Hedging LIBOR Derivatives in a Field Theory Model of Interest Rates

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Abstract

We investigate LIBOR-based derivatives using a parsimonious field theory interest rate model capable of instilling imperfect correlation between different maturities. Delta and Gamma hedge parameters are derived for LIBOR Caps and Floors against fluctuations in underlying forward rates. An empirical illustration of our methodology is also conducted to demonstrate the influence of correlation on the hedging of interest rate risk.

1 Introduction

LIBOR-based derivatives such as Caps and Floors are important financial contracts involving a sequence of quarterly payments ranging from one to ten years. Consequently, pricing and hedging such derivatives requires the modeling of multiple LIBOR rates.

In an economy where LIBOR rates are perfectly correlated across different maturities, a single volatility function is sufficient. However, non-parallel movements in the LIBOR term structure introduce an important complication. To reduce the number of necessary inputs, volatility parameters
within certain time intervals are often assumed to be identical. However, this assumption represents a serious compromise, and longer maturity options still require a large number of volatility parameters even after such aggregation.

In light of this issue, we utilize field theory models introduced by Baaquie [2] to instill imperfect correlation between LIBOR maturities as a parsimonious alternative to the existing theory. We derive the corresponding hedge parameters for LIBOR Caplets for applications to risk management. We then demonstrate the ease at which our formulation is implemented and the implications of correlation on the hedge parameters.

Hedge parameters that minimize the risk associated with a finite number of random fluctuations in forward rates is provided in Baaquie, Srikant, and Warachka [3]. Previously, field theory research has focused on applications involving traditional Heath, Jarrow, and Morton [10] forward rates, and on the pricing of LIBOR-based derivatives [5]. This paper extends the concept of stochastic delta hedging developed in [2] to the hedging of LIBOR derivatives.

The remainder of this paper starts with the review of the field theory model for pricing LIBOR derivatives. Section 3 then investigates their corresponding hedge parameters, while Section 4 details their empirical implementation. The conclusion follows in Section 5.

2 Field Theory Model

The introduction of imperfect correlation between all underlying LIBOR rates is accomplished by the specification of a propagator for interest rate dynamics. In terms of notation, \( L(t, T) \) denotes the LIBOR rate at the current time \( t \) between time \( T \) and \( T + \ell \) in the future where \( \ell = 1/4 \) year denotes the standard 3-month time interval between payoffs.

Since forward rates are the basis for LIBOR rates, we first detail the Lagrangian underlying the evolution of forward rates. Let \( A(t, x) \) be a two dimensional field driving the evolution of forward rates \( f(t, x) \) through time

\[
\frac{\partial f(t, x)}{\partial t} = \alpha(t, x) + \sigma(t, x)A(t, x)
\]  

(1)

where \( \sigma(t, x) \) and \( \alpha(t, x) \) denote their volatility and drift velocity respectively.

Following Baaquie and Bouchaud [4], the Lagrangian of the field is defined by three parameters.
Definition 2.1 The Lagrangian which describes the evolution of instantaneous forward rates equals
\[ L[A] = -\frac{1}{2} \left \{ A^2(t, z) + \frac{1}{\mu^2} \left ( \frac{\partial A(t, z)}{\partial z} \right )^2 + \frac{1}{\lambda^4} \left ( \frac{\partial^2 A(t, z)}{\partial^2 z} \right )^2 \right \}, \]  
(2)
where psychological future time is defined by \( z = (x - t)^\eta \).

The Lagrangian in Definition 2.1 contains a squared Laplacian term that describes the stiffness of the forward rate curve. Baaquie and Bouchaud [4] demonstrate that this formulation is empirically able to account for the phenomenology of interest rate dynamics. Ultimately, our pricing formulae for Caps and Floors stems from a volatility function and correlation parameters \( \mu, \lambda \) and \( \eta \) contained in the propagator, as well as the initial term structure.

The associated Action \( S[A] \) of the Lagrangian is defined as \( \int_t^\infty dt \int_\infty^\infty dA \mathcal{L}[A] \). In addition, a normalizing constant equal to the path integral \( Z = \int DA e^{S[A]} \) is employed in our subsequent analysis.

These forward rate dynamics are ultimately invoked for the pricing of Caps and Floors after expressing derivatives on interest rates in terms of their counterparts on bonds.

