Noncommutative 6D Gauge Higgs Unification Models

J. C. López-Domínguez and O. Obregón
Instituto de Física de la Universidad de Guanajuato,
P.O. Box E-143, 37150 León Gto., México.

C. Ramírez and J. J. Toscano
Facultad de Ciencias Físico Matemáticas,
Universidad Autónoma de Puebla, P.O. Box 1364, 72000 Puebla, México.

(Dated: April 3, 2006)

Abstract

The influence of higher dimensions in noncommutative field theories is considered. For this purpose, we analyze the bosonic sector of a recently proposed 6 dimensional SU(3) orbifold model for the electroweak interactions. The corresponding noncommutative theory is constructed by means of the Seiberg-Witten map in 6D. We find in the reduced bosonic interactions in 4D theory, couplings which are new with respect to other known 4D noncommutative formulations of the Standard Model using the Seiberg-Witten map. Phenomenological implications due to the noncommutativity of extra dimensions are explored. In particular, assuming that the commutative model leads to the standard model values, a bound $-5.63 \times 10^{-8} \text{GeV}^{-2} < \theta^{45} < 1.06 \times 10^{-7} \text{GeV}^{-2}$ on the corresponding noncommutativity scale is derived from current experimental constraints on the $S$ and $T$ oblique parameters. This bound is used to predict a possibly significant impact of noncommutativity effects of extra dimensions on the rare Higgs boson decay $H \rightarrow \gamma\gamma$.

PACS numbers: 11.10.Nx, 12.10.-g, 12.60.-i
1. INTRODUCTION

A renewed interest in theories in 6D has recently emerged [1]. An anomaly free gauged supergravity in $D = 6$, the Salam-Sezgin model [2], has been considered. This model is compactified on a 2-sphere and in four dimensions gives a $SU(2) \times U(1)$ gauge theory [3]. In particular, it has been argued that these theories with 3-Branes could point out towards solving the cosmological constant problem [4]. Also, in [5] it is shown that chaotic inflation consistent with constraints coming from the amplitude of the cosmic microwave anisotropies can be naturally realized.

In the search for a unified theory of elementary particles, the incorporation of the Higgs field in the standard model (SM) of electroweak interactions has motivated various proposals in 6D [6]. These are 6D pure gauge theories, in which after dimensional reduction the Higgs field naturally arises. Recently new proposals have been made, considering orbifold compactifications, in [7], a $U(3) \times U(3)$ model has been considered. In this work the Higgs mass term is generated radiatively, with a finite value at one loop as the quadratic divergences are suppressed by the six dimensional gauge symmetry. Further, a $SU(3)$ model of this type has been developed in [8, 9], with one Higgs doublet and a predicted W-boson mass of half the Higgs mass. In this case the weak angle has a non realistic value, although it can be improved by an extended gauge group as in [10], or by the introduction of an $U(1)$ factor as done in [8].

Noncommutativity in field theories has been the subject of an important number of works in the last few years. In particular, the Seiberg-Witten construction [11] and its generalization for any gauge group [12] have been studied. This construction allows to express the noncommutative gauge fields in terms of the usual ones and their derivatives, maintaining the same degrees of freedom. It has been extended for noncommutative matter fields, which can also be generated in terms of the commutative matter fields and the gauge fields of interest [12]. By this procedure, noncommutative versions of the standard model and consequently the electroweak interaction sector have been given [13] (see also [14]). As a consequence new interactions among the fields of the theory are predicted.

In this work, we will investigate the noncommutative generalization of the bosonic sector of Gauge Higgs unification models in 6D based on the $SU(3)$ gauge group compactified on $T^2/Z_N$ [9]. The noncommutative extension is obtained by means of the Seiberg-Witten map.
We calculate, for the bosonic sector, the resulting first order corrections and compare them with the results obtained in other works. Assuming noncommutativity only between the extra dimensions, the phenomenological consequences are considered in the framework of the effective theories technique. First we compute the new physics effects corrections to the S and T oblique parameters. Further, from the experimental constraints we get a bound for the noncommutativity parameter $\theta^{45}$. This bound allows us to calculate the correction to the decay width for the rare decay of the Higgs boson into two photons.

In section 2 we review the model of reference [9], in section 3 the Seiberg-Witten map and its generalization to nonabelian groups is presented in some detail. In section 4 we present the noncommutative formulation of the model [9] and show that our results differ from those calculated directly in 4D. The phenomenological consequences of the noncommutativity between extra dimensions are discussed in section 5. Section 6 is devoted to conclusions.

2. THE 6-DIMENSIONAL MODEL

2.1. Gauge Fields in a 6-Dimensional Space-Time

Let us consider a Yang-Mills theory in 6-dimensional space-time with a $SU(3)$ gauge group, the Lagrangian of the theory is

$$\mathcal{L} = -\frac{1}{2} \text{Tr} F_{mn} F^{mn},$$

the field strength tensor is defined by

$$F_{mn} = \partial_m A_n - \partial_n A_m - ig_6 [A_m, A_n],$$

and $g_6$ is the coupling constant in 6D. This action is interpreted by a dimensional reduction on an orbifold $T^2/Z_N$ for $N = 3, 4, 6$ [9], by a separation of the connection in its 4-dimensional space-time part $A_\mu$, and the other two components, $A_z$ and $A_{\bar{z}}$ which in 4D will play the role of scalars, with $z = \frac{1}{\sqrt{2}} (x^4 + ix^5)$ and $\bar{z} = \frac{1}{\sqrt{2}} (x^4 - ix^5)$. These fields are the zero modes of Kaluza-Klein and depend only on the four space-time coordinates $x^\mu$. The result of this reduction is given by

$$\mathcal{L} = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + 2 \text{Tr} D_\mu A_{\bar{z}} D^\mu A_z - g^2 \text{Tr} [A_z, A_{\bar{z}}]^2,$$  \hspace{1cm} (1)
where \( g = g_6 \sqrt{V} \) is the gauge coupling of the 4-dimensional effective theory, \( V \) is the volume of the two extra dimensions and

