Simultaneous Flavor Transformation of Neutrinos and Antineutrinos with Dominant Potentials from Neutrino-Neutrino Forward Scattering

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In astrophysical environments with intense neutrino fluxes, neutrino-neutrino forward scattering contributes both diagonal and off-diagonal potentials to the flavor-basis Hamiltonian that governs neutrino flavor evolution. We examine a special case where adiabatic flavor evolution can produce an off-diagonal potential from neutrino-neutrino forward scattering that dominates over both the corresponding diagonal term and the potential from neutrino-matter forward scattering. In this case, we find a solution that, unlike the ordinary Mikheyev-Smirnov-Wolfenstein scenario, has both neutrinos and antineutrinos maximally mixed in medium over appreciable ranges of neutrino and antineutrino energy. Employing the measured solar and atmospheric neutrino mass-squared differences, we identify the conditions on neutrino fluxes that are required for this solution to exist deep in the supernova environment, where it could affect the neutrino signal, heavy-element nucleosynthesis, and even the revival of the supernova shock. We speculate on how this solution might or might not be attained in realistic supernova evolution. Though this solution is ephemeral in time and/or space in supernovae, it may signal the onset of subsequent appreciable flavor mixing for both neutrinos and antineutrinos. A similar solution may also exist in an early universe with significant net neutrino-lepton numbers.

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I. INTRODUCTION

In this paper we study the problem of coherent nonlinear flavor evolution of active neutrinos in environments where neutrino-neutrino forward scattering makes a significant contribution to the effective neutrino mass in medium. In particular, we examine a special case where the off-diagonal potential from neutrino-neutrino forward scattering becomes the dominant term in the flavor-basis Hamiltonian that governs neutrino flavor evolution.

In both post-core-bounce supernovae and an early universe with net lepton numbers, local net neutrino number densities can exceed electron and baryon number densities. These large neutrino number densities or fluxes, sometimes referred to as a “neutrino background,” require that the usual Mikheyev-Smirnov-Wolfenstein (MSW) formalism for calculating the evolution of neutrino flavors be modified to include the effects of neutrino-neutrino forward scattering. Though the resulting problem of neutrino flavor evolution can be complicated, we can identify a key parameter governing the relevant physics: the ratio of the product of neutrino energy and the off-diagonal potential from neutrino-neutrino forward scattering relative to the difference of the squares of the vacuum neutrino mass eigenvalues. If this ratio becomes very large as a result of adiabatic neutrino flavor evolution, we can even find an interesting solution, which has neutrinos and antineutrinos simultaneously maximally mixed in medium. This is different from the usual MSW case where, at a given location or time, neutrino flavor mixing in medium is maximal for a specific range of neutrino energies (this range is narrow for small vacuum mixing angles), while antineutrino mixing is suppressed; or vice versa. As we discuss below, the off-diagonal potential from neutrino-neutrino forward scattering plays a unique role: it can alter neutrino flavor evolution into a form that is utterly unlike the MSW case.

In the presence of a significant neutrino background, neutrino flavor histories can be followed by solving a mean-field Schrödinger-like equation in the modified-MSW format. This is a long-standing and vexing problem. In fact, it has defied general and complete solution for the supernova environment, even with sophisticated numerical treatments. The existence and importance of the flavor-diagonal potential from neutrino-neutrino forward scattering and how it might modify MSW-like neutrino flavor evolution in supernovae were pointed out early on \cite{I} (see Ref. \cite{II} for a subsequent formal treatment). However, the existence of the corresponding off-diagonal potential was established only later \cite{III}. This latter discovery may prove to be a watershed event in supernova neutrino physics.

There have been several attempts to elucidate how flavor-diagonal and/or off-diagonal potentials from neutrino-neutrino forward scattering can affect active-active neutrino flavor transformation in the post-core-bounce supernova regime, especially as regards shock re-heating \cite{IV, V, VI} and r-process nucleosynthesis \cite{VII, VIII, IX, X}. The rationale for these studies was that the energy spectra and/or the fluxes of the various neutrino flavors could differ on emergence from the neutron star surface or neutrino sphere, and therefore, flavor inter-conversion above
this surface could alter these spectra and/or fluxes to change supernova dynamics and nucleosynthesis or the neutrino signal in a detector. If the energy distribution functions and the associated net energy luminosities are the same for all flavors of neutrinos and antineutrinos, then, obviously, flavor transformation will have no effect. However, on account of core de-leptonization and concomitant changes in composition, size, and equation of state, this is unlikely to be the case over the entire post-core-bounce period of ~20 s during which neutrino fluxes are appreciable.

If at any epoch in the supernova environment there develops a hierarchy of average neutrino energies or luminosities among the different neutrino flavors, then flavor conversion could alter the rates of electron neutrino and antineutrino capture on free nucleons:

\[
\nu_e + n \rightarrow p + e^{-}, \quad (1) \\
\bar{\nu}_e + p \rightarrow n + e^{+}. \quad (2)
\]

These are the processes principally responsible for depositing energy in the material behind the shock after core bounce. Therefore, altering their rates by, for example, swapping flavor labels between possibly less energetic electron neutrinos and more energetic mu and/or tau neutrinos could significantly affect the prospects for a supernova explosion. Also, the competition between these processes and their reverse reactions sets the neutron-to-proton ratio in neutrino-heated material. In turn, this ratio is sometimes a crucial parameter for r-process and other heavy-element nucleosynthesis associated with slow neutrino-heated outflows.

Most of the studies cited above posited the existence of neutrino mass-squared differences \(\geq 0.2 \text{eV}^2\). This was required for normal MSW resonances to occur in the high-density regions most relevant for supernova shock re-heating and r-process nucleosynthesis. These regions lie above but relatively close to the neutron star, generally within a few hundred kilometers. Without the hypothesized high neutrino mass-squared differences, conventional MSW evolution in these regions would not result in any significant neutrino flavor conversion.

Although we do not know the absolute vacuum neutrino mass eigenvalues, \(m_1\), \(m_2\), and \(m_3\), the two independent differences of their squares are now measured to be \(\delta m^2 \approx 7 \times 10^{-5} \text{eV}^2\) and \(3 \times 10^{-3} \text{eV}^2\) by observations of solar and atmospheric neutrinos, respectively. The lower \(\delta m^2\) has also been measured directly by the KamLAND reactor experiment. As these \(\delta m^2\) values are certainly small compared with the scale previously believed to be most relevant for supernovae, one may tend to conclude that they have no consequence for supernova dynamics and nucleosynthesis. However, as we will discuss below, neutrino background effects could alter this conclusion dramatically.

Similar to the supernova case, when there are net lepton numbers residing in the neutrino sector in the early universe, neutrino flavor conversion can be important in, for example, setting the neutron-to-proton ratio and, hence, the \(^4\text{He}\) abundance yield in primordial nucleosynthesis. It was recognized in Ref. 13 that the flavor-diagonal potential from neutrino-neutrino forward scattering affects neutrino propagation in the coherent limit of the problem. However, for reasons that will become clear, a complete treatment of active-active neutrino flavor conversion in the early universe requires a coupled calculation including both flavor-diagonal and off-diagonal potentials from neutrino-neutrino forward scattering as well as general inelastic neutrino scattering. In fact, the seminal numerical work in Refs. 14-16 shows that the measured neutrino mass-squared differences and mixing angles result in an “evening up” of the initially disparate lepton numbers residing in neutrinos of different flavors. The flavor oscillations in these calculations exhibit “synchronization” in time and space and can correspond to near-maximal flavor mixing for neutrinos and antineutrinos. We will argue below that this numerical result is closely related to our solution for the special case where the off-diagonal potential from neutrino-neutrino forward scattering dominates.