2.1 LIBOR Dynamics

The following relationship between the forward interest rates and the LIBOR term structure
\[ L(t, T) = e^{\int_t^{T+\ell} dx f(t, x)} - 1. \]  
(3)

In the original Heath, Jarrow, and Morton model [10], the martingale measure is defined by discounting Treasury Bonds denoted \( B(t, T) \) by the money market account \( R(t, t^*) \), defined as
\[ R(t, t^*) = e^{\int_{t^*}^t r(t) dt}, \]  
(4)
for the spot rate of interest denoted \( r(t) \). In this paper, all computations are carried out using the LIBOR measure for which LIBOR rates evolve as martingales. In other words, for \( t^* > t \)
\[ L(t, T_n) = E_L [L(t^*, T_n)]. \]  
(5)

Following the material in Baaquie [5], the drift \( \alpha_L(t, x) \) that corresponds to the LIBOR martingale condition is given by
\[ \alpha_L(t, x) = -\sigma(t, x) \int_{T_n}^x dx^{' \cdot} D(x, x^{' \cdot}; t)\sigma(t, x^{' \cdot}); \quad T_n \leq x < T_{n+\ell}. \]  
(6)
As proved in Baaquie [5], a money market numeraire entails more complex calculations but
arrives at identical prices if one instead uses the LIBOR measure. For the remainder of this paper,
the subscript of $L$ is suppressed with all expectations performed under the LIBOR measure.

2.2 Pricing an Individual Caplet

The existing literature justifies the Black model for pricing Caps and Floors by modifying risk
neutral Heath, Jarrow, and Morton [10] forward rates to yield LIBOR dynamics under the forward
measure. Brace, Gatarek, and Musiela [6] is the seminal paper in this area, with additional details
found in Musiela and Rutkowski [12].

We review the field theory pricing formula for a Caplet for both a general volatility function
$\sigma(t, T)$ and propagator $D(x, x'; t)$ underlying risk neutral forward rates [2]. Denote the principal
amount of the Cap as $V$. If the Caplet is exercised at time $T$, the payment is made in arrears at
time $T + \ell$. Hence the payoff function at time $T + \ell$ is given by

$$g(T + \ell) = \ell V (L(T, T) - K)_+$$

where $K$ denotes the strike rate of the Caplet. Note that before discounting the payoff at time $T$,
we first discount from $T + \ell$ back to time $T$. The entire expression for the Caplet price is given by

$$\text{Caplet}(t, T) = B(t, T) E_{[t,T]} [B(T, T + \ell)g(T + \ell)]$$

$$= \left[ \frac{V}{X} \right] B(t, T) E_{[t,T]} \left[ (X - e^{-\int_{T+\ell}^{T} df(t,x)})_+ \right]$$

according to equation (8) and for $X \equiv \frac{1}{1+\ell K}$. Observe that invoking the forward measure involves
multiplying by the bond $B(t, T)$ with only the random forward rate term structure from $T$ to $T + \ell$.
Then,

$$\text{Caplet}(t, T) = \int_{-\infty}^{+\infty} dG \Psi(G, T, T + \ell)(X - e^{-G})_+$$

where, as the derivation in Baaquie [2], $\Psi(G, T, T + \ell)$ equals

$$\left[ \frac{V}{X} \right] B(t, T) \sqrt{\frac{1}{2\pi q^2(T-t)}} \exp \left\{ -\frac{1}{2q^2(T-t)} \left( G - \int_{T}^{T+\ell} df(t,x) - \frac{q^2(T-t)}{2} \right)^2 \right\}.$$  

(11)

The above result leads to the next proposition for Caplet pricing.
Proposition 2.1 The price of a Caplet with strike $K$ which matures at time $T$ equals

$$Caplet(t, T, T + \ell) = \left[ \frac{V}{X} \right] B(t, T) [X N(d_+) - FN(d_-)]$$

for $X = \frac{1}{1 + tK}$, $B(t, T) = \frac{1}{1 + tL(t, T)}$, and the following definitions

$$F = \frac{1}{1 + tL(t, T)}$$

$$d_\pm = \frac{1}{q\sqrt{T - t}} \left[ \ln \left( \frac{F}{X} \right) \pm \frac{q^2 (T - t)}{2} \right]$$

$$q^2 = \frac{1}{T - t} \int_t^T dt \int_T^{T + \ell} dx d' \sigma(t, x) D(x, x'; t) \sigma(t, x').$$

Observe that the propagator for forward rates are elements of the Caplet price. The price of an at-the-money Caplet is then defined for $X = F$, which yields $d_\pm = \pm \frac{\sqrt{T - t}}{2}$, implying an associated price of

$$Caplet(t, T, T + \ell) = VB(t, T) [N(-d_+) - N(-d_-)]$$

$$= VB(t, T) \left[ N \left( \frac{\sqrt{T - t}}{2} \right) - N \left( -\frac{\sqrt{T - t}}{2} \right) \right].$$

3 Stochastic Delta Hedging

Stochastic hedging of interest rate derivatives has been introduced by Baaquie [2], where the specific case of hedging Treasury Bonds is considered in detail. We focus on applying this technique to the hedging of LIBOR Caplets. Consider the hedging of a Cap against fluctuations in the forward rate $f(t, x)$. A portfolio $\Pi(t)$ composed of a $Cap(t_0, t_*, T_n)$ and a LIBOR futures contract chosen to ensure fluctuations in the value of the portfolio are minimized is studied.