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu],
\]

\[
D_\mu A_z = \partial_\mu A_z - ig [A_\mu, A_z] = F_{\mu z},
\]

\[
D_\mu A_\bar{z} = \partial_\mu A_\bar{z} - ig [A_\mu, A_\bar{z}] = F_{\mu \bar{z}}. \tag{2}
\]

The orbifold reduction \cite{9} for the gauge fields \( A_m \) leads to: the 4-dimensional \( A_\mu \), that contains four electroweak bosons, \( W_\mu \in SU(2) \) and \( B_\mu \in U(1) \),

\[
A_\mu = \begin{pmatrix} W_\mu & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2 \sqrt{3}} \begin{pmatrix} B_\mu I & 0 \\ 0 & -2B_\mu \end{pmatrix},
\]

and the two complex components of the scalar boson doublet (Higgs), which are contained in the \( A_z \) and \( A_\bar{z} \) gauge fields,

\[
A_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix}, \quad A_\bar{z} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ \phi^\dagger & 0 \end{pmatrix}.
\]

Substituting these expressions in the Lagrangian \cite{11} we find

\[
\mathcal{L} = -\frac{1}{2} \text{Tr} F_{\mu\nu}(W) F^{\mu\nu}(W) - \frac{1}{4} F_{\mu\nu}(B) F^{\mu\nu}(B) + (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi), \tag{3}
\]

where \( D_\mu \phi = (\partial_\mu - \frac{1}{2} i g W_\mu^a \tau_a - \frac{1}{2} i g \tan \theta_W B_\mu) \phi \), \( \tan \theta_W = \sqrt{3} \) and \( V(\phi) = g^2 |\phi|^4 \).

Thus, this Lagrangian has a \( SU(2) \times U(1) \) invariance, with a scalar massless doublet with a quartic potential. However, as shown in \cite{10}, quantum fluctuations induce corrections to the potential \( V(\phi) \) which can trigger radiative symmetry breaking. The leading terms in the one-loop effective potential for the Higgs are,

\[
V_{\text{eff}}(\phi) = -\mu^2 |\phi|^2 + \lambda |\phi|^4.
\]

Assuming \( \mu^2 > 0 \), so that electroweak symmetry breaking can occur, we have that \( \langle |\phi| \rangle = \nu / \sqrt{2} \) with \( \nu = \mu / \sqrt{3} \). Note that the value of the electroweak angle in \cite{3} is too large. However, as mentioned in the introduction, the model can be extended in such a way that it correctly reproduces the SM value.
3. NONCOMMUTATIVE GAUGE THEORIES

3.1. Noncommutative Space-Time

Noncommutative space-time incorporates coordinates \( \hat{x}^\mu \), given by operators that satisfy the following relations,

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu},
\]

where \( \theta^{\mu\nu} = -\theta^{\nu\mu} \) are real numbers. To construct a field theory in this space, it is more convenient to consider usual fields, which are functions. This is allowed by the Weyl-Wigner-Moyal correspondence, which establishes an equivalence between the Heisenberg algebra of the operators \( \hat{x}^\mu \) and the function algebra in \( \mathbb{R}^m \). It has an associative and noncommutative star product, the Moyal \( \star \)-product, given by,

\[
f(x) \star g(x) \equiv \left[ \exp\left( \frac{i}{2} \frac{\partial}{\partial \varepsilon^\alpha} \frac{\partial}{\partial \eta^\beta} \right) f(x + \varepsilon) g(y + \eta) \right]_{\varepsilon=\eta=0}
\]

\[
= fg + \frac{i}{2} \theta^{\alpha\beta} \partial_\alpha f \partial_\beta g + O(\theta^2).
\]

Under complex conjugation it satisfies \( (f \star g) = g \star f \). Since we will work with a nonabelian gauge group, our functions are matrix valued, and the corresponding matrix Moyal product is denoted by an \( \star \). In this case the hermitian conjugation is given by \( (f \star g)^\dagger = g^\dagger \star f^\dagger \). Under the integral, for closed manifolds, this product has the cyclic property \( \text{Tr} \int f_1 \star f_2 \star \cdots \star f_n = \text{Tr} \int f_n \star f_1 \star f_2 \star \cdots \star f_{n-1} \). In particular \( \text{Tr} \int f_1 \star f_2 = \text{Tr} \int f_1 f_2 \). Therefore, a theory on the noncommutative space of the \( \hat{x} \), is equivalent to a theory of usual fields, where the function product is substituted by the Moyal \( \star \)-product.

This suggests that any theory can be converted into a noncommutative one by replacing the ordinary function product with the \( \star \)-product.

3.2. The Seiberg-Witten Map

In order to build noncommutative Yang-Mills theories it is necessary, first of all, that a commutative limit exists and that a perturbative study of the noncommutative theory is possible. In this case the solutions of such a theory must depend on the noncommutativity parameter \( \theta \) in the form of a power series expansion.
For an ordinary Yang-Mills theory, the gauge field and the strength field tensor transformations can be written as:

$$\begin{align*}
\delta A_\mu &= \partial_\mu \lambda + i \lambda A_\mu - i A_\mu \lambda, \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i A_\mu A_\nu + i A_\nu A_\mu, \\
\delta F_{\mu\nu} &= i \lambda F_{\mu\nu} - i F_{\mu\nu} \lambda.
\end{align*}$$

(6)