In Sec. II we will outline how neutrino flavor transformation proceeds in the coherent limit when neutrino backgrounds are non-negligible. In Sec. III we discuss the particular limit of domination by the off-diagonal potential from neutrino-neutrino forward scattering and the corresponding solution. We also speculate under what conditions and to what extent this solution could be attained. Similar issues for a lepton-degenerate early universe are discussed in Sec. IV. Conclusions are given in Sec. V.

II. COHERENT FLAVOR EVOLUTION WITH NEUTRINO BACKGROUNDS

Here we give a brief synopsis of coherent neutrino flavor amplitude development in the supernova and early universe environments. In the supernova core and the dense environment immediately above it, and in the early universe prior to weak decoupling, non-forward neutrino scattering can result in neutrino flavor conversion through de-coherence. We will ignore this in what follows and instead concentrate on the purely coherent evolution of the neutrino fields. It should always be kept in mind, however, that our considerations may need to be modified at high density or in high neutrino flux regimes.

Even in the purely coherent limit, following the effects of neutrino-neutrino forward scattering in the most general case is daunting in scope. Part of the difficulty in following neutrino flavor evolution in the supernova environment is geometric: flavor evolution histories on different neutrino trajectories are coupled. This is because two neutrino states will experience quantum entanglement to the future of a forward scattering event occurring at the intersection of their world lines. In light of this entan-
glement there has been considerable speculation about whether neutrino flavor evolution can be modeled adequately by a mean-field treatment with Schrödinger equations. The mean field in this case is the potential seen by a neutrino by virtue of forward scattering on particles in the environment that carry weak charge. Here we will follow the conclusions of Ref. \[17\] and take the mean-field treatment as sufficient in a statistical sense. This seems reasonable for the early universe and supernova environments because a statistically large number of neutrino scattering events and entanglements occur in these places.

Further complicating the supernova problem is the non-isotropic nature of the neutrino fields above the neutrino sphere. Neutrinos traveling along trajectories nearly tangential to the neutrino sphere may have quite different flavor amplitude histories from those moving radially or near radially. For now we will ignore this feature of flavor development and instead approximate all neutrinos as evolving the way radially propagating neutrinos evolve. This approximation is not a good one (it has been made in all previous numerical work \[6, 10\]), but it will suffice in our analytic arguments here.

### A. Overview

In vacuum, the flavor (weak interaction) eigenstates of neutrinos are related to the mass (energy) eigenstates by a unitary transformation $U_m$:

$$
\begin{pmatrix}
|\nu_e\rangle \\
|\nu_\mu\rangle \\
|\nu_\tau\rangle
\end{pmatrix} = U_m
\begin{pmatrix}
|\nu_1\rangle \\
|\nu_2\rangle \\
|\nu_3\rangle
\end{pmatrix},
$$

where the mass eigenstates $|\nu_1\rangle$, $|\nu_2\rangle$, and $|\nu_3\rangle$ correspond to the vacuum mass eigenvalues, $m_1$, $m_2$, and $m_3$, respectively. The unitary transformation $U_m$ can be written in terms of a sequence of rotations,

$$
U_m = U_{23} U_{13} U_{12}.
$$

A convenient representation for these rotations is

$$
U_{23} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos\theta_{23} & \sin\theta_{23} \\
0 & -\sin\theta_{23} & \cos\theta_{23}
\end{pmatrix},
$$

$$
U_{13} = \begin{pmatrix}
\cos\theta_{13} & 0 & e^{i\delta}\sin\theta_{13} \\
0 & 1 & 0 \\
-e^{-i\delta}\sin\theta_{13} & 0 & \cos\theta_{13}
\end{pmatrix},
$$

$$
U_{12} = \begin{pmatrix}
\cos\theta_{12} & \sin\theta_{12} & 0 \\
-\sin\theta_{12} & \cos\theta_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

In the above representation, the mixing angles, $\theta_{12}$ and $\theta_{23}$, have been measured by observations of solar and atmospheric neutrinos and related experiments. In particular, the best fit for the vacuum mixing of the mu and tau neutrinos is very near maximal, which gives $\theta_{23} \approx \pi/4$. However, the mixing angle, $\theta_{13}$, and the CP-violating phase, $\delta$, have not been measured yet.

Here we consider a neutrino mixing scenario where $m_3 > m_2 > m_1$ with $\delta m_{12}^2 \equiv m_2^2 - m_1^2 \approx 7 \times 10^{-5}$ eV$^2$ and $\delta m_{13}^2 \equiv m_3^2 - m_1^2 \approx 3 \times 10^{-3}$ eV$^2$, $\theta_{23} = \pi/4$, and $\delta = 0$. With the definitions

$$
|\nu_\mu^*\rangle \equiv \frac{|\nu_\mu\rangle - |\nu_\tau\rangle}{\sqrt{2}},
$$

$$
|\nu_\tau^*\rangle \equiv \frac{|\nu_\mu\rangle + |\nu_\tau\rangle}{\sqrt{2}},
$$

it is straightforward to show that

$$
\begin{pmatrix}
|\nu_e\rangle \\
|\nu_\mu^*\rangle \\
|\nu_\tau^*\rangle
\end{pmatrix} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13} \\
-s_{12} & c_{12} & 0 \\
-c_{12}s_{13} & -s_{12}s_{13} & c_{13}
\end{pmatrix}
\begin{pmatrix}
|\nu_1\rangle \\
|\nu_2\rangle \\
|\nu_3\rangle
\end{pmatrix},
$$

where for example, $c_{12} \equiv \cos\theta_{12}$ and $s_{12} \equiv \sin\theta_{12}$. The states $|\nu_\mu^*\rangle$ and $|\nu_\tau^*\rangle$ are still useful in medium. This is because in the supernova medium the mu and tau neutrinos have very nearly the same interactions, so that matter effects on mixing and effective mass for these species are nearly identical (likewise for the mu and tau antineutrinos). This will also be true for the early universe if the net muon and tau lepton numbers are identical. For the sake of our arguments here, we will take the symmetry between mu and tau neutrinos and that between their antiparticles to be rigorously true so that $|\nu_\mu^*\rangle$ and $|\nu_\tau^*\rangle$ are effective flavor eigenstates in medium. Since $\delta m_{13}^2 \gg \delta m_{12}^2$, the regions of neutrino flavor mixing governed by these parameters should be well separated. As neutrinos propagate outward from the neutrino sphere at high density in supernovae, $\delta m_{12}^2 \approx 3 \times 10^{-3}$ eV$^2$ becomes relevant first. We will focus on neutrino flavor mixing with this parameter, for which $|\nu_\mu^*\rangle$ is effectively decoupled [see Eq. (3)] and we only need consider mixing of $\nu_e$ and $\nu_\mu^*$. Thus, the general problem of 3ν mixing in medium is reduced to one of 2ν mixing in our scenario.

With the above simplification we can hereafter follow the notation of Ref. \[18\]. In particular, we now simply refer to $|\nu_\tau^*\rangle$ as $|\nu_\tau\rangle$ and write the effective 2ν unitary transformation in vacuum as

$$
\begin{align*}
|\nu_e\rangle & = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \\
|\nu_\tau\rangle & = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle,
\end{align*}
$$

where $|\nu_1\rangle$ and $|\nu_2\rangle$ refer generically to the light and heavy mass eigenstates, respectively, and $\theta$ is the effective 2ν vacuum mixing angle. The relevant vacuum mass-squared difference is $\delta m_{12}^2 \approx 3 \times 10^{-3}$ eV$^2$. The corresponding effective vacuum mixing angle is $\theta \sim \theta_{13}$, with the current reactor experiment limit being $\sin^2 2\theta_{13} < 0.1$ (see e.g., Ref. \[12\]).

