r) \) which are defined as the following linear combinations of \( g_j \) and \( b_k \):

\[
h_j = \sum_{k=j}^{N-2} b_k, \quad c_j = 1 - g_j - g_{j+1}, \quad j = 0, \ldots, N-2,
\]

(19)

where \( g_{N-1} = 0 \).

Next we will describe in detail the gravitating monopoles obtained from our harmonic map ansatz for the simplest cases of \( SU(2) \) and \( SU(3) \). The situation for general \( SU(N) \) will then become clear.

### III. GRAVITATING MONOPOLES IN \( SU(2) \)

For \( N = 2 \) there are two profile functions, \( b_0, c_0 \) and our ansatz (15) reduces equations (10) and (11) to the following set of second order nonlinear ordinary differential equations:

\[
\begin{align*}
\frac{1}{A} \left( AB c_0' \right)' &= \frac{1}{4} b_0^2 c_0 + \frac{1}{r^2} c_0 \left( c_0^2 - 1 \right), \\
\frac{1}{Ar^2} \left( r^2 AB b_0' \right)' &= \frac{2}{r^2} b_0 c_0^2 + \frac{\lambda}{2} b_0 \left( b_0^2 - 4v^2 \right).
\end{align*}
\]

(20)

(21)

Finally the Einstein equations (12) take the form:

\[
\begin{align*}
\frac{2}{r^2} m' &= 8\pi G \left[ \frac{B}{8} b_0^2 + \frac{1}{4r^2} b_0^2 c_0^2 + \frac{1}{r^2} Bc_0^2 + \frac{1}{2r^4} \left( 1 - c_0^2 \right)^2 + \frac{\lambda}{16} \left( b_0^2 - 4v^2 \right)^2 \right], \\
\frac{2}{r} A' &= 8\pi G \left( \frac{1}{4} b_0^2 + \frac{2}{r^2} c_0^2 \right),
\end{align*}
\]

(22)

(23)

where \( m(r) \) is given by (7). These equations have previously been studied in great detail in [11] (after the rescale of the Higgs profile function \( b_0 \rightarrow 2b_0 \)). We will not repeat the numerical calculations here, but refer the reader to the mentioned papers.

### IV. GRAVITATING MONOPOLES IN \( SU(3) \)

For \( N = 3 \) there are four profile functions, \( b_0, b_1, c_0, c_1 \) and our ansatz (15) reduces equations (10) and (11) to the following set of second order nonlinear ordinary differential
equations:
\[ \frac{1}{A} (AB c_j)' = \frac{1}{4} b_j^2 c_j - \frac{1}{r} c_j (1 - 2c_j^2 + c_k^2) , \]  
\[ \frac{1}{Ar^2} (r^2 AB b_j')' = \frac{2}{r^2} (2b_j c_j^2 - b_k c_k^2) + \frac{2\lambda}{3} b_j \left( b_j^2 + b_k b_j + b_k^2 - \frac{9}{2} v^2 \right) \].

Here the indices are chosen from the set \( \{0, 1\} \), \( k \neq j \), and we assume the symmetry under the interchange of indices \( 0 \leftrightarrow 1 \) when applied to both the \( b_j \) and \( c_j \) functions. Note that in the flat limit, ie for \( A = B = 1 \), and for \( \lambda = 0 \) the equations of [9] are recovered.

Finally, the Einstein equations (12) take the form:
\[ \frac{2}{r^2} m' = 8\pi G \left[ \frac{B}{6} (b_0^2 + b_0' b_1 + b_1^2) + \frac{1}{2r^2} (b_0^2 c_0^2 + b_1^2 c_1^2) + \frac{2B}{r^2} (c_0^2 + c_1^2) \right. \\
\left. + \frac{2}{r^2} (1 - c_0^2 - c_1^2 + c_0^4 + c_1^4 - c_0^2 c_1^2) + \frac{\lambda}{18} \left( b_0^2 + b_0 b_1 + b_1^2 - \frac{9}{2} v^2 \right)^2 \right] , \]
\[ \frac{2 A'}{r^2} = 8\pi G \left[ \frac{1}{3} (b_0^2 + b_0' b_1 + b_1^2) + \frac{4}{r^2} (c_0^2 + c_1^2) \right] . \]

These equations correspond to the \((\tau_1, \tau_2, \tau_3) = (\lambda_7, -\lambda_5, \lambda_2)\) embedding of \( SU(2) \) into \( SU(3) \) (see section below). The \( \tau_i \)'s here are the \( SU(2) \) (or \( SO(3) \)) generators, while the \( \lambda_i \)'s denote the Gell-Mann matrices.