For the noncommutative gauge theory, we use the same equations in the gauge field and strength field tensor transformations, except that the matrix multiplications are replaced by the * product. Then the gauge field and the strength field tensor transformations are [11]:

$$\begin{align*}
\hat{\delta} \lambda \hat{A}_\mu &= \partial_\mu \hat{\lambda} + i \hat{\lambda} * \hat{A}_\mu - i \hat{A}_\mu * \hat{\lambda}, \\
\hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i \hat{A}_\mu * \hat{A}_\nu + i \hat{A}_\nu * \hat{A}_\mu, \\
\hat{\delta} \lambda \hat{F}_{\mu\nu} &= i \hat{\lambda} * \hat{F}_{\mu\nu} - i \hat{F}_{\mu\nu} * \hat{\lambda},
\end{align*}$$

(7)

from which the original Yang-Mills theory (6) results in the limit $\theta \to 0$. Notice that equations (7) are valid even for abelian gauge fields. Due to the form of the Moyal product (5), the noncommutative theory has the structure of a nonlocal theory. However, if we consider it as an effective theory, its energy scale gives us a cutoff and nonlocality is not a problem. Further, as shown by Kontsevich [15], at the level of the physical degrees of freedom there is a one to one relation between the commutative and the noncommutative theories. However both theories are quite different, as noncommutativity generates new couplings.

Let us consider the noncommutative gauge transformations of an abelian theory,

$$\begin{align*}
\delta \lambda \hat{A}_\mu &= \partial_\mu \hat{\lambda} + i \hat{\lambda} * \hat{A}_\mu - i \hat{A}_\mu * \hat{\lambda},
\end{align*}$$

(8)

we see that they look like nonabelian ones, although they continue to depend on only one generator. For nonabelian groups, things are more complicated [12],

$$\begin{align*}
\delta \lambda \hat{A}_\mu &= \partial_\mu \hat{\lambda} + i \hat{\lambda} * \hat{A}_\mu - i \hat{A}_\mu * \hat{\lambda} \\
&= \partial_\mu \hat{\lambda}^a \Lambda_a + i \hat{\lambda}^a \Lambda_a * \hat{A}_\mu^b \Lambda_b - i \hat{A}_\mu^b \Lambda_b * \hat{\lambda}^a \Lambda_a \\
&= \partial_\mu \hat{\lambda}^a \Lambda_a + \frac{i}{2} \left\{ \hat{\lambda}^a * \hat{A}_\mu^b \right\} [\Lambda_a, \Lambda_b] + \frac{i}{2} \left[ \hat{\lambda}^a * \hat{A}_\mu^b \right] \{\Lambda_a, \Lambda_b\}.
\end{align*}$$

(9)
Now the transformation algebra is generated by commutators and anticommutators, which amounts to the universal enveloping algebra of the original algebra $U(g, R)$, where $R$ is the corresponding representation. The generators of this algebra satisfy,

$$[\Lambda_A, \Lambda_B] = i f_{ABC} \Lambda_C, \quad \{\Lambda_A, \Lambda_B\} = d_{ABC} \Lambda_C,$$

(10)

where $f_{ABC} = -f_{BAC}$ and $d_{ABC} = d_{BAC}$ are the structure constants.

These transformations are satisfied order by order on $\theta$, and all coefficients of the higher terms can be used to fix the gauge degrees of freedom of $\hat{A}_\mu$. In such a gauge fixing, the only remaining freedom of the transformation parameters $\hat{\lambda}$ are the ones of the commutative theory, so they should depend only on $\lambda^a$ and their derivatives. In this case consistency implies that an infinitesimal commutative gauge transformation $\delta A_\mu = \partial_\mu \lambda + i \lambda A_\mu - i A_\mu \lambda$, will induce the noncommutative one,

$$\hat{A}_\mu(A + \delta\lambda A) = \hat{A}_\mu(A) + \hat{\delta}_\lambda \hat{A}_\mu(A).$$

(11)

This is the so called Seiberg-Witten map.

The solution to (11) can be obtained by setting $\hat{A}_\mu = A_\mu + A'_\mu(A)$ and $\hat{\lambda} = \lambda + \lambda'(\lambda, A)$, where $A'_\mu$ and $\lambda'$ are local functions of $\lambda$ and $A_\mu$ of first order in $\theta$. Then substituting in (11) and expanding to first order,

$$A'_\mu(A + \delta\lambda A) - A'_\mu(A) - \partial_\mu \lambda' - i[\lambda', A_\mu] - i[\lambda, A'_\mu] = -\frac{1}{2} \theta^{\alpha\beta}(\partial_\alpha \lambda \partial_\beta A_\mu + \partial_\beta A_\mu \partial_\alpha \lambda).$$

(12)

One solution of this equation is given by [11],

$$\hat{A}_\mu(A) = A_\mu + A'_\mu(A) = A_\mu - \frac{1}{4} \theta^{\alpha\beta} \{A_\alpha, \partial_\beta A_\mu + F_{\beta\mu}\} + O(\theta^2),$$

(13)

$$\hat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A) = \lambda + \frac{1}{4} \theta^{\alpha\beta} \{\partial_\alpha \lambda, A_\beta\} + O(\theta^2),$$

(14)

from which it turns out that,

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4} \theta^{\alpha\beta} (2 \{F_{\mu\alpha}, F_{\nu\beta}\} - \{A_\alpha, (D_\beta + \partial_\beta) F_{\mu\nu}\}) + O(\theta^2).$$

(15)

These equations (13, 14, 15) are the explicit form of the Seiberg-Witten map, which in this way can be constructed for any Lie algebra of transformations [12].

As shown the noncommutative generators $\hat{\lambda}$ take values in the enveloping algebra. In the case of the fundamental representation of unitary groups $U(N)$, they coincide with their enveloping algebras. For the algebra of $SU(N)$ in the fundamental representation, the enveloping algebra incorporates, through the anticommutators of the generators, the identity matrix $\Lambda_0 = \frac{1}{\sqrt{2N}} I_{N \times N}$, and is then given by $U(N)$.
4. THE NONCOMMUTATIVE MODEL

As previously mentioned, our purpose is the construction of a noncommutative version of the 6-dimensional $SU(3)$ gauge theory presented in Section 2. The fact that we are considering noncommutativity in 6D, means that we only need the Seiberg-Witten map for gauge fields. Thus the effects of noncommutativity on the Higgs field and its interactions will arise after dimensional reduction, in particular from the Seiberg-Witten map of the gauge fields $A_z$ and $A_{\bar{z}}$.