Consider a neutrino of initial flavor $\alpha = e$ or $\tau$. As it propagates outward from the neutrino sphere in supernovae, the evolution of its state, the ket $|\Psi_{\nu_\alpha}(t)\rangle$, can be described as

$$
|\Psi_{\nu_\alpha}(t)\rangle = a_{e\alpha}(t)|\nu_e\rangle + a_{\tau\alpha}(t)|\nu_\tau\rangle,
$$

where $a_{e\alpha}(t)$ and $a_{\tau\alpha}(t)$ are the survival amplitudes.
where \(a_{\nu\alpha}(t)\) or \(a_{\tau\alpha}(t)\) is the amplitude for the neutrino to be a \(\nu_\alpha\) or \(\tau_\alpha\), respectively, at time \(t\). (Note that \(t\) could be any affine parameter such as radius along the neutrino’s world line.) Alternatively, the evolution of \(|\Psi_{\nu\alpha}(t)\rangle\) can be described as

\[
|\Psi_{\nu\alpha}(t)\rangle = a_{1\alpha}(t)|\nu_1(t)\rangle + a_{2\alpha}(t)|\nu_2(t)\rangle, \tag{12}
\]

where \(|\nu_1(t)\rangle\) and \(|\nu_2(t)\rangle\) are the instantaneous mass (energy) eigenstates in medium, and \(a_{1\alpha}(t)\) and \(a_{2\alpha}(t)\) are the corresponding amplitudes. The flavor eigenstates are related to \(|\nu_1(t)\rangle\) and \(|\nu_2(t)\rangle\) as

\[
|\nu_\alpha\rangle = \cos \theta_M(t)|\nu_1(t)\rangle + \sin \theta_M(t)|\nu_2(t)\rangle, \tag{13}
\]

\[
|\nu_\tau\rangle = -\sin \theta_M(t)|\nu_1(t)\rangle + \cos \theta_M(t)|\nu_2(t)\rangle, \tag{14}
\]

where \(\theta_M(t)\) is the effective 2ν mixing angle in medium at time \(t\). In matrix form, the ket \(|\Psi_{\nu\alpha}(t)\rangle\) can be represented by

\[
\Psi_f \equiv \begin{bmatrix} a_{\nu\alpha}(t) \\ a_{\tau\alpha}(t) \end{bmatrix} \tag{15}
\]

in the flavor basis and by

\[
\Psi_M \equiv \begin{bmatrix} a_{1\alpha}(t) \\ a_{2\alpha}(t) \end{bmatrix} \tag{16}
\]

in the energy basis. In analogous fashion we will employ a 2ν scheme to follow separately the flavor evolution of the antineutrino sector.

### B. Characterizing Neutrino Densities

The single neutrino density operator at time \(t\) projected into the energy basis is

\[
|\Psi_{\nu\alpha}(t)\rangle\langle\Psi_{\nu\alpha}(t)| = |a_{1\alpha}(t)|^2|\nu_1(t)\rangle\langle\nu_1(t)| + |a_{2\alpha}(t)|^2|\nu_2(t)\rangle\langle\nu_2(t)|
+ a_{1\alpha}(t)a_{2\alpha}(t)|\nu_1(t)\rangle\langle\nu_2(t)| + a_{1\alpha}^*(t)a_{2\alpha}(t)|\nu_2(t)\rangle\langle\nu_1(t)|. \tag{17}
\]

The second line of Eq. \((17)\) contains cross terms which in general have complex coefficients. However, these cross terms vanish in the limit where neutrino flavor evolution is adiabatic. This is because a neutrino evolving adiabatically is always in a single energy state. For example, in this limit we might have \(|a_{1\alpha}(t)| = 1\), which would imply that \(|a_{2\alpha}(t)| = 0\) due to the normalization condition

\[
|\langle \Psi_{\nu\alpha}(t)|\Psi_{\alpha}(t)\rangle|^2 = |a_{1\alpha}(t)|^2 + |a_{2\alpha}(t)|^2 = 1. \tag{18}
\]

The density operator for the neutrinos or antineutrinos with momentum centered around \(p\) in a pencil of neutrino or antineutrino momenta and directions \(d^3p\) can be defined as in Ref. [6]:

\[
\hat{\rho}_{\nu}(t)d^3p \equiv \sum_\alpha \bar{\rho}_{\nu\alpha}(t)d^3p = \int \rho_{\nu\alpha}(t)d^3p, \tag{19}
\]

\[
\hat{\rho}_{\nu}(t)d^3p \equiv \sum_\alpha \bar{\rho}_{\nu\alpha}(t)d^3p = \int \rho_{\nu\alpha}(t)d^3p. \tag{20}
\]

Note that the traces of these operators over neutrino flavor do not give unity but rather the total number density of neutrinos or antineutrinos of all kinds in the pencil.

We assume that neutrinos and antineutrinos of all flavors are emitted from the same sharp neutrino sphere of radius \(R_\nu\) in supernovae. (This is not a particularly good approximation for neutrinos very near the neutron star surface, but it will suffice for our arguments.) At a radius \(r > R_\nu\), the neutrino sphere subtends a solid angle of

\[
\Delta \Omega_\nu(r) = 2\pi\left(1 - \sqrt{1 - R_\nu^2/r^2}\right). \tag{21}
\]

Within this solid angle, the number density of \(\nu_\alpha\) in a pencil of directions and momenta is

\[
dn_{\nu\alpha} = \frac{L_{\nu\alpha}}{\pi R_\nu^2} \int dE_\nu \left(\frac{d\Omega_\nu}{4\pi}\right) f_{\nu\alpha}(E_\nu) dE_\nu, \tag{22}
\]

where \(L_{\nu\alpha}\) is the energy luminosity of \(\nu_\alpha\), \(d\Omega_\nu\) is the pencil of directions, \(E_\nu\) is the neutrino energy, \(f_{\nu\alpha}(E_\nu)\) is the normalized energy distribution function for \(\nu_\alpha\), and \(\langle E_{\nu\alpha}\rangle\) is the corresponding average \(\nu_\alpha\) energy. Here and in the rest of this paper we assume that neutrinos have relativistic kinematics and employ natural units where \(\hbar = c = 1\). The function \(f_{\nu\alpha}(E_\nu)\) can be fitted to the results from supernova neutrino transport calculations and is commonly taken to be of the form

\[
f_{\nu\alpha}(E_\nu) = \frac{1}{T_{\nu\alpha}^3 F_2(\eta_{\nu\alpha})} \frac{E_\nu^2}{e^{E_\nu/T_{\nu\alpha} - \eta_{\nu\alpha}} + 1}, \tag{23}
\]

where \(T_{\nu\alpha}\) and \(\eta_{\nu\alpha}\) are two fitting parameters and \(F_2(\eta_{\nu\alpha})\) is the Fermi integral of order 2 and argument \(\eta_{\nu\alpha}\). The Fermi integral of order \(k\) and argument \(\eta\) is defined as

\[
F_k(\eta) \equiv \int_0^\infty \frac{x^k dx}{e^{x - \eta} + 1}. \tag{24}
\]
In terms of these integrals, the average $\nu_\alpha$ energy is

$$\langle E_{\nu_\alpha} \rangle = \int_0^\infty E_\nu f_{\nu_\alpha}(E_\nu) dE_\nu = T_{\nu_\alpha} \frac{E_\nu^3}{2} \left( \frac{\eta_{\nu_\alpha}}{\eta_{\nu_\alpha}} \right).$$

We assume that all neutrino species have thermal, Fermi-Dirac energy distribution functions in the early universe, so the number density of $\nu_\alpha$ in a pencil of directions and momenta is

$$dn_{\nu_\alpha} = \frac{1}{2\pi^2} \left( \frac{d\Omega_\nu}{4\pi} \right) e^{E_\nu/T_{\nu_\alpha} - \eta_{\nu_\alpha} + 1}.$$  