We begin by forming the portfolio

$$\Pi(t) = Cap(t, t_*, T_n) + n_1(t) F(t, T_{n1}),$$

where $n_1(t)$ represents the hedge parameter for the futures contract. The LIBOR futures and Cap prices are denoted by

$$F(t, T_{n1}) = V [1 - \ell L(t, T_{n1})]$$

$$Cap(t, t_*, T_n) = \tilde{V} B(t, T_n) \int_{-\infty}^{+\infty} \frac{dG}{\sqrt{2\pi q^2}} e^{-\frac{1}{2q^2} (G - \int_{T_n}^{T_n+\ell} dx f(t, x) - \frac{q^2}{2})^2} (X - e^{-G})_+$$

This is a more general expression for a Cap referred to as the midcurve Cap.
From equation 16, we have
\[ \Pi(t) = \text{Cap}(t, t_*, T_n) + Vn_1(t)(1 - \ell L(t, T_{n1})). \]

For the sake of brevity, we suppress \( Vn_1 \) which is irrelevant for hedging from above equation, and change the negative sign before the LIBOR futures to positive,
\[ \Pi(t) = \text{Cap}(t, t_*, T_n) + Vn_1(t)\ell L(t, T_{n1}) \]
\[ = \text{Cap}(t, t_*, T_n) + Vn_1(t)\left(e^{\int_{T_{n1}}^{T_{n1}} f(t, x)\, dt} - 1\right). \]

(19)

The portfolio is required to be independent of small changes in the forward rate. Thus, Delta hedging this portfolio requires
\[ \frac{\partial}{\partial f(t)} \Pi(t) = 0. \]

(20)

In field theory, for each time \( t, \) there are infinitely many random variables driving forward rates, and one can never Delta hedge by satisfying equation 20. The best alternative is to Delta hedge on average, and this scheme is referred to as **stochastic Delta** hedging, as detailed in [2]. To implement stochastic Delta hedging, one considers the conditional expectation value of the portfolio \( \Pi(t), \) conditioned on the occurrence of some specific value of the forward rate \( f(t, x_h), \) namely \( E[\Pi(t)|f(t, x_h)]. \) Finite time Delta hedging can be defined by hedging against the fluctuations of the forward rate \( f(t_h, x_h) \equiv f_h \) in the future \( t_h > t. \)

Define the conditional probability of a Cap and a LIBOR futures by
\[ \tilde{\text{Cap}}(t_h, t_*, T_n; f_h) = E[\text{Cap}(t_h, t_*, T_n)|f_h] \]
\[ \tilde{L}(t_h, T_{n1}; f_h) = E[L(t_h, T_{n1})|f_h]. \]

(21)

Stochastic Delta hedging is defined by approximating equation 20 as
\[ \frac{\partial}{\partial f_h} E[\Pi(t_h)|f_h] = 0. \]

(22)

Hence, from equation 22, stochastic Delta hedging yields
\[ n_1 = -\frac{\partial \tilde{\text{Cap}}(t_h, t_*, T_n; f_h)}{\partial f_h} \frac{\partial \tilde{L}(t_h, T_{n1}; f_h)}{\partial f_h}. \]

(23)

\(^2\)The maturity \( x_h \) can be any future time provided \( t_h < t_* \) since the Cap expires at \( t_* \).
As can be seen from above, changes in the hedged portfolio $\Pi(t_h)$, for Delta hedging in field theory, are only on the average sensitive to the fluctuation in the forward rate $f(t_h, x_h)$.

The hedging weight $n_1$ is evaluated explicitly for the field theory forward rates in the Appendix which contains the relevant notation. The final result, from equation (34) is given by

$$n_1 = C \cdot \tilde{C}ap(t, t_*, T_n; f_h) - B \cdot \chi \cdot \tilde{V} \cdot \left[ XN'(d_+)/Q + e^{-G_0+\frac{Q^2}{2}}N(d_+) - e^{-G_0+\frac{Q^2}{2}}N'(d_-)/Q \right] e^{G_1+\frac{Q^2}{2}} \cdot B_1 \tag{24}$$

The HJM limit of the hedging functions is also analyzed in the Appendix.