The noncommutative action is given by:

$$\hat{S}_{NC} = -\frac{1}{2} \text{Tr} \int d^6 x \hat{F}_{mn} \hat{F}^{mn}, \quad (16)$$

where

$$\hat{F}_{mn} = F_{mn} + \frac{1}{4} \theta^{kl} (2 \{ F_{mk}, F_{nl} \} - \{ A_k, (D_l + \partial_l) F_{mn} \}) + O(\theta^2). \quad (17)$$

Here the indexes $m, n, k$ and $l$ take the values $0, ..., 3, z$ and $\bar{z}$. Thus the noncommutative parameter $\theta^{kl}$ can be: $\theta^{\mu\nu}$ (noncommutativity among the 4-dimensional space-time coordinates), $\theta^{\mu z}$, $\theta^{\mu \bar{z}}$ (noncommutativity among the 4-dimensional space-time coordinates and the extra dimensions coordinates) and $\theta^{z \bar{z}}$ (noncommutativity between the extra dimensions). Therefore, after inserting the noncommutative field strength (17) into (16), the noncommu-
tative action gets the following first order corrections,

\[
- \frac{\theta^{\alpha \beta}}{4} \text{Tr} \left\{ \begin{array}{l}
2 \{ F_{\mu \alpha}, F_{\nu \beta} \} - \{ A_\alpha, (D_\beta + \partial_\beta) F_{\mu \nu} \} \bigg] F_{\mu \nu} \\
+ 2 \{ F_{\mu \alpha}, F_{z \beta} \} - \{ A_\alpha, (D_\beta + \partial_\beta) F_{\mu z} \} \bigg] F_{\mu z} \\
+ 2 \{ F_{\mu \alpha}, F_{\bar{z} \beta} \} - \{ A_\alpha, (D_\beta + \partial_\beta) F_{\mu \bar{z}} \} \bigg] F_{\mu \bar{z}} \\
+ 2 \{ F_{2 \alpha}, F_{\bar{z} \beta} \} - \{ A_\alpha, (D_\beta + \partial_\beta) F_{2 \bar{z}} \} \bigg] F_{2 \bar{z}} \\
\end{array} \right\}
\]

\[
- \frac{\theta^{\alpha i}}{4} \text{Tr} \left\{ \begin{array}{l}
4 \{ F_{\mu \alpha}, F_{i \mu} \} - \{ A_\alpha, (D_i + \partial_i) F_{\mu \nu} \} + \{ A_i, (D_\alpha + \partial_\alpha) F_{\mu \nu} \} \bigg] F_{\mu \nu} \\
+ 2 \{ F_{\mu \alpha}, F_{ji} \} - \{ A_\alpha, (D_i + \partial_i) F_{\mu j} \} + \{ A_i, (D_\alpha + \partial_\alpha) F_{\mu j} \} \bigg] F_{\mu j} \\
+ 2 \{ F_{i \alpha}, F_{\bar{z} \bar{z}} \} - \{ A_\alpha, (D_\bar{z} + \partial_\bar{z}) F_{i \bar{z}} \} + \{ A_{\bar{z}}, (D_\alpha + \partial_\alpha) F_{i \bar{z}} \} \bigg] F_{i \bar{z}} \\
\end{array} \right\}
\]

\[
- \frac{\theta^{z \bar{z}}}{4} \text{Tr} \left\{ \begin{array}{l}
4 \{ F_{\mu z}, F_{\nu \bar{z}} \} - \{ A_\nu, (D_z + \partial_z) F_{\mu \nu} \} + \{ A_z, (D_\nu + \partial_\nu) F_{\mu \nu} \} \bigg] F_{\mu \nu} \\
+ 2 \{ F_{\mu z}, F_{z \bar{z}} \} - \{ A_\nu, D_z F_{\mu \nu} \} + \{ A_z, D_\nu F_{\mu \nu} \} \bigg] F_{\mu z} \\
+ 2 \{ F_{z \bar{z}}, F_{\mu \bar{z}} \} - \{ A_\nu, D_\bar{z} F_{\mu \nu} \} + \{ A_{\bar{z}}, D_\nu F_{\mu \nu} \} \bigg] F_{\mu \bar{z}} \\
+ 2 \{ F_{z \bar{z}}, F_{z \bar{z}} \} - \{ A_\nu, D_\bar{z} F_{z \bar{z}} \} + \{ A_{\bar{z}}, D_\nu F_{z \bar{z}} \} \bigg] F_{z \bar{z}} \\
\end{array} \right\},
\]

(18)

where \( \mu, \nu, \alpha, \beta = 0, \ldots, 3, i = z, \bar{z} \).

After somewhat cumbersome computations, we obtain the following expression for these corrections in terms of the \( SU(2) \) and \( U(1) \) field strengths \( W^{\mu \nu} \) and \( B^{\mu \nu} \) respectively, the
corresponding gauge fields $W^\mu$ and $B^\mu$ and the Higgs field $\phi$,

$$\hat{\mathcal{L}}_{NC} = -\frac{1}{2} \text{Tr} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - \frac{g^2}{2} |\phi|^4$$

$$- \frac{1}{4} g^{\alpha\beta} \left\{ \frac{1}{2\sqrt{3}} \text{Tr} \left[ 4\{W_{\alpha\beta}, B_{\nu\beta}\} W^{\mu\nu} + 2\{W_{\alpha\beta}, W_{\nu\beta}\} B^{\mu\nu} I - \{W_\alpha, D_\beta W_{\mu\nu}\} B^{\mu\nu} I \right] \right.$$