Note that although the above equation uses the same symbols $T_{\nu_\alpha}$ and $\eta_{\nu_\alpha}$ as Eq. (24), the physical meanings of these symbols are very different. In Eq. (24), $T_{\nu_\alpha}$ and $\eta_{\nu_\alpha}$ are simple parameters used to fit the energy distribution functions obtained from supernova neutrino transport calculations, while in Eq. (26), $T_{\nu_\alpha}$ is the temperature and $\eta_{\nu_\alpha}$ is the degeneracy parameter of the $\nu_\alpha$ gas in the early universe. However, as the assumed neutrino energy distribution functions for the supernova and early universe environments have the same functional form, we use the same symbols in both cases for convenience. For the homogeneous and isotropic neutrino gas in the early universe, the local proper number density of $\nu_\alpha$ is

$$n_{\nu_\alpha} = \frac{T_{\nu_\alpha}^3}{2\pi^2} F_2(\eta_{\nu_\alpha}).$$

Equation (26) can be rewritten as

$$dn_{\nu_\alpha} = n_{\nu_\alpha} \left( \frac{d\Omega_\nu}{4\pi} \right) f_{\nu_\alpha}(E_\nu) dE_\nu.$$  

In terms of the scaled neutrino energy $\epsilon \equiv E_\nu/T_{\nu_\alpha}$, we have

$$f_{\nu_\alpha}(E_\nu) dE_\nu = f_{\nu_\alpha}(\epsilon) d\epsilon = \frac{1}{F_2(\eta_{\nu_\alpha})} \frac{\epsilon^3 d\epsilon}{e^{\epsilon - \eta_{\nu_\alpha}} + 1}.$$  

In analogy to the baryon-to-photon ratio $\eta \equiv (n_{b} - n_{\gamma})/n_{\gamma} \approx 6 \times 10^{-10}$, we define a $\nu_\alpha$-to-photon ratio

$$\ell_{\nu_\alpha} \equiv \frac{n_{\nu_\alpha} - n_{\nu_\beta}}{n_{\gamma}} \approx \frac{\pi^2}{12\zeta(3) \left( T_{\nu_\alpha}/T_{\gamma} \right)^3} \left( \frac{\eta_{\nu_\alpha}}{\eta_{\nu_\alpha}} \right),$$

where $T_{\gamma}$ is the photon temperature, $\zeta(3) = 1.20206$, and we have used $T_{\nu_\alpha} = T_{\nu_\beta}$ and $\eta_{\nu_\beta} = -\eta_{\nu_\alpha}$ to obtain the last identity. For $T_{\nu_\alpha} = T_{\gamma}$ and small $\ell_{\nu_\alpha}$, $\eta_{\nu_\beta} \approx 1.46 \ell_{\nu_\alpha}$. Current limits on all lepton numbers are $\ell_{\nu_\alpha} < 0.1$ (cf. Ref. [10]).

C. Neutrino Propagation in Medium

For a neutrino originating as a $\nu_\alpha$ at $t = 0$, its subsequent flavor evolution along a radially-directed trajectory with Affine parameter $t$ is described by

$$\frac{\partial}{\partial t} |\Psi_{\nu_\alpha}\rangle = \left( \hat{H}_{\nu_{\nu_\alpha}} + \hat{H}_{\nu_{\nu_\beta}} + \hat{H}_{\nu_{\nu_\gamma}} \right) |\Psi_{\nu_\alpha}\rangle,$$

where we have decomposed the overall evolution Hamiltonian into contributions from vacuum neutrino masses and from mean-field ensemble averages for neutrino-electron and neutrino-neutrino forward scattering. These contributions are discussed individually below.

For a neutrino with energy $E_\nu$ and vacuum mass $m \ll E_\nu$, we have $E_\nu = \sqrt{p^2 + m^2} \approx p + m^2/(2p)$, where $p$ is the magnitude of the neutrino momentum $p$. In this limit, the vacuum-mass contribution to the flavor evolution Hamiltonian is

$$\hat{H}_{\nu_{\nu_\nu}} \approx p\hat{I} + \frac{1}{2p} \left[ m_1^2 |\nu_1\rangle \langle \nu_1| + m_2^2 |\nu_2\rangle \langle \nu_2| \right],$$

where $\hat{I}$ is the identity operator.

Electron neutrinos and antineutrinos can forward scatter on electrons and positrons through exchange of $W^\pm$. In contrast, there is no such charged-current forward scattering for $\nu_\mu$, $\nu_\tau$, and their antiparticles due to the absence of $\mu^\pm$ and $\tau^\pm$ in the environments of interest here. Consequently, the effective contribution from charged-current neutrino-electron forward scattering to the flavor evolution Hamiltonian is

$$\hat{H}_{\nu_{\nu\nu}} = \sqrt{2} G_F (n_{\nu_e} - n_{\bar{\nu}_e}) \approx \sqrt{2} G_F n_e Y_e.$$  

In the above equation, $n_{\nu_e}$, $n_{\bar{\nu}_e}$, and $n_b$ are the proper number densities of electrons, positrons, and baryons, respectively, at the position corresponding to time $t$, and $Y_e = (n_{\nu_e} - n_{\bar{\nu}_e})/n_b$ is the net electron number per baryon, or electron fraction.

For a specific neutrino with momentum $p$, the effective neutral-current neutrino-neutrino forward scattering contribution $\hat{H}_{\nu_{\nu_{\nu}}}$ to the flavor evolution Hamiltonian is

$$\hat{H}_{\nu_{\nu_{\nu}}} = \sqrt{2} G_F \int \left( 1 - \cos \theta_{pq} \right) \left[ \rho_{pq}(t) - \bar{\rho}_{pq}(t) \right] d^3q,$$

where $q$ is the momentum of the background neutrinos and $\cos \theta_{pq} = p \cdot q/pq$. The term $\int (1 - \cos \theta_{pq})$ stems from the structure of the weak current [20]. This can be seen from the limit where completely relativistic neutrinos are traveling in the same direction along the same spacetime path. In this limit $1 - \cos \theta_{pq} = 0$ and neutrinos never forward scatter on one another. Obviously, for the homogeneous and isotropic neutrino distribution functions characteristic of the early universe, $\cos \theta_{pq}$ averages to zero and the ensemble average of $(1 - \cos \theta_{pq})$ is unity. In the supernova environment the term $(1 - \cos \theta_{pq})$ will be largest close to the neutron star, where the neutrino trajectories can intersect at high angles. At sufficiently large radii above the neutron star, the neutrino-neutrino forward-scattering contribution to the flavor evolution Hamiltonian will scale as $r^{-4}$. As the neutrino-electron forward-scattering contribution will scale roughly as $r^{-3}$,
it may be dominated by the neutrino-neutrino forward-scattering contribution at small to moderate distances from the neutron star.