To hedge against the $\Gamma = \partial^2 \Pi(t)/\partial f^2$ fluctuations, one needs to form a portfolio with two LIBOR futures contracts that minimizes the change in the value of $E[\Pi(t_h)|f_h]$ through Delta and Gamma hedging. These parameters are solved analytically, with empirical results presented in Section 4.

Suppose a Cap needs to be hedged against the fluctuations of $N$ forward rates, namely $f(t_h, x_i)$ for $i = 1, 2, \ldots, N$. The conditional probabilities for the Cap and LIBOR futures, with $N$ forward rates fixed at $f(t_h, x_i) = f_i$

$$\tilde{C}ap(t_h, t_*, T_n; f_1, f_2, \ldots, f_N) = E[\tilde{C}ap(t_h, t_*, T_n)|f_1, f_2, \ldots, f_N]$$

$$\tilde{L}(t_h, T_{n1}; f_1, f_2, \ldots, f_N) = E[\tilde{L}(t_h, T_{n1})|f_1, f_2, \ldots, f_N].$$

A portfolio of LIBOR futures contracts with varying maturities $T_{ni} \neq T$ is defined as

$$\Pi(t) = Cap(t, t_*, T_n) + \sum_{i=1}^{N} n_i(t)L(t, T_{ni}), \tag{25}$$

and the stochastic Delta hedging conditions are given by

$$\frac{\partial}{\partial f_j} E[\Pi(t_h)|f_1, f_2, \ldots, f_N] = 0 \text{ for } j = 1, 2, \ldots, N.$$}

One can solve the above system of $N$ simultaneous equations to determine the $N$ hedge parameters denoted $n_i$. The volatility of the hedged portfolio is reduced by increasing $N$.

To illustrate Delta hedging against more than 1 forward rate, we construct a portfolio with 3 LIBOR futures maturities. Thus, we have equation (25) with $N = 3$.

Clearly, the three hedging parameters are fixed by Delta hedging twice

$$\frac{\partial}{\partial f_j} E[\Pi(t_h)|f_1, f_2] = 0 \text{ for } j = 1, 2$$
and an additional cross Gamma term
\[
\frac{\partial^2}{\partial f_1 \partial f_2} E[\Pi(t_h)|f_1, f_2] = 0.
\]
These hedge parameters are evaluated explicitly in the Appendix. Intuitively, we expect the portfolio to be hedged more effectively with the inclusion of the cross Gamma parameter.

Analytically, Delta hedge parameters for two different forward rates differs only by a prefactor. Thus, all three parameters cannot be uniquely solved. Therefore, we construct a portfolio with two LIBOR futures maturities, then fix the parameters by Delta hedging and cross Gamma hedging once. This environment is studied numerically in the next section.

Until now, we get the parameter for each choice of the LIBOR futures and forward rates being hedged. Furthermore, we can minimize the following

\[
\sum_{i=1}^{N} |n_i|
\]

(26)
to find the minimum portfolio. This additional constraint finds the most effective futures contracts, where effectiveness is measured according to the least amount of required buying or selling.

In general, stochastic Delta hedging against \(N\) forward rates for large \(N\) is complicated, and closed-form solutions are difficult to obtain.

## 4 Empirical Implementation

This section illustrates the implementation of our field theory model and provides preliminary results for the impact of correlation on the hedge parameters. The correlation parameter for the propagator of LIBOR rates are estimated from historical data on LIBOR futures and at-the-money options. We calibrate the term structure of the volatility, \(\sigma(\theta)\), (see [7], [8]) and the propagator with the parameters \(\lambda\) and \(\mu\) as in Baaquie and Bouchaud [4].

Stochastic hedging only mitigates the risk of fluctuations in specified forward rates. The focus of this section is on the stochastic hedge parameters, with the best strategy chosen to ensure the LIBOR futures portfolio involves the smallest possible long and short positions. As an illustration, fig 1 plot the hedge parameters against the LIBOR futures maturity, and the forward rate being hedged.
We first study a portfolio with one LIBOR futures and one Cap to hedge against a single term structure movement. Hedge parameters for different LIBOR futures contract maturities, and the maturity of the forward rate, are shown in fig 1. This figure describes the selection of the LIBOR futures in the minimum portfolio that requires the fewest number of long and short positions.