The equations corresponding to the \((\tau_1, \tau_2, \tau_3) = (\lambda_1/2, \lambda_2/2, \lambda_3/2)\) embedding of \( SU(2) \) into \( SU(3) \) have been studied in [12] and are not included in our approach as can be seen by setting either \( c_0 \) or \( c_1 \) equal to zero and comparing our equations with those of [12]. This is due to the fact that in our construction the corresponding Higgs and gauge fields are non-embeddings of the \( SU(2) \) ones.

A. Comparison with the spherically symmetric ansatz

The spherically symmetric ansatz used in [7, 8, 10] is given as follows:
\[ \Phi = F_1(r) Y + F_2(r) \left( Y^2 - \frac{2}{3} \right) , \]  
where \( Y = \hat{r} \cdot \vec{\Lambda} \), \( \vec{\Lambda} = (\lambda_7, -\lambda_5, \lambda_2) \) (or in components \( Y_{ab} = -i\epsilon_{abc}\hat{r}_c \)) and
\[ A_i = \frac{G(r)}{2r} (\hat{r} \times \Lambda)_i + \frac{H(r)}{2r} \left[ (\hat{r} \times \vec{\Lambda})_i, \hat{r} \cdot \vec{\Lambda} \right]_+ , \quad A_0 = 0 , \]  
where \([,]_+\) denotes the anti-commutator.
After some algebra, it can be seen that the above Higgs field can be rewritten in terms of hermitian and orthogonal projectors constructed out of the matrix $Y$ (note that $Y^3 = Y$):

$$
\Phi = (b_0 + b_1) \left( P_0 - \frac{1}{3} \right) + b_1 \left( P_1 - \frac{1}{3} \right),
$$

(30)

with

$$
P_0 = \frac{Y + Y^2}{2}, \quad P_1 = Y^2 - 1.
$$

(31)

With these projectors, the gauge fields can be constructed in an analogous way. Finally, the full correspondence between the harmonic map ansatz and the spherically symmetric ansatz can be established by observing that

$$
F_1 = \frac{1}{2} (b_0 + b_1), \quad F_2 = \frac{1}{2} (b_0 - b_1), \quad \tilde{G} \equiv G - 2 = c_0 - c_1, \quad H = c_0 + c_1.
$$

(32)

B. Numerical results

As in [11], without any loss of generality, we set the vacuum expectation value $v = 1$ (different $v$’s can always be accommodated by rescaling the radial coordinate and the Higgs field functions) and we also define $\alpha^2 \equiv 4\pi G$. Then the dimensionless mass of the solutions is defined as: $M = \frac{m}{\alpha^2}$. Here we will consider only the $\lambda = 0$ case and leave $\lambda \neq 0$ to a further study.

In this section we construct the monopole solutions of equations (24)-(27) numerically using a collocation method for the boundary-value ordinary differential equations [17]. In this procedure the set of non-linear coupled differential equations is solved using a damped Newton method of quasi-linearization.

The boundary conditions for the metric profiles read:

$$
m(r = 0) = 0, \quad A(r = \infty) = 1
$$

(33)

and we also assume that $A(r = 0)$ is finite.

Since the equations (24)-(27) are very similar to those of the $SU(2)$ case, we expect that their solutions will bifurcate into extremal Reissner-Nordström solutions at some critical value of the gravitational coupling constant $\alpha$. Prior to describing our numerical results, it is worth mentioning that the charge of the limiting Reissner-Nordström solution can be given in terms of the asymptotic values of the gauge fields $\tilde{c}_0 \equiv c_0(\infty)$, $\tilde{c}_1 \equiv c_1(\infty)$. Due
to (26) and taking into account the asymptotic behaviour of the various fields, we find that the mass function \( m_{RN}(r) \) of the Reissner-Nordström solution is given by:

\[
m_{RN}(r) = m_{\infty,RN} - \frac{\alpha^2 Q^2}{2r}, \quad Q^2 = 4 \left(1 - \tilde{c}_0^2 - \tilde{c}_1^2 + \tilde{c}_0^4 + \tilde{c}_1^4 - \tilde{c}_0^2 \tilde{c}_1^2\right).
\]

(34)

Note that for the extremal Reissner-Nordström solution \( m_{\infty,RN} = \alpha Q \).