In the above equation, \( \Delta \) gives only an overall phase to the neutrino state \( s \), and is therefore unimportant in neutrino flavor conversion. In the above equation, \( \Delta \equiv \delta m^2/2E_{\nu} \), and \( \alpha_{\nu}, B, \text{ and } B_{e\tau} \) are the potentials from neutrino-neutrino forward scattering. Specifically,

\[
\begin{align*}
\alpha_{\nu} &= \frac{\sqrt{2} G_F}{2} \int (1 - \cos \theta_{pq}) \left( [\hat{\rho}_q(t) - \hat{\rho}_q(t)]_{ee} + [\hat{\rho}_q(t) - \hat{\rho}_q(t)]_{\tau\tau} \right) d^3q, \\
B &= \frac{\sqrt{2} G_F}{2} \int (1 - \cos \theta_{pq}) \left( [\hat{\rho}_q(t) - \hat{\rho}_q(t)]_{ee} - [\hat{\rho}_q(t) - \hat{\rho}_q(t)]_{\tau\tau} \right) d^3q, \\
B_{e\tau} &= 2\sqrt{2} G_F \int (1 - \cos \theta_{pq}) \left[ \hat{\rho}_q(t) - \hat{\rho}_q(t) \right]_{e\tau} d^3q,
\end{align*}
\]

where the matrix elements of the density operators are defined as

\[
\begin{align*}
[\hat{\rho}_q(t) - \hat{\rho}_q(t)]_{ee} d^3q &= \langle \nu_e | \hat{\rho}_q(t) d^3q | \nu_e \rangle - \langle \bar{\nu}_e | \hat{\rho}_q(t) d^3q | \bar{\nu}_e \rangle, \\
[\hat{\rho}_q(t) - \hat{\rho}_q(t)]_{\tau\tau} d^3q &= \langle \nu_\tau | \hat{\rho}_q(t) d^3q | \nu_{\tau} \rangle - \langle \bar{\nu}_\tau | \hat{\rho}_q(t) d^3q | \bar{\nu}_\tau \rangle, \\
[\hat{\rho}_q(t) - \hat{\rho}_q(t)]_{e\tau} d^3q &= \langle \nu_e | \hat{\rho}_q(t) d^3q | \nu_{\tau} \rangle - \langle \bar{\nu}_e | \hat{\rho}_q(t) d^3q | \bar{\nu}_{\tau} \rangle.
\end{align*}
\]

The physical interpretation of these matrix elements is straightforward, even if the notation is cumbersome. For example, \([\hat{\rho}_q(t) - \hat{\rho}_q(t)]_{ee} d^3q \) gives the expectation value for the net \( \nu_e \) number density in the pencil of momenta and directions \( d^3q \) centered on \( q \). Note that the off-diagonal matrix element vanishes and makes no contribution to \( B_{e\tau} \) if neutrinos remain in their initial flavor states. This is evident if we expand out the first term in Eq. (12):

\[
\langle \nu_e | \hat{\rho}_q(t) d^3q | \nu_e \rangle = \sum_\alpha d\nu_\alpha \langle \nu_e | \Psi_{\nu_\alpha} \rangle \langle \Psi_{\nu_\alpha} | \nu_e \rangle.
\]

In the above equation, one or the other amplitude in the sum on the right-hand side will be zero unless some neutrino flavor transformation has occurred at the time \( t \) when this matrix element is evaluated.

For real \( B_{e\tau} = B_{\tau e} \), it is convenient to define the effective mixing angle \( \theta_{M} \) in medium by

\[
\begin{align*}
\cos 2\theta_{M} (t) &= (\Delta \cos 2\theta - A - B)/\Delta_{\text{eff}}, \\
\sin 2\theta_{M} (t) &= (\Delta \sin 2\theta + B_{\tau e})/\Delta_{\text{eff}},
\end{align*}
\]

where

\[
\Delta_{\text{eff}} = \sqrt{(\Delta \cos 2\theta - A - B)^2 + (\Delta \sin 2\theta + B_{\tau e})^2}.
\]

With the term proportional to the identity matrix dropped, Eq. (36) can be transformed to the instantaneous energy basis to give

\[
\begin{align*}
\frac{i}{\hbar} \frac{\partial \Psi_M}{\partial t} &= \begin{bmatrix} -\Delta_{\text{eff}}/2 & -i\dot{\theta}_{M} (t) \\ i\dot{\theta}_{M} (t) & \Delta_{\text{eff}}/2 \end{bmatrix} \Psi_M,
\end{align*}
\]

where \( \dot{\theta}_{M} (t) = d\theta_{M}/dt \). In the limit where \( |\dot{\theta}_{M} (t)| \ll \Delta_{\text{eff}}/2 \), Eq. (47) becomes two decoupled equations and flavor amplitude evolution is adiabatic. The flavor evolution equations and the corresponding effective mixing angle \( \theta_{M} \) in medium for antineutrinos can be obtained from those for neutrinos by replacing \( A, B, \text{ and } B_{e\tau} \) in the latter with \( -A, -B, \text{ and } -B_{e\tau} \), respectively.

The condition \( |\dot{\theta}_{M} (t)| \ll \Delta_{\text{eff}}/2 \) for adiabatic neutrino flavor evolution is most stringent when \( \Delta_{\text{eff}} \) reaches the minimum value \( |\Delta \sin 2\theta + B_{e\tau}| \) at an MSW resonance corresponding to

\[
\Delta \cos 2\theta = A + B.
\]

At resonance, the effective in-medium mixing angle is \( \theta_{M} (t_{\text{res}}) = \pi/4 \) and mixing is maximal with \( \sin^2 2\theta_{M} = 1 \). We can define an adiabaticity parameter

\[
\gamma = \frac{\Delta_{\text{eff}} (t_{\text{res}})}{2|\dot{\theta}_{M} (t_{\text{res}})|} = \frac{(\Delta \sin 2\theta + B_{e\tau})^2}{\Delta \cos 2\theta} \mathcal{H},
\]

where

\[
\mathcal{H} = \frac{V}{V_{\text{res}}} = \frac{|A + B|}{|A + B|_{\text{res}}}
\]
is the scale height for the total potential \( V = A + B \) at resonance with \( V = dV/dt \). We can gain more insight into the adiabaticity parameter by further defining a resonance region corresponding to \( 1/2 \leq \sin^2 2\theta_M \leq 1 \). In this region the change in \( V \) around the resonance value \( \Delta \cos 2\theta \) is \( \delta V = |\Delta \sin 2\theta + B_{\tau\tau}| \), so the width of this region is

\[
(\delta t)_{res} = \frac{\delta V}{|V|_{res}} = \frac{\Delta \sin 2\theta + B_{\tau\tau}}{\Delta \cos 2\theta} \mathcal{H}.
\]  

(51)

As the oscillation length at resonance is

\[
L_{res} = \frac{2\pi}{\Delta_{eff}(t_{res})} = \frac{2\pi}{|\Delta \sin 2\theta + B_{\tau\tau}|},
\]  

(52)

we have

\[
\gamma = 2\pi \frac{(\delta t)_{res}}{L_{res}}.
\]  

(53)

In summary, large \( B_{\tau\tau} \) increases \( \gamma \) in two ways: (1) by increasing the resonance width \( (\delta t)_{res} \) and; (2) by decreasing the oscillation length \( L_{res} \).

Clearly, neutrino flavor evolution will be adiabatic for \( \gamma \gg 1 \). For the small effective vacuum mixing angle \( \theta \ll 1 \) of interest here, neutrino flavor conversion will be complete in this limit. For arbitrary \( \gamma \), the probability of neutrino flavor conversion after propagation through resonance is well approximated by \( 1 - P_{LZ} \), where \( P_{LZ} = \exp(-\pi \gamma/2) \) is the Landau-Zener probability for a neutrino to jump from one energy eigenstate to the other in traversing the resonance region.