$$\left. + \{B_\alpha, D_\beta W_{\mu\nu}\} W^{\mu\nu} \right] + \frac{1}{2\sqrt{3}} B_\alpha \partial_\beta B_{\mu\nu} B^{\mu\nu} - \frac{1}{2\sqrt{3}} B_{\mu\alpha} B_{\nu\beta} B^{\mu\nu}$$

$$+ 2(D^\mu \phi)^\dagger \left( W_{\mu\alpha} - \frac{1}{2\sqrt{3}} B_{\mu\alpha} I \right) (D^\alpha \phi) + h.c.$$ 

$$+ (D^\mu \phi)^\dagger \left( W_\alpha - \frac{1}{2\sqrt{3}} B_\alpha I \right) \left( \nabla_\beta + D_\beta \right) (D_\mu \phi)$$

$$+ (D^\mu \phi)^\dagger \left( \nabla_\beta + D_\beta \right) \left( W_\alpha - \frac{1}{2\sqrt{3}} B_\alpha I \right) (D_\mu \phi)$$

$$+ ig \left[ \phi^\dagger (D_\alpha \phi)(D_\beta \phi)^\dagger \phi - (D_\beta \phi)^\dagger (D_\alpha \phi) \phi^\dagger \phi \right] - ig^3 \phi^\dagger \phi \phi^\dagger W_\beta W_\alpha \phi$$

$$- g^2 \left[ \phi^\dagger \left( W_\alpha + \frac{1}{2\sqrt{3}} B_\alpha I \right) \partial_\beta (\phi \phi^\dagger) \phi - \frac{2}{\sqrt{3}} B_\alpha \partial_\beta (\phi \phi^\dagger) \phi^\dagger \phi \right.$$ 

$$+ \phi^\dagger \partial_\beta (\phi \phi^\dagger) \left( W_\alpha + \frac{1}{2\sqrt{3}} B_\alpha I \right) \phi \right] \right\}$$

$$+ i \frac{g}{2} \theta^\alpha_\beta \left\{ - 2i (D_\mu \phi)^\dagger \left( W^{\mu\nu} + \frac{1}{\sqrt{3}} B^{\mu\nu} I \right) (D_\nu \phi) \right.$$ 

$$+ g \left[ \phi^\dagger \phi (D_\mu \phi)^\dagger (D^\mu \phi) - \phi^\dagger \phi (D_\mu \phi)^\dagger (D^\mu \phi) \right]$$

$$- g \phi^\dagger \left( W_{\mu\nu} W^{\mu\nu} + \frac{1}{\sqrt{3}} W_{\mu\nu} B^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \right) \phi \right\}, \quad (19)$$

In this equation there are new interactions with respect to the ones found in the 4D noncommutative formulations of the SM [13, 16]. For instance the interactions between the weak gauge fields and the electromagnetic field which appear in the first terms that multiply the four dimensional noncommutativity parameter $\theta^{\alpha\beta}$. Note that there are not corrections linear in the noncommutativity parameter $\theta^{\alpha\beta}$, they turn out to be identically zero, as a consequence of the orbifold symmetries. Of particular interest are the corrections corresponding to noncommutativity between the extra dimensions, i.e. the terms multiplied by $\theta^{\alpha\beta}$, given by interactions among the Higgs and the gauge bosons, and also higher order
Higgs self-interactions. Considering only these sort of corrections, we have,

\[ \hat{\mathcal{L}}_{NC} = -\frac{1}{2} \text{Tr} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_{\mu}\phi)^\dagger (D^\mu \phi) - \frac{g^2}{2} |\phi|^4 \\
+ \frac{i}{2} \theta^{45} \left\{ -2i (D_{\mu}\phi)^\dagger \left( W^{\mu\nu} + \frac{1}{\sqrt{3}} B^{\mu\nu} I \right) (D_{\nu}\phi) \\
+ \frac{g}{2} \left[ \phi^\dagger \phi (D_{\mu}\phi)^\dagger (D^\mu \phi) - (D_{\mu}\phi)^\dagger \phi \phi^\dagger (D^\mu \phi) \right] \\
- g \phi^\dagger \left( W_{\mu\nu} W^{\mu\nu} + \frac{1}{\sqrt{3}} W_{\mu\nu} B^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \right) \phi \right\} . \] (20)

The noncommutative corrections in this Lagrangian are dimension-six operators, well known from the electroweak effective Lagrangian technique [17], a scheme in which the effects of these terms can be studied in a model-independent manner. It is interesting to consider the extension of the \( SU(3) \) six dimensional gauge group by an \( U(1) \) factor. Its gauge field is invariant under the orbifold symmetries [8] and as it does not mix with the \( SU(3) \) gauge field, the noncommutative corrections (17) and in (20) will not be affected.

5. PHENOMENOLOGICAL IMPLICATIONS

In this section we will analyze the phenomenological implications of noncommutativity between extra dimensions, in the case of Lagrangian (20). Thus, we fix our attention on the terms multiplied by \( \theta^{45} \) (\( \theta^{45} = i \theta^{\pi} \)). These terms contain new interactions relative to the SM and its noncommutative versions [13, 16]. As mentioned, the model we are considering predicts a too large weak angle. However we can expect that the kind of noncommutative terms considered here will also arise from a more realistic theory, that would reproduce the SM in the limit of vanishing \( \theta \) parameters. For instance, as mentioned end of the last section, in the case of an \( U(1) \) extension, the noncommutative corrections which affect the electroweak gauge fields are not modified. Accordingly, in the following we will consider these terms as representing deviations of the genuine standard electroweak Lagrangian. In order to analyze their effects, we observe that these new interactions are given by well-known dimension-six operators, which have been already studied in, by example, the electroweak effective Lagrangian approach [17], a scheme appropriate to investigate in a model-independent manner physics lying beyond the Fermi scale. For this purpose, it is
convenient to divide the above operators into three sets, as follows