Fig 2 shows how the hedge parameters depend on $x_h$ for a fixed $T_{n1}$. Two limits $T_{n1} = \delta = \frac{1}{4}$ (3 months) and $T_{n1} = 16\delta$ are chosen. We also find that $x_h = \delta$ is always the most important forward rate to hedge against. Another graph describing the parameter dependence on $T_{n1}$ is given in fig 3 with $x_h = \delta$. For greater generality, we also hedge $Cap(t, t_*, T_n)$ for different $t_*$ and $T_n$ values, and find that although the value of the parameter changes slightly, the shape of the parameter surface is almost identical.

One advantage of the field theory model is that, in principle, a hedge strategy against the movements of infinitely many correlated forward rates is available. To illustrate the contrast between our field theory model and a single-factor HJM model, we plot the identical hedge portfolio as above when $D = 1$, which has been shown to be the HJM limit of field theory models in [2]. From fig 4 the hedge parameters are invariant to maturity, which is expected since all forward rates are perfectly correlated in a single-factor HJM model. Therefore, it makes no difference which of the
In fig 5, we investigate hedging with two LIBOR futures by employing both Delta and cross Gamma hedging. From the previous case, we can hedge against $f(t, δ)$ in order to obtain a minimum portfolio involving the least amount of short and long positions. The diagonal reports that two LIBOR futures with the same maturity reduces to Delta hedging with one LIBOR futures. Selling 38 contracts of $L(t, t + 6δ)$ and buying 71 $L(t, t + δ)$ contracts identifies the minimum portfolio.

In addition, we consider hedging fluctuations in two forward rates. Specifically, we study a portfolio comprised of two LIBOR futures and one Caplet where the parameters are fixed by Delta hedging and cross Gamma hedging. The result is displayed in fig 6 where we hedge against two short maturity forward rates, such as $f(t, δ)$ and $f(t, 2δ)$. Buying 45 contracts of $L(t, t + 15δ)$ and selling 25 $L(t, t + 3δ)$ contracts forms the minimum portfolio. Fig 5 and fig 6 result from the summation of hedge parameters (as in equation 26) which depends on the maturities of the LIBOR futures. The corresponding empirical results are consistent with our earlier discussion.\(^3\)

\(^3\)If we choose the hedged portfolio by minimizing $\sum_{i=1}^{N} n_i$, we find that the minimum portfolio requires 1500 contracts (long the short maturity and short their long maturity counterparts).
Figure 3: Hedge parameter for stochastic hedging of $\text{Cap}(t,1,4)$ against $f(t,t+\delta)$ where $\delta = 3/12$.

5 Conclusion

LIBOR-based Caps and Floors are important financial instruments for managing interest rate risk. However, the multiple payoffs underlying these contracts complicates their pricing as the LIBOR term structure dynamics are not perfectly correlated. A field theory model which allows for imperfect correlation between every LIBOR maturity overcomes this difficulty while maintaining model parsimony.

Furthermore, hedge parameters for the field theory model are provided for risk management applications. Although the field theory model implies an incomplete market since hedging cannot be conducted with an infinite number of interest rate dependent securities in practice, the correlation structure between LIBOR rates is exploited to minimize risk. An empirical illustration demonstrates the implementation of our model.
Figure 4: Hedge parameter for stochastic hedging of $Cap(t, 1, 4)$ when $D = 1$ (forward rates perfectly correlated).

6 Acknowledgment

The data in our empirical tests was generously provided by Jean-Philippe Bouchaud of Science and Finance, and consists of daily closing prices for quarterly Eurodollar futures contracts as described in Bouchaud, Sagna, Cont, El-Karoui and Potters [7] as well as Bouchaud and Matacz [8].

A Conditional Probability of the First Portfolio

Follow Baaquie [2] and equation [11], we have the conditional probability of a Cap given by

$$C\tilde{a}p(t_h, t_x, T_n; f_h) = \tilde{V} \int_{-\infty}^{\infty} dG \{ (x - e^G)_{+} \Psi(G|f_h) \}$$

(27)

$$\Psi(G|f_h) = \frac{\int_{-\infty}^{\infty} df e^{-\frac{f^2}{2}} e^{ipG-\frac{p^2}{2}} \int Df e^{-\int_{t_h}^{t_x} f(t_h, x) e^{ip} f_{t_n}^{T_n+\delta} dx f(t_h, x)} \delta(f(t_h, x_h) - f)e^{S}}{\int Df \delta(f(t_h, x_h) - f)e^{S}},$$
Figure 5: LIBOR futures portfolio when Delta and cross Gamma hedging \( \text{Cap}(t, 1, 4) \).

while the conditional probability of a LIBOR futures is

\[
\tilde{L}(t_h, T_{n1}; f_h) = \int_{-\infty}^\infty dG e^{G} \Phi(G|f; t_h, T_{n1})
\]