### D. Neutrino Potentials in the Adiabatic Limit

Due to the cross terms in the single neutrino density operator in Eq. (17), \( B_{\tau\tau} \) is generally complex. If these cross terms are unimportant, then both \( B \) and \( B_{\tau\tau} \) are real and their expressions in Eqs. (38) and (39) can be simplified as

\[
B = -\sqrt{2}G_F \sum_\alpha \int (1 - \cos \theta_{pq}) \left[ \cos 2\theta_M \left(1 - 2|a_{1\alpha}|^2\right) dn_{\nu_\alpha} - \cos 2\bar{\theta}_M \left(1 - 2|\bar{a}_{1\alpha}|^2\right) dn_{\bar{\nu}_\alpha} \right],
\]  

(54)

\[
B_{\tau\tau} = \sqrt{2}G_F \sum_\alpha \int (1 - \cos \theta_{pq}) \left[ \sin 2\theta_M \left(1 - 2|a_{1\alpha}|^2\right) dn_{\nu_\alpha} - \sin 2\bar{\theta}_M \left(1 - 2|\bar{a}_{1\alpha}|^2\right) dn_{\bar{\nu}_\alpha} \right].
\]  

(55)

As mentioned in Sec. II B, the cross terms in Eq. (17) vanish if neutrino states evolve adiabatically. In this limit, the above expressions of \( B \) and \( B_{\tau\tau} \) can be simplified further. As the electron and neutrino number densities at the neutrino sphere are far above those satisfying the resonance condition, \( \nu_e \) and \( \bar{\nu}_\tau \) are born essentially as the energy eigenstates \( |\nu_2\rangle \) and \( |\nu_1\rangle \), respectively. For adiabatic evolution, \( |a_{1\tau}|^2 \approx 0 \) and \( |a_{1\alpha}|^2 \approx 1 \) for all subsequent time \( t \). For antineutrinos, adiabatic evolution gives \( |\bar{a}_{1\alpha}|^2 \approx 1 \) and \( |\bar{a}_{1\tau}|^2 \approx 0 \). Thus, we have

\[
B \approx \sqrt{2}G_F \int (1 - \cos \theta_{pq}) \left[ (dn_{\nu_e} - dn_{\bar{\nu}_e}) \cos 2\theta_M + (dn_{\nu_\tau} - dn_{\bar{\nu}_\tau}) \cos 2\bar{\theta}_M \right],
\]  

(56)

\[
B_{\tau\tau} \approx \sqrt{2}G_F \int (1 - \cos \theta_{pq}) \left[ (dn_{\nu_e} - dn_{\bar{\nu}_e}) \sin 2\theta_M + (dn_{\nu_\tau} - dn_{\bar{\nu}_\tau}) \sin 2\bar{\theta}_M \right].
\]  

(57)

### III. FLAVOR MIXING WITH LARGE OFF-DIAGONAL POTENTIAL

We now focus on adiabatic neutrino flavor evolution, for which the potentials from neutrino-neutrino forward scattering are given by Eqs. (56) and (57). Our main concern is the effects of these potentials on neutrino flavor evolution in supernovae. In Sec. II C, we have outlined this evolution for a specific neutrino as it propagates outward from the neutrino sphere. At a given radius \( r \) with potentials \( A \) and \( B \), the resonance condition in Eq. (48) will be met for a particular neutrino energy \( E_{res} \), i.e.,

\[
\frac{\delta m^2}{2E_{res}} \cos 2\theta = A + B.
\]  

(58)

As \( B \) and \( B_{\tau\tau} \) at a given radius involve integration of \( \cos 2\theta_M, \cos 2\bar{\theta}_M, \sin 2\theta_M \), and \( \sin 2\bar{\theta}_M \) over the neutrino energy distribution functions, it is convenient to write

\[
\cos 2\theta_M = \frac{1 - E_\nu/E_{res}}{\sqrt{(1 - E_\nu/E_{res})^2 + |\tan 2\theta + (E_\nu/E_{res})(2E_{res}B_{\tau\tau}/(\delta m^2 \cos 2\theta))^2}|}.
\]  

(59)
Let us now consider the limits of the above expressions for the in-medium mixing angles when \( \theta \ll 1 \) and \( B_{\pi} \) is positive and so large that the second terms in the square root of these expressions dominate the first terms. In this limit we have

\[
\begin{align*}
\cos 2\theta_M &\to 0, \\
\sin 2\theta_M &\to 1, \\
\cos 2\bar{\theta}_M &\to 0, \\
\sin \bar{\theta}_M &\to -1,
\end{align*}
\]

for which both neutrinos and antineutrinos have maximal in-medium mixing with

\[
\theta_M \to \frac{\pi}{4}, \quad \bar{\theta}_M \to \frac{3\pi}{4}.
\]

A large negative \( B_{\pi} \) clearly changes the signs of the limits in Eqs. (63)–(66), and the in-medium mixing angles in this case are \( \theta_M \to 3\pi/4 \) and \( \bar{\theta}_M \to \pi/4 \).

In general, we see that large in-medium mixing will occur simultaneously for neutrinos and antineutrinos over a broad range of energies if adiabatic flavor evolution results in

\[
|B_{\pi}| \gg \frac{\delta m^2}{2E_{\text{res}}} \cos 2\theta
\]

at some radius above the neutrino sphere. Using Eq. (63), we can rewrite the above equation as \( |B_{\pi}| \approx A + B \). As \( \cos 2\theta_M \to 0 \) and \( \cos 2\bar{\theta}_M \to 0 \) when this is achieved, Eq. (65) gives \( B \to 0 \). Therefore, Eq. (69) reduces to

\[
|B_{\pi}| \approx A.
\]

Note that even a \( |B_{\pi}| \) only as large as \( \frac{\delta m^2}{2E_{\text{res}}} \cos 2\theta \) already has important effects on the in-medium mixing of neutrinos and antineutrinos. While a neutrino with resonance energy \( E_\nu = E_{\text{res}} \) has maximal in-medium mixing independent of \( B_{\pi} \), the energy range over which neutrinos have large in-medium mixing with \( 1/2 \leq \sin^2 2\theta_M \leq 1 \) is strongly affected by \( B_{\pi} \). For \( B_{\pi} = 0 \), this energy range corresponds to \( E_{\text{res}}(1 - \tan 2\theta) \leq E_\nu \leq E_{\text{res}}(1 + \tan 2\theta) \), which is very narrow for \( \theta \ll 1 \). In contrast, for example, with \( B_{\pi} = \frac{\delta m^2}{2E_{\text{res}}} \cos 2\theta \), all neutrinos with \( E_\nu \geq E_{\text{res}}/2 \) have \( 1/2 \leq \sin^2 2\theta_M \leq 1 \) even for \( \theta \ll 1 \). Furthermore, \( B_{\pi} \) also affects the in-medium mixing for antineutrinos, which is strongly suppressed (\( \sin^2 2\theta_M \ll 1 \)) in the absence of neutrino-antineutrino forward scattering. For \( B_{\pi} = \frac{\delta m^2}{2E_{\text{res}}} \cos 2\theta \) and \( \theta \ll 1 \), antineutrinos with \( E_\nu \geq E_{\text{res}} \) have substantial in-medium mixing with \( 1/5 \leq \sin^2 2\theta_M \leq 1/2 \).

A. Towards a Self-Consistent Solution with a Large \( B_{\pi} \)

Here we outline a possible self-consistent \( B_{\pi} \)-dominant solution (BDS) which meets two criteria: (1) \( |B_{\pi}| \gg A \); and (2) adiabaticity, \( \gamma \gg 1 \). An immediate question is: can adiabatic neutrino flavor evolution ever produce a large \( B_{\pi} \) as in Eq. (63)? Obviously, the answer is yes if one can demonstrate that this result is obtained under some conditions. Such demonstration requires following the flavor evolution of neutrinos with a wide range of energies covered by their energy distributions. As mentioned in Sec. III, this process may sound straightforward but turns out to be computationally difficult. On the other hand, if adiabatic flavor evolution can indeed give rise to a large \( B_{\pi} \), then the conditions required for this to occur must depend on the following: the neutrino mixing parameters \( \delta m^2 \) and \( \sin^2 2\theta \), the profile of electron number density that gives the potential \( A \) and the neutrino luminosities and energy distribution functions that are related to the potentials \( B \) and \( B_{\pi} \). Our goal here is to examine these dependences. In so doing, we will not be able to answer the question posed at the beginning of this paragraph, but we will be able to provide a range of conditions that can guide future numerical calculations in search of a complete solution for neutrino flavor mixing.