\[ O_{\phi W} = \frac{g}{2} \theta^{45} (\phi^\dagger W_{\mu\nu} W^{\mu\nu} \phi), \]  
(21)

\[ O_{\phi B} = -\frac{g}{8} \theta^{45} (\phi^\dagger B_{\mu\nu} B^{\mu\nu} \phi), \]  
(22)

\[ O_{WB} = \frac{g}{2} \sqrt{3} \theta^{45} (\phi^\dagger W_{\mu\nu} B^{\mu\nu} \phi), \]  
(23)

\[ O_{(1) \phi} = -g \theta^{45} (\phi^\dagger \phi) (D_\mu \phi)^\dagger (D^\mu \phi), \]  
(24)

\[ O_{(3) \phi} = g \theta^{45} [(D_\mu \phi)^\dagger \phi] [\phi^\dagger (D^\mu \phi)], \]  
(25)

\[ O_{DW} = i \theta^{45} (D_\mu \phi)^\dagger W^{\mu\nu} (D_\nu \phi), \]  
(26)

\[ O_{DB} = \frac{i}{\sqrt{3}} \theta^{45} (D_\mu \phi)^\dagger B^{\mu\nu} (D_\nu \phi). \]  
(27)

First we observe that there are potential modifications induced by these operators on the quadratic SM Lagrangian, as they can alter some tree level relations which are experimentally constrained, such as the kinetic energy part of the $W$ and $Z$ bosons. In particular, they can give tree–level contributions to the $S$ and $T$ oblique parameters\cite{18}. Let us focus our attention on those interactions which affect the quadratic part of the SM gauge sector.

After spontaneous symmetry breaking, all the above operators induce new nonrenormalizable interactions, as well as renormalizable ones, which modify those predicted by the dimension–four theory. In particular, the first two sets of operators induce bilinear terms that can eventually modify the SM parameters \cite{17,19}. On the other hand, although the last set of operators are potentially interesting from the phenomenological point of view \cite{20}, they are not important for our purposes, as they do not introduce modifications in the quadratic Lagrangian. Concerning the first set, it is easy to see that $O_{\phi W}$ and $O_{\phi B}$ modify the canonical form of the kinetic terms $W_{\mu\nu} W^{\mu\nu}$ and $B_{\mu\nu} B^{\mu\nu}$, respectively. However, these effects are unobservable indeed, since they can be absorbed in a finite renormalization of the gauge fields and the coupling constant $g$. As to the $O_{WB}$, $O_{(1) \phi}$, and $O_{(3) \phi}$ operators, they introduce nontrivial modifications in the quadratic Lagrangian. In particular, as we will see below, the first and the last of these operators are sensitive to the low–energy data, as they contribute to the $S$ and $T$ parameters at the tree level. Up to some surface terms, the quadratic part of the effective Lagrangian, i.e., the SM and new contributions, can
conveniently be written as

\[ L_{\text{Kinetic}} = \frac{1}{2} W^{\mu\nu} \left\{ \left[ \square + \frac{g^2 v^2}{4} \left( 1 - \frac{\alpha^{(1)}_\phi}{2} \right) \right] g_{\mu\nu} - \partial_\mu \partial_\nu \right\} W_{\alpha\beta} + \frac{1}{2} B^{\mu\nu} \left\{ \left[ \square + \frac{g^2 v^2}{4} \left( 1 - \frac{\alpha^{(1)}_\phi}{2} \right) \right] g_{\mu\nu} - \partial_\mu \partial_\nu \right\} B^{\mu\nu} \\
+ W_{3\mu} \left( \frac{g^2 v^2}{16} \alpha^{(3)}_\phi \right) g_{\mu\nu} W_{3\nu} + W_{3\mu} \left[ \alpha_{\text{WB}} \left( \square g_{\mu\nu} - \partial_\mu \partial_\nu \right) \right] B^{\nu}, \tag{28} \]

where the unobservable effects arising from the \( \mathcal{O}_{\phi W} \) and \( \mathcal{O}_{\phi B} \) operators were ignored. In addition, in order to identify the origin of each contribution, we have introduced the definitions: \( \alpha_{\text{WB}} = g v^2 \theta^{45} / 2\sqrt{3} \) and \( \alpha^{(1)}_\phi = \alpha^{(3)}_\phi = g v^2 \theta^{45} \), with \( v \) the Fermi scale. The new ingredients in this expression with respect to the standard result, is the mixing between the field strengths \( W^{3\mu\nu} \) and \( B_{\mu\nu} \) induced by the \( \mathcal{O}_{\text{WB}} \) operator, as well as the presence of a quadratic term in \( W^{3\mu} \) generated by the \( \mathcal{O}^{(3)}_\phi \) operator. As we will see below, the \( W^{3\mu\nu}_\mu B^{\mu\nu} \) mixing given by the \( \mathcal{O}_{\text{WB}} \) operator contribute to the \( S \) parameter at the tree level, as it involves derivatives. Also, it is important to notice that while \( \mathcal{O}^{(1)}_\phi \) affects with the same intensity both the \( W \) and \( Z \) masses (see first and second terms in (28)), the \( \mathcal{O}^{(3)}_\phi \) operator modifies only the \( Z \)–mass, as it is evident from the term proportional to \( W^{3\mu}_\mu W^{3\nu}_\nu \). This asymmetric contribution to the gauge field masses is the responsible for deviations from the SM value of the \( \rho \)–parameter \( \rho = \alpha T \left[ 17, 21 \right] \). This contributions is associated with a violation of the custodial \( SU(2) \) symmetry \left[ 22 \right], which as it is well known, guarantees the tree level value \( \rho = 1 \) in the SM. The diagonalization of the resultant kinetic energy sector and its impact on the SM parameters have been studied in the literature in the more general context of electroweak effective Lagrangians \left[ 17, 19 \right]. The tree level contribution of the \( \mathcal{O}_{\text{WB}} \) and \( \mathcal{O}^{(3)}_\phi \) operators to the \( S \) and \( T \) parameters has already been studied in this more general context by Hagiwara et al. \left[ 23 \right]. For our purposes, it is convenient to follow the approach introduced by these authors. The oblique parameters characterize the influence of physics beyond the Fermi scale. They are given as linear combinations of the transverse components of the gauge–boson vacuum polarizations