Following the discussions in [9], we note that there are three different types of solutions which seem to be of particular interest. They can be distinguished from each other by whether they satisfy, or not, the Bogomolnyi equation and whether the symmetry-breaking (SB) is maximal (unbroken group \( U(1) \times U(1) \)) or minimal (unbroken group \( U(2) \)).

First, we will discuss in detail the case of non-Bogomolnyi maximal SB solutions which satisfy the following boundary conditions for the fields:

\[
b_0(r = 0) = 0, \quad b_1(r = 0) = 0, \quad c_0(r = 0) = 1, \quad c_1(r = 0) = 1
\]

at the origin and

\[
b_0(r = \infty) = -2, \quad b_1(r = \infty) = 4, \quad c_0(r = \infty) = 0, \quad c_1(r = \infty) = 0
\]

at infinity. Any of such solutions has a magnetic charge \((0, 2)\) and according to the discussion given there can be interpreted as a superposition of two monopoles and two pairs of monopoles-antimonopoles. In the flat limit, this solution has a mass \( M = 4.5 \). When gravity is minimally coupled to the system, the solution gets smoothly deformed by it. The metric function \( B(r) \) develops a minimum which gets deeper as \( \alpha \) increases. At a critical value of \( \alpha \) the solution develops a double zero at some finite value of \( r = r_h \), which can be interpreted as the horizon of the extremal Reissner-Nordström solution. Thus, the limiting solution can be described by this field for \( r \geq r_h \), while it is non-trivial and non-singular for \( 0 \leq r < r_h \). This is illustrated in Fig. 1 (respectively 2), where we present the profiles of the metric and gauge functions (respectively, metric and Higgs functions) for the flat limit \( \alpha = 0.0 \) and \( \alpha \approx \alpha_{cr} = 0.615 \).

As in the \( SU(2) \) case, the numerical results indicate that the gravitating solution exists up to a maximal value of \( \alpha \), \( \alpha_{max} \approx 0.625 \). By decreasing \( \alpha \) from \( \alpha_{max} \), another branch of solutions can be constructed and \( B_{min} = 0 \) is reached at the critical value \( \alpha_{cr} \approx 0.615 \). This is illustrated in Fig. 3, where plots of \( \frac{m_{\infty}}{\alpha} \) and of the minimal value \( B_{min} \) in terms of \( \alpha \) are presented. Note that at \( \alpha = \alpha_{cr} \), the quantity \( \frac{m_{\infty}}{\alpha} \) equals two. Since for the
Reissner-Nordström solution $\frac{m_{\infty,RN}}{\alpha} = Q$, this implies that the solution indeed bifurcates into a charge-two Reissner-Nordström solution (as can be calculated from (34)).

Next we studied the gravitating analogue of the Burzlaff solution [7]. In this case $b_0 = -b_1$ while $c_0 = c_1$ and the solution corresponds to a non-Bogomolnyi non-maximal SB case with charge $(0, [2])$. [Here the notation is that magnetic weights are defined by square brackets [9].] We choose the following boundary conditions for the fields:

$$b_0(r = 0) = 0, \quad b_1(r = 0) = 0, \quad c_0(r = 0) = 1, \quad c_1(r = 0) = 1$$ (37)

at the origin and

$$b_0(r = \infty) = \sqrt{3}, \quad b_1(r = \infty) = -\sqrt{3}, \quad c_0(r = \infty) = 0, \quad c_1(r = \infty) = 0$$ (38)

at infinity. The choice of the boundary conditions for the Higgs field at infinity is not fixed for a vanishing potential; however, following [8] we have chosen our conditions as if the potential was present. This solution, due to (34), should bifurcate producing a charge-two Reissner-Nordström solution as confirmed in Fig. 3. Again, we find a back-bending since the gravitating Burzlaff solution exists up to $\alpha_{\text{max}} \approx 1.106$ and reaches $B_{\text{min}} = 0$ at $\alpha = \alpha_{\text{cr}} \approx 1.063$. In the flat limit, the mass of the solution is $M = 2.5$. Since the solutions are less heavy than the ones in the non-Bogomolnyi maximal SB case, it is clear that the critical value of the gravitational coupling should be bigger in the Burzlaff case (which is confirmed by our numerical results).