We start with the basic input for our discussion. As explained in Sec. II A the mixing parameters of interest here are \( \delta m^2 \approx 3 \times 10^{-3} \, \text{eV}^2 \) and \( \sin^2 2\theta < 0.1 \). To characterize the potential \( A \), we need the electron number density \( n_e = Y_e n_b \). We note that the envelope above the post-core-bounce neutron star can be approximated as a quasi-static configuration with a constant entropy per baryon \( S \) in the gravitational field of the neutron star. In this case, the enthalpy per baryon, \( T S \), is roughly the gravitational binding energy of a baryon, so that the temperature \( T \) scales with radius as

\[
T \approx \frac{M_{\text{NS}} m_p n_b}{m_{\text{pl}}} S^{-1} r^{-1},
\]
where $m_{p1} \approx 1.221 \times 10^{22}$ MeV is the Planck mass, $m_p$ is the proton mass, and $M_{\text{NS}}$ is the neutron star mass. At late times relevant for $r$-process nucleosynthesis, the environment above the neutron star is radiation-dominated, so

$$S \approx \frac{2\pi^2}{45} \frac{g_*}{n_b} \frac{T^3}{r_b}$$

in units of Boltzmann constant $k_B$ per baryon. In the above equation, $g_*$ is the statistical weight in relativistic particles: $g_* \approx 11/2$ when $e^-\bar{\nu}$-pairs are abundant and $g_* \approx 2$. Combining Eqs. (71) and (72), we obtain the run of baryon number density for the $r$-process epoch as

$$n_b \approx \frac{2\pi^2}{45} g_* \left( \frac{M_{\text{NS}} m_p}{m_{p1}^2} \right)^3 S^{-4} r^{-3}.$$  (73)

The potential $A$ is given by

$$A = \sqrt{2} G_F Y_e n_b \approx \frac{2\sqrt{2\pi^2}}{45} g_* Y_c G_F \left( \frac{M_{\text{NS}} m_p}{m_{p1}^2} \right)^3 S^{-4} r^{-3}$$

$$\approx (5.2 \times 10^{-13} \text{MeV}) g_* Y_e \left( \frac{M_{\text{NS}}}{1.4 M_\odot} \right)^3 S^{-4} r^{-3}.$$  (74)

where $S_{100}$ is $S$ in units of $100 k_B$ per baryon and $r_6$ is $r$ in units of $10^6$ cm. The $r$-process epoch corresponds to a time post-core-bounce $t_{pb} > 3$ s. This is a relatively long time after core bounce, at least compared with the time scale of the shock re-heating epoch and the time scale for evolution of neutrino emission characteristics such as luminosities and average energies. The potential $A$ in Eq. (74) can also be used to describe crudely the shocked regions of the envelope above the core in the shock reheating epoch $t_{pb} < 1$ s if we take $g_* \approx 1$ and employ a low entropy $T$. To evaluate $B$ and $B_{\text{cr}}$, we need the differential number density of each neutrino species at radius $r > R_\nu$ above the neutrino sphere. The differential $\nu_e$ number density in the absence of flavor evolution is given by Eq. (72), which depends on the luminosity $L_{\nu_e}$ and the average energy $\langle E_{\nu_e} \rangle$. For some illustrative numerical estimates we will assume that all neutrino species have the same luminosity,

$$L_\nu \equiv L_{\nu_e} = L_{\bar{\nu}_e} = L_{\nu_\mu} = L_{\bar{\nu}_\mu},$$

and take the average neutrino energies to be

$$\langle E_{\nu_e} \rangle = 10 \text{ MeV}, \quad \langle E_{\bar{\nu}_e} \rangle = 15 \text{ MeV},$$

$$\langle E_{\nu_\mu} \rangle = \langle E_{\bar{\nu}_\mu} \rangle = 27 \text{ MeV}.$$  (76)

Assuming that adiabatic flavor evolution up to some radius $r > R_\nu$ results in a BDS described by the criterion in Eq. (70), we now examine the implications of this criterion for supernova conditions. For definiteness, we discuss the case of a large positive $B_{\text{cr}}$, which gives $\sin^2 \theta_M \to 1$ and $\sin \theta_M \to -1$. In this case, Eq. (77) gives

$$B_{\text{cr}} \approx \sqrt{2} G_F \int (1 - \cos \theta_{pq}) (dn_{\nu_e} - dn_{\bar{\nu}_e}),$$  (77)

where we have assumed that the luminosities and energy distribution functions for $\nu_e$ and $\bar{\nu}_e$ are very nearly the same in the supernova environment. This is a good approximation because these species experience nearly identical interactions both in the dense environment of the core and in the more tenuous outer regions. For a radially-propagating test neutrino, the intersecting angles of the background neutrinos, $\theta_{pq}$, are coincident with the polar angle in the integration over $d\Omega_\nu$ for the test neutrino. Assuming that neutrinos of all flavors originate on the same neutrino sphere and using $dn_{\nu_e}$ and $dn_{\bar{\nu}_e}$ of the form in Eq. (22), $B_{\text{cr}}$ in Eq. (77) can be evaluated as

$$B_{\text{cr}} \approx \frac{\sqrt{2} G_F}{4\pi R_\nu^2} \left[ 1 - \sqrt{1 - R_\nu^2/r^2} \right]^{2} \left( \frac{L_{\nu_e}}{\langle E_{\nu_e} \rangle} - \frac{L_{\bar{\nu}_e}}{\langle E_{\bar{\nu}_e} \rangle} \right)$$

$$\approx \left( 2.1 \times 10^{-10} \text{MeV} \right) R_\nu^2 \left[ 1 - \sqrt{1 - R_\nu^2/r^2} \right]^{2} \times \left[ \frac{L_{\nu_e} \bar{\nu}_e}{\langle E_{\nu_e} \rangle/(10 \text{MeV})} - \frac{L_{\bar{\nu}_e} \nu_e}{\langle E_{\bar{\nu}_e} \rangle/(10 \text{MeV})} \right].$$  (78)

where $R_\nu \equiv R_\nu/(10^6 \text{cm})$, $L_{\nu_e} \bar{\nu}_e \equiv L_{\nu_e}/(10^{52} \text{ergs s}^{-1})$, and $L_{\bar{\nu}_e} \nu_e \equiv L_{\bar{\nu}_e}/(10^{52} \text{ergs s}^{-1})$. For $r \gg R_\nu \sim 10^6$ cm and the assumptions in Eqs. (76) and (77), we obtain

$$B_{\text{cr}} \approx (1.8 \times 10^{-11} \text{MeV}) L_{\nu_e} \bar{\nu}_e R_\nu^2 R_\nu^2.$$  (79)

Using Eqs. (72) and (78), we can rewrite the criterion $B_{\text{cr}} \gg A$ as

$$\frac{R_\nu^3/r^3}{\left( 1 - \sqrt{1 - R_\nu^2/r^2} \right)^2} \ll \frac{45}{8\pi^3} \left( \frac{m_{p1}^2}{M_{\text{NS}}} \right)^3 \left[ \frac{R_\nu S_{100}^4}{g_* Y_e} \right] \left[ \frac{L_{\nu_e}}{\langle E_{\nu_e} \rangle} - \frac{L_{\bar{\nu}_e}}{\langle E_{\bar{\nu}_e} \rangle} \right] \approx (398) \left( \frac{1.4 M_\odot}{M_{\text{NS}}} \right)^3 \left[ \frac{R_{\nu_6} S_{100}^4}{g_* Y_e} \right] \left[ \frac{L_{\nu_e} \bar{\nu}_e}{\langle E_{\nu_e} \rangle/(10 \text{MeV})} - \frac{L_{\bar{\nu}_e} \nu_e}{\langle E_{\bar{\nu}_e} \rangle/(10 \text{MeV})} \right].$$  (80)
For $r \gg R_{\nu}$ and the assumptions in Eqs. (73) and (74), we obtain

$$r_6 \ll 33 \left(\frac{1.4 M_\odot}{M_{NS}}\right)^3 \left(\frac{S_{100}^4 L_{\nu_{52}} R_{e6}^2}{g_s Y_e}\right).$$  \hspace{1cm} (81)