\[ \Pi_{ij}^{\mu\nu}(p) = \Pi_{ij}(p^2) g^{\mu\nu} + (p^\mu p^\nu \text{ terms}), \tag{29} \]

where \( ij \) stands for \( aa, YY, \) and \( 3Y \), where \((a, Y)\) are \( SU(2) \times U_Y(1) \) indices. In particular,
the $S$ and $T$ parameters are defined by

$$\alpha S = \frac{2e^2}{m_Z^2} [\Pi_{3Y}(0) - \Pi_{3Y}(m_Z^2)], \quad (30)$$

$$\alpha T = \frac{2e^2}{m_W^2} \text{Re}[\Pi_{11}(0) - \Pi_{33}(0)]. \quad (31)$$

It is not difficult to see from (28), that $O_{WB}$ contributes to $S$ but not to $T$, whereas $O_{\phi}^{(3)}$ contributes to $T$ but not to $S$. It is also evident from (28) that $O_{\phi}^{(1)}$ does not contribute to these parameters. Considering only the new physics contribution, one finds,

$$S_{NP} = 2c_W \sqrt{\frac{\pi}{3\alpha}} (v^2 \theta^{45}), \quad (32)$$

$$T_{NP} = -\frac{1}{s_W} \sqrt{\frac{\pi}{\alpha}} (v^2 \theta^{45}), \quad (33)$$

where $s_W$ and $c_W$ stand for $\sin\theta_W$ and $\cos\theta_W$, respectively. On the other hand, the current experimental data give the next values of $S$ and $T$ which can be induced by new physics effects

$$S_{NP}^{Exp} = -0.13 \pm 0.10(-0.08), \quad (34)$$

$$T_{NP}^{Exp} = -0.17 \pm 0.12(+0.09), \quad (35)$$

where the central values assume $m_H = 117$ GeV. The change for $m_H = 300$ GeV is shown in parenthesis. Assuming the first value for the Higgs mass and using the values for the SM parameters reported in [25], one finds at 95% C.L.

$$-5.63 \times 10^{-8} \text{ GeV}^{-2} < \theta^{45} < 1.91 \times 10^{-7} \text{ GeV}^{-2}, \quad (36)$$

$$-3.18 \times 10^{-7} \text{ GeV}^{-2} < \theta^{45} < 1.06 \times 10^{-7} \text{ GeV}^{-2}, \quad (37)$$

which leads to the following bound for the new physics scale

$$-5.63 \times 10^{-8} \text{ GeV}^{-2} < \theta^{45} < 1.06 \times 10^{-7} \text{ GeV}^{-2}. \quad (38)$$

To conclude this part, it is worth comparing this bound with those obtained in the literature for the noncommutativity scale of four dimensional theories. To this respect, bounds of order of one TeV have been obtained from collider physics [26]. On the other hand, more stringent bounds of order $(10 \text{ TeV})^{-2}$ [27] or higher [28] have been derived from low–energy tests of Lorentz violation. However, as it has been recently argued [29], that these bounds are extremely model dependent and should be taken with some care.
We are now in position to discuss some phenomenological implications of these new interactions. Motivated by the fact that new physics effects would be more evident in those processes which are forbidden or strongly suppressed in the SM, we will consider the rare Higgs boson decay into two photons, which is an one–loop prediction of the model and thus is naturally suppressed \[30\]. Due to its phenomenological importance, this decay has been the subject of permanent interest in the literature. Apart from providing a good signature for the Higgs boson search at hadron colliders with mass in the intermediate range \(120 \text{ Gev} < m_H < 2m_Z\) \[31\], the decay width of this process is also of great interest because it determines the cross section for Higgs production in \(\gamma\gamma\) collisions \[32\]. Due to the fact that the \(H\gamma\gamma\) coupling is generated by loop effects of charged particles, its sensitivity to new heavy charged particles has been studied in many well motivated extensions of the SM, as the two Higgs doublet model (THDM) \[33\], the minimal supersymmetric standard model (MSSM) \[34\], the left–right symmetric models (LRM) \[35\], and the Littlest Higgs model (LHM) \[36\]. Many of its properties have also been studied in a model–independent manner using the effective Lagrangian framework \[20, 37\]. In our model, the \(H\gamma\gamma\) vertex (as well as \(H\gamma Z\) one) is induced at the tree level by the set of operators given in eqs.(21-23). The corresponding Lagrangian can be written as follows:

\[
\mathcal{L}_{H\gamma\gamma} = \frac{\alpha_W}{4}m_W\theta^{45}HF_{\mu\nu}F^{\mu\nu},
\]

(39)

where \(\alpha_W = 4s_W^2 - c_W^2 - 2s_2W / \sqrt{3}\). The total decay width \(\Gamma_{NC}\) can be conveniently written in terms of the SM width \(\Gamma_{SM}\), as follows:

\[
\Gamma_{NC}(H \rightarrow \gamma\gamma) = \Gamma_{SM}(H \rightarrow \gamma\gamma)\left[1 + \frac{\mathcal{A}_{NC}}{\mathcal{A}_{SM}}\right]^2,
\]

(40)

where \(\mathcal{A}_{NC} = \alpha_Wm_W^2\theta^{45}\), whereas \(\mathcal{A}_{SM}\) represents the charged fermion and \(W\) boson loop contributions, which is given by

\[
\mathcal{A}_{SM} = \frac{\alpha^{3/2}}{\sqrt{4\pi s_W}}\left[\sum_f N_{Cf}Q_f^2F_f + F_W\right].
\]

(41)