(28)

\[
\Phi(G|f; t_h, T_{n1}) = \frac{\int D f \delta(G - \int_{T_{n1}}^{t_h+\ell} f(t_h, x)dx)\delta(f(t_h, x_h) - f)e^S}{\int D f \delta(f(t_h, x_h) - f)e^S}.
\]

(29)

Using the results of the Gaussian models in Baaquie \cite{2}, after a straightforward but tedious calculation, the following results

\[
\Psi(G|f_h) = \frac{\chi}{\sqrt{2\pi Q^2}} \exp \left[ -\frac{1}{2Q^2}(G - G_0)^2 \right]
\]

(30)

\[
\Phi(G|f; t_h, T_{n1}) = \frac{1}{\sqrt{2\pi Q_1^2}} \exp \left[ -\frac{1}{2Q_1^2}(G - G_1)^2 \right].
\]

(31)
Figure 6: LIBOR futures portfolio for stochastic hedging against two forward rates, with both Delta and cross Gamma hedging of $Cap(t, 1, 4)$.

The results are shown as follow

$$X = \frac{1}{1 + \ell k} ; \quad \tilde{V} = (1 + \ell k)V$$

$$\chi = \exp \left\{ - \int_{t_h}^{T_n} dx f(t_0, x) - \int_{M_1} \alpha(t, x) + \frac{1}{2} E + \frac{C}{A} (f(t_0, x_h) + \int_{t_0}^{t_h} dt \alpha(t, x_h) - f - \frac{C}{2}) \right\}$$

$$d_+ = (\ln x + G_0)/Q ; \quad d_- = (\ln x + G_0 - Q^2)/Q$$

$$G_0 = \int_{T_n}^{T_n + \ell} dx f(t_0, x) - F - \frac{B}{A} (f(t_0, x_h) - C - f + \int_{t_0}^{t_h} dt \alpha(t, x_h)) + \frac{q^2}{2}$$

$$Q^2 = q^2 - \frac{B^2}{A}$$

$$G_1 = \int_{T_{n1}}^{T_{n1} + \ell} dx f(t_0, x) + \int_{M_2} \alpha(t, x) - \frac{B_1}{A} (f(t_0, x_h) - \int_{t_0}^{t_h} dt \alpha(t, x_h) - f)$$

$$Q_1^2 = D - \frac{B_1^2}{A}$$

$$A = \int_{t_0}^{t_h} dt \sigma(t, x_h)^2 D(t, x_h, x_h; T_{FR})$$

$$B = \int_{M_2} \sigma(t, x_h) D(t, x_h, x; T_{FR}) \sigma(t, x)$$
\[ B_1 = \int_{M_1} \sigma(t,x_h)D(t,x_h,x;T_{FR})\sigma(t,x) \]
\[ C = \int_{M_2} \sigma(t,x_h)D(t,x_h,x;T_{FR})\sigma(t,x) \]
\[ D = \int_{Q_1} \sigma(t,x)D(t,x,x';T_{FR})\sigma(t,x') \]
\[ q^2 = \int_{Q_2+Q_4} \sigma(t,x)D(t,x,x';T_{FR})\sigma(t,x') \]
\[ E = \int_{Q_4} \sigma(t,x)D(t,x,x';T_{FR})\sigma(t,x') \]
\[ F = \int_{t_0}^{t_h} dt \int_{t_h}^{T_n} dx \int_{T_n}^{T_n+\ell} dx' \sigma(t,x)D(t,x,x';T_{FR})\sigma(t,x'). \]

The domain of integration is given in figs 7 and 8. It can be seen that the unconditional probability distribution for the Cap and LIBOR futures yields volatilities \( q^2 \) and \( D \) respectively. Hence the conditional expectation reduces the volatility of Cap by \( B_2 A \), and by \( B_2 A \) for the LIBOR futures. This result is expected since the constraint imposed by the requirement of conditional probability reduces the allowed fluctuations of the instruments.

It could be the case that there is a special maturity time \( x_h \) that causes the largest reduction of the conditional variance. The answer is found by minimizing the conditional variance

\[ C \cdot \text{Cap}(t_h, t_*, T_n; f_h) = \chi \bar{V}(xN(d_+) - e^{-G_0 + \frac{Q^2}{2}}N(d_-)) \]  \hspace{1cm} (32)
\[ \bar{L}(t_h, T_{n1}; f_h) = e^{G_1 + \frac{Q^2}{2}}. \]  \hspace{1cm} (33)