Finally, we constructed the gravitating analogues of the Bais solution [6] by choosing the boundary conditions (again) as if the potential were present [8]. The boundary conditions were

$$b_0(r = 0) = 0, \quad b_1(r = 0) = 0, \quad c_0(r = 0) = 1, \quad c_1(r = 0) = 1$$ (39)

at the origin and

$$b_0(r = \infty) = -\sqrt{3}, \quad b_1(r = \infty) = 0, \quad c_0(r = \infty) = 0, \quad c_1(r = \infty) = \frac{1}{\sqrt{2}}$$ (40)

at infinity. These solutions correspond to the Bogomolnyi non-maximal SB solutions with charge $(2, [1])$. Due to (34) the branch of solutions should bifurcate producing a branch of Reissner-Nordström solutions with charge $\sqrt{3}$ as shown in Fig. 3. In fact the gravitating Bais solutions exist up to a maximal value of $\alpha$: $\alpha_{\text{max}} \approx 1.446$. We were not able to find a back-bending and it is likely that there is none since $B_{\text{min}} \approx 4.7 \cdot 10^{-3}$ at $\alpha = \alpha_{\text{max}}$. If
there were a back-bending, then the branch of solutions would be very small. Since, in the flat limit, the mass of the gravitating Bais solution (again) is smaller than in the other two cases studied, the maximal value of $\alpha$ for the former is larger than for the latter.

V. CONCLUSIONS

We have studied gravitating $SU(N)$ monopoles by relying on the harmonic map ansatz. In the $SU(2)$ case, we have recovered the equations studied previously in great detail in [11]; while in the $SU(3)$ one, we have found the gravitating analogues of the solutions obtained by the embedding of $SO(3)$ into $SU(3)$. Since in the case of the vanishing potential (considered here), the boundary conditions of the Higgs profile functions at infinity are not fixed, they can be chosen at will. In fact they have been chosen so that the gravitating analogues of the solutions discussed in [9] can be constructed. These solutions correspond to monopole-antimonopole configurations and not to single monopoles as constructed in [12].

In all three cases studied here, it has been found that the solutions bifurcate producing a branch of extremal Reissner-Nordström ones with charge $Q$, which is fixed by the asymptotic values of the gauge fields. Interestingly, our numerical results indicate that a second branch of solutions, which extends backwards from the maximal possible value of the gravitational coupling, exists only in the case of non-maximal symmetry breaking. In the case of maximal symmetry breaking (even in the limit of vanishing Higgs coupling) no second branch was found. This is in contrast with the case of single $SU(2)$ [11] and $SU(3)$ [12] monopoles where the second branch of solutions exists for vanishing or small Higgs coupling.

In this paper we have not constructed solutions for non-vanishing potential or non-abelian black hole solutions. A systematic study of these solutions is left to a future work [18]. In particular, we would like to study the corresponding configurations for intermediate Higgs coupling constants. It is known that for the single $SU(2)$ and $SU(3)$ monopoles the so-called “Lue-Weinberg” [19] phenomenon was observed; ie for intermediate values of the Higgs coupling the solutions develop a second “inner” horizon and in the limit of critical gravitational coupling describe “hairy” black holes. It will be interesting to see whether this phenomenon persists for monopole-antimonopole configurations.

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FIG. 1: The profiles of the metric \( A(r) \) and \( B(r) \) as well as of the gauge fields \( G(r) = c_0(r) - c_1(r) + 2 \) and \( H(r) = c_0(r) + c_1(r) \) are presented for the non-Bogomolnyi maximal SB solutions in the flat limit \( \alpha = 0.0 \) (dashed) and close to the critical limit \( \alpha = \alpha_{\text{cr}} \) (solid). Note that for \( \alpha = 0.0 \), \( A(r) \) and \( B(r) \) are constant since \( A(r) = B(r) = 1 \).
FIG. 2: Same as Fig. 1 for the profile functions of the Higgs field $F_1(r) = \frac{1}{2}(b_0(r) + b_1(r))$ and $F_2(r) = \frac{1}{2}(b_0(r) - b_1(r))$. For comparison, we show again the metric functions $A(r)$ and $B(r)$. 
FIG. 3: The quantity $\alpha M = \frac{m_{\infty}}{\alpha}$ and the minimal value of the metric function $B(r)$, $B_{\min}$, in terms of $\alpha$ is plotted for a) the Bogomolnyi non-maximal SB case (gravitating Bais solution), b) the non-Bogomolnyi non-maximal SB case (gravitating Burzlaff solution) and c) the non-Bogomolnyi maximal SB case.