In this expression, \(f\) stands for quarks or leptons, \(N_{Cf}\) is the color index, and \(Q_f\) is the electric charge of the fermion in units of the charge of the positron. In addition, \(F_f\) and \(F_W\) are the fermion and \(W\) boson loop amplitudes, respectively, which can be found in ref. \[33\]. Though the bound for \(\theta^{45}\) was estimated for a value \(m_H = 117\) GeV, for illustration purposes we will present results that contemplate larger values of the Higgs mass. In Fig.1
the variation of the normalized decay width $R = \Gamma_{NP}/\Gamma_{SM}$ is displayed as function of the Higgs mass. From this figure, it can be appreciated that $R$ is sensitive to the sign of the $\theta_{45}$ parameter. A constructive or destructive effect corresponds to $\theta_{45} < 0 (\theta_{45} > 0)$, which can increase or decrease the standard model prediction $\Gamma_{SM}$ up to by 27% and 35%, respectively, for $m_H$ in the range 120 $-$ 200 GeV.

It is interesting to compare this result with those obtained from other models. In general, theories beyond the SM require more complicated Higgs sectors, i.e. they incorporate new neutral and charged Higgs bosons. However, in most cases, it is always possible to identify in an appropriate limit a SM–like Higgs boson, that is a CP–even neutral scalar whose couplings to pairs of $W$ and $Z$ bosons coincide with those given in the minimal SM. Furthermore, new contributions coming from new charged scalar, fermion, and vector particles are expected. We briefly review the results for a SM–like Higgs boson decaying into two photons, for the models mentioned previously. In the THDM, the only new charged particle is the $H^\pm$ Higgs boson, but its loop contribution to this decay is very small compared with the dominant $W$ contribution. Thus, in this model, the $\gamma\gamma$ decay width is essentially the same as for the SM [33]. In the case of the MSSM, this decay gets new contributions from superpartner loops. The $\gamma\gamma$ width tends to be lower than the one of the SM due to cancelations between the $W$ loop and the supersymmetric chargino loops. In this case, if the charginos are heavy, the decay width can be quite near to the SM width [34]. Further, it was found in ref. [35] that the contribution of a new $W$ boson, like the one predicted by LRM, is quite suppressed, because the corresponding loop amplitude is related to the SM $W$ boson amplitude as
\[ F_{W_R} = (m_{W_L}/m_{W_R})^2 F_{W_L}. \]

Taking into account the existing \( W_R \)-mass bounds \[25\], the \( \gamma\gamma \) width can be enhanced up to 5% at best. As to the prediction of the Littlest Higgs model is concerned, in \[36\] it was found that the \( \gamma\gamma \) width is reduced by 5 – 7% compared to the SM value. From these results, we can see that the impact of noncommutative extra dimensions on the \( H \rightarrow \gamma\gamma \) decay may be significantly more important than that predicted by some of the most popular SM extensions.

6. CONCLUSIONS

In this work we explore the consequences of noncommutativity in a 6-dimensional model, by means of the Seiberg-Witten map. We consider the \( SU(3) \) gauge Higgs unification model of the electroweak interactions of \[10\] and \[9\], compactified to 4D on an orbifold \( T^2/Z_N \) for \( N = 3, 4, 6 \). We analyze noncommutativity among all the 6-dimensional coordinates. As a consequence of the orbifold symmetries, it turns out that there are no corrections to the model due to noncommutativity among the 4D coordinates and the two-extra dimensions. We find that the corrections we obtain corresponding to noncommutativity among the 4D coordinates, differ from the ones of noncommutative models calculated directly in 4D, also by means of the Seiberg-Witten map \[13\]. On the other side, the corrections corresponding to noncommutativity of the extra dimensions have interesting phenomenological consequences, as we emphasize below.

As well as in the commutative model, the spontaneous symmetry breaking should arise dynamically, from first order quantum corrections. This step can be done in the noncommutative theory, as far as the expected Higgs mass is much less than the noncommutativity scale. Thus it would be interesting to include matter and to study the corresponding noncommutative corrections, which could be done following \[12\], progress in this direction will be reported elsewhere.

As mentioned in the introduction, the model we are considering here has a too high value for the weak angle. However, as noted in \[9\], there are various ways to solve this problem, in particular by an extension by a \( U(1) \) factor. Furthermore, the noncommutative Seiberg-Witten map of the corresponding gauge field will not mix with the already present noncommutative corrections. Thus we can expect that in a noncommutative version of this extended model, the kind of corrections presented here will still be present, in particular
those corresponding to noncommutativity between extra dimensions. Under this working assumption, we have studied the corrections due to noncommutativity of the extra dimensions by means of the effective lagrangian techniques. First, by the observation that these terms are quite sensitive to the $S$ and $T$ oblique parameters, we could obtain the bound to $\theta^{45}$ given by (38). In four dimensional noncommutative models, there are bounds obtained e.g. from low–energy tests of Lorentz violation, which are extremely model dependent [29]. We think that, in the framework of our working assumption, our bound has a less speculative nature, as it was obtained directly from the experimental constraints on the oblique parameters, without additional assumptions. With this bound established, we have looked at the impact of our corrections on the rare Higgs decay into two photons. It turns out that the effect depends on the signature of the noncommutativity parameter, increasing or decreasing respectively the value of the SM decay width $\Gamma_{SM}$, with a net effect which could be significantly more important than that predicted by some of the most popular SM extensions [25, 34, 35, 36].

Finally, from the results of the particular noncommutative model we started with, which could be interesting on its own, we can conclude that noncommutativity in higher dimensional models can have interesting consequences and phenomenological effects beyond those of four dimensional noncommutative theories. The study of more realistic models, including matter fields, is in progress.

Acknowledgments

We thank H. García-Compeán for discussions. This work has been supported by CONACYT grant 47641 and Projects by PROMEP, UG and VIEP-BUAP 13/I/EXC/05.