Recall the hedging parameter is given by equation 23. Using equation 33 and setting \( t_0 = t \), \( t_h = t + \epsilon \), we get an (instantaneous) stochastic Delta hedge parameter \( \eta_1(t) \) equal to

\[ \frac{C \cdot \text{Cap}(t, t_*, T_n; f_h) - B \cdot \chi \cdot \bar{V} \cdot \left[ xN'(d_+)/Q + e^{-G_0 + \frac{Q^2}{2}}N(d_-) - e^{-G_0 + \frac{Q^2}{2}}N'(d_-)/Q \right]}{e^{G_1 + \frac{Q^2}{2}} \cdot B_1}. \]  \hspace{1cm} (34)

B HJM Limit of Hedging Function

The HJM-limit of the hedging functions is analyzed for the specific exponential function considered by Jarrow and Turnbull [9]

\[ \sigma_{hjm}(t,x) = \sigma_0 e^{\beta(x-t)}, \]  \hspace{1cm} (35)
which sets the propagator $D(t, x, x'; T_{FR})$ equal to one. It can be shown that

$$A = \frac{\sigma^2_0}{2\beta} e^{-2\beta x_h} (e^{2\beta t_h} - e^{2\beta t_0})$$

$$B = \frac{\sigma^2_0}{2\beta^2} e^{-\beta x_h} (e^{-\beta T_n} - e^{-\beta T_n + \ell})(e^{2\beta t_h} - e^{2\beta t_0})$$

$$B_1 = \frac{\sigma^2_0}{2\beta^2} e^{-\beta T_n} (e^{-\beta T_n} - e^{-\beta T_n + \ell})(e^{2\beta t_h} - e^{2\beta t_0})$$

$$C = \frac{\sigma^2_0}{2\beta^2} e^{-\beta x_h} (e^{-\beta T_n + \ell})(e^{2\beta t_h} - e^{2\beta t_0})$$

$$D = \frac{\sigma^2_0}{2\beta^3} (e^{-\beta T_n - \ell} - e^{-\beta T_n})^2 (e^{2\beta t_h} - e^{2\beta t_0})$$

$$E = \frac{\sigma^2_0}{2\beta^3} (e^{-\beta T_n} - e^{-\beta t_h})^2 (e^{2\beta t_h} - e^{2\beta t_0})$$

$$F = \frac{\sigma^2_0}{2\beta^3} (e^{-\beta T_n} - e^{-\beta t_h})(e^{-\beta T_n} - e^{-\beta t_h})(e^{2\beta t_h} - e^{2\beta t_0})$$

The exponential volatility function given in equation 35 has the remarkable property, similar to the case found for the hedging of Treasury Bonds [2], that

$$Q^2_1(h, m) = D_{h, m} - \frac{B_{1h, m}^2}{A_{h, m}} \equiv 0. \quad (36)$$

Hence, the conditional probability for the LIBOR futures is deterministic. Indeed, once the forward rate $f_h$ is fixed, the following identity is valid

$$\tilde{L}_{h, m}(t_h, T_{n1}; f_h) \equiv L(t_h, T_{n1}). \quad (37)$$

In other words, for the volatility function in equation 35, the LIBOR futures for the HJM model is exactly determined by one of the forward rates.

But the conditional probability for the Cap is not deterministic since the volatility from $t_h$ to $t_s$, before the Cap’s expiration, is not compensated for by fixing the forward rate.

C Conditional Probability of the Second Portfolio

As detailed in the Appendix, when hedging against 2 forward interest rates, from equation 27 and 28 we have the conditional probability of a Cap given by

$$\Psi(G | f_1, f_2) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{q^2}{2p^2}} e^{ip(G - \frac{q^2}{2})} \int Df e^{-\int_{t_{h}}^{T_{n1}} f(t_h, x) dx} e^{ip \int_{t_{h}}^{T_{n1}} dx f(t_h, x) \prod_{i=1}^{2} \delta(f(t_h, x_i) - f_i) e^{S}} \int Df \prod_{i=1}^{2} \delta(f(t_h, x_i) - f_i) e^{S}, \quad (38)$$
and the conditional probability of LIBOR being

\[
\Phi(G|f_1, f_2, T_{nj}) = \frac{\int Df \delta(G - \int_{T_{nj}}^{T_{nj}+\ell} f(t_h, x)dx) \prod_{i=1}^{2} \delta(f(t_h, x_i) - f_i)e^{S}}{\int Df \prod_{i=1}^{2} \delta(f(t_h, x_i) - f_i)e^{S}} \quad j = 1, 2
\]  

(39)

yielding the following results

\[
\Psi(G|f_1, f_2) = \frac{\chi}{\sqrt{2\pi Q^2}} \exp \left[ -\frac{1}{2Q^2} (G - G_0)^2 \right] 
\]  

(40)

\[
\Phi(G|f_1, f_2, T_{nj}) = \frac{\chi}{\sqrt{2\pi \tilde{Q}_j^2}} \exp \left[ -\frac{1}{2\tilde{Q}_j^2} (G - \tilde{G}_j)^2 \right] \quad j = 1, 2.
\]  

(41)

The results are shown as follows

\[
X = \frac{1}{1 + \ell k} \quad ; \quad \tilde{V} = (1 + \ell k)V
\]

\[
\chi = \exp \left\{ -\int_{t_h}^{T_n} df(t_0, x) - \int_{M} \alpha(t, x) + \frac{1}{2}E + \frac{C_{12}}{A_{12}}(R_{12} - C_{12}) \right\}
\]

\[
d_+ = (\ln x + G_0)/Q \quad ; \quad d_- = (\ln x + G_0 - Q^2)/Q
\]

\[
G_0 = \int_{T_n}^{T_{n}+\ell} df(t_0, x) - F - \frac{B_{12}}{A_{12}}(R_{12} - C_{12}) + \frac{q^2}{2}
\]

\[
Q^2 = q^2 - \frac{B_{12}^2}{A_{12}}
\]

\[
\tilde{G}_j = \int_{T_{nj}}^{T_{nj}+\ell} df(t_0, x) + \int_{M_j} \alpha(t, x) - \frac{\tilde{B}_{12j}}{A_{12}}R_{12} \quad j = 1, 2
\]

\[
\tilde{Q}_j^2 = D_j - \frac{\tilde{B}_{12j}^2}{A_{12}} \quad j = 1, 2
\]

\[
R_i = f(t_0, x_i) + \int_{t_0}^{t_h} d\alpha(t, x_i) - f_i \quad i = 1, 2
\]

\[
R_{12} = R_1 - \frac{A_{12}}{A_2}R_2
\]

\[
A_i = \int_{t_0}^{t_h} d\sigma(t, x_i)^2 D(t, x_i, x_i; T_{FR}) \quad i = 1, 2
\]

\[
A_{12} = \int_{t_0}^{t_h} d\sigma(t, x_1) D(t, x_1, x_2; T_{FR})\sigma(t, x_2)
\]

\[
\tilde{A}_{12} = A_1 - \frac{A_{12}}{A_2}
\]  

(42)
\[ B_i = \int_{M_2} \sigma(t, x_i) D(t, x_i, x; T_{FR}) \sigma(t, x) \quad i = 1, 2 \]

\[ B_{12} = B_1 - \frac{A_{12}}{A_2} B_2 \]

\[ \tilde{B}_{ij} = \int_{M_j} \sigma(t, x_i) D(t, x_i, x; T_{FR}) \sigma(t, x) \quad i = 1, 2; \quad j = 1, 2 \]

\[ \tilde{B}_{12j} = \tilde{B}_{1j} - \frac{A_{12}}{A_2} \tilde{B}_{2j} \quad j = 1, 2 \ldots, 5 \]

\[ C_i = \int_{M_1} \sigma(t, x_i) D(t, x_i, x; T_{FR}) \sigma(t, x) \quad i = 1, 2 \]

\[ C_{12} = C_1 - \frac{A_{12}}{A_2} C_2 \]

\[ D_j = \int_{\tilde{Q}_j} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x') \quad j = 1, 2 \]

\[ q^2 = \int_{Q_2 + Q_4} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x') \]

\[ E = \int_{Q_1} \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x') \]

\[ F = \int_{t_0}^{t_h} dt \int_{t_h}^{T_n} dx \int_{T_n}^{T_n + \ell} dx' \sigma(t, x) D(t, x, x'; T_{FR}) \sigma(t, x') . \]

The domain of integration is given in figs 7 and 8.

---

Figure 7: Domain of integration \( M_1, M_2 \) and integration cube \( Q_1, Q_2, Q_4 \) where the \( x' \) axis has the same limit as its corresponding \( x \) axis.
Figure 8: Domain of integration $\tilde{M}_j$ and integration cube $\tilde{Q}_j$ where the $x'$ axis has the same limit as its corresponding $x$ axis.

Furthermore, an $N$-fold constraint on the instruments would clearly further reduce the variance of the instruments,

\[ C\tilde{a}(t_h, t_*, T_n; f_1, f_2) = \chi \tilde{V}(xN(d_+) - e^{-G_0 + \frac{Q^2}{2}} N(d_-)) \]  \hspace{1cm} (43)
\[ \tilde{L}(t_h, T_n; f_1, f_2) = e^{G_j + \frac{Q^2}{2}}. \]  \hspace{1cm} (44)

References