We have assumed adiabatic neutrino flavor evolution in the above discussion. Though the adiabaticity of the general flavor evolution can be ascertained only with a sophisticated numerical treatment, the BDS clearly will not be self-consistent if flavor evolution is not adiabatic at the radius where $B^{\text{BDS}}_{e\tau} \gg A$ is achieved. On the other hand, if we can show that at this radius the adiabaticity parameter for the neutrino with energy $E_{\text{res}}$ satisfies $\gamma^{\text{BDS}} \gg 1$, then the BDS is more likely to be obtained. With $\mathcal{H} \approx |A/A| \approx r/3$, this criterion [see Eq. (49)] can be rewritten as

$$\gamma^{\text{BDS}} \approx \frac{B^{\text{BDS}}_{e\tau}}{A} \mathcal{H}$$

$$\approx \frac{15}{16\sqrt{2}r^4} \left(\frac{m_{\nu_1}^2}{M_{NS} T_{\nu_1}}\right)^3 \left(1 - \frac{1 - R_e^2/r^2}{R_e/r}\right)$$

$$\times \left(\frac{G_F S^4}{g_s Y_e}\right) \left(\frac{L_{\nu_e}}{E_{\nu_e}} - \frac{L_{\bar{\nu}_e}}{E_{\bar{\nu}_e}}\right)^2 \gg 1,$$  \hspace{1cm} (82)

which reduces to

$$\gamma^{\text{BDS}} \approx 10^7 \left(\frac{1.4 M_\odot}{M_{NS}}\right)^3 \left(\frac{S_{100}^4 L_{\nu_{52}}^2}{g_s Y_e}\right) \frac{R_{e6}^4}{r_6^6} \gg 1$$  \hspace{1cm} (83)

for $r \gg R_{\nu}$ and the assumptions in Eqs. (73) and (74).

The criteria in Eqs. (73) and (74) can be met in some regions with significant scale and duration above the neutron star during both the $r$-process and shock reheating epochs. For example, an $r$-process environment with modest entropy might have $Y_e \approx 0.4$, $R_{e6} \approx 1$, $g_s \approx 11/2$, $S_{100} \approx 1.5$, and $L_{\nu_{52}} \approx 0.1$. For these parameters Eqs. (73) and (74) would give $r_6 \ll 8$ and 22, respectively, so the BDS may be obtained over an extended region above the neutrino sphere. For a higher entropy, $S_{100} = 2.5$, but with all the other parameters remaining the same, Eqs. (73) and (74) would give $r_6 \ll 60$ and 40, respectively. We can put these limits in perspective by noting the temperature at which salient events or processes occur above the neutron star. The radius corresponding to a temperature $T_0$ (measured in units of $10^9$ K) is very roughly $r_6 \approx 22.5/(T_0 S_{100})$ [see Eq. (74)]. Weak freeze-out, where the neutron-to-proton ratio is set, occurs at $T_0 \sim 10$. The neutron capture regime in the $r$-process is typically further out, occurring between $T_0 \approx 3$ and $T_0 \approx 1$. Therefore, the limits on the radius discussed above are so generous that maximal in-medium mixing for both neutrinos and antineutrinos associated with the BDS could affect important weak interaction processes in the envelope, the $r$-process, and the neutrino signal.

Taking $Y_e = 0.4$, $g_s = 11/2$, $M_{NS} = 1.4 M_\odot$, $R_e = 10$ km, $L_{\nu_e} = L_{\bar{\nu}_e} = L_{\nu}, \langle E_{\nu_e}\rangle = 10$ MeV, and $\langle E_{\bar{\nu}_e}\rangle = 15$ MeV, we use Eq. (73) to calculate the combinations of $L_{\nu}$ and $S$ for which the criterion $B^{\text{BDS}}_{e\tau} \gg A$ can be met below a fixed radius. The results are shown in Fig. 1 as contours labelled by the limiting radius. Except for the region corresponding to the larger values of the limiting radius, these results are generally more stringent than those from $\gamma^{\text{BDS}} \gg 1$. The chosen range of parameters is meant to be characteristic of the $r$-process epoch. The horizontal axis is entropy in units of $100 g_B$ per baryon, while the vertical axis is neutrino luminosity $L_{\nu}$, in units of $10^{52}$ erg s$^{-1}$. Here we take $Y_e = 0.4$, $g_s = 11/2$, $M_{NS} = 1.4 M_\odot$, and $R_e = 10$ km. We also assume that $L_{\nu_e} = L_{\nu_\mu} = L_{\nu_\tau}, \langle E_{\nu_e}\rangle = 10$ MeV, $\langle E_{\nu_\mu}\rangle = 15$ MeV, and that all mu and tau neutrinos and antineutrinos have identical luminosities and energy spectra.

![FIG. 1: Contours of limiting radius (in units of $10^6$ cm) beneath which $B^{\text{BDS}}_{e\tau} \gg A$ may be obtained. Except for the region corresponding to the larger values of the limiting radius, these results are generally more stringent than those from $\gamma^{\text{BDS}} \gg 1$. The chosen range of parameters is meant to be characteristic of the $r$-process epoch. The horizontal axis is entropy in units of $100 g_B$ per baryon, while the vertical axis is neutrino luminosity $L_{\nu}$, in units of $10^{52}$ erg s$^{-1}$. Here we take $Y_e = 0.4$, $g_s = 11/2$, $M_{NS} = 1.4 M_\odot$, and $R_e = 10$ km. We also assume that $L_{\nu_e} = L_{\nu_\mu} = L_{\nu_\tau}, \langle E_{\nu_e}\rangle = 10$ MeV, $\langle E_{\nu_\mu}\rangle = 15$ MeV, and that all mu and tau neutrinos and antineutrinos have identical luminosities and energy spectra.](image-url)
This limit becomes maximal neutrino flavor mixing associated with the BDS. The early shock re-heating process could be affected by the neutrino signal from the neutronization burst and cases the limit from Eq. (82) is much weaker. Therefore, we define the effective net number of neutrinos per baryon through \( L_{\nu_b} \), and taking \( L_{\nu_e52} = 10 \), \( (E_{\nu_e}) = 10 \text{ MeV}, R_{\nu_6} = 4, Y_e = 0.35, g_s = 1.5, \) and \( S_{100} = 0.15 \), we find that \( B^{\text{BDS}} \gg A \) may be obtained for \( r_6 \ll 15 \) [Eq. \( 80 \)]. This limit becomes \( r_6 \ll 50 \) if \( S_{100} = 0.2 \). For both cases the limit from Eq. \( 82 \) is much weaker. Therefore, the neutrino signal from the neutronization burst and the early shock re-heating process could be affected by maximal neutrino flavor mixing associated with the BDS.

### B. Is the BDS Ever Attained?

Achieving the BDS is dependent on a number of conditions, many of which are unlikely to strictly and generically obtain in environments in nature with high neutrino fluxes. The essence of the BDS is the dominance of the flavor off-diagonal potential and, in particular, \( 2E_{\nu_e}B_{\tau\tau} \gg \delta m^2 \cos 2\theta \). Since the measured neutrino mass-squared differences are small, it will not take a large flavor off-diagonal potential to force the system into something like the BDS.

However, as Eq. \( 83 \) shows, a necessary condition for \( B_{\tau\tau} \) to be non-zero at some time/position is that *some* neutrinos must have transformed their flavors there. This can be problematic because in both the early universe and the post-shock supernova environment a fluid element will evolve from conditions of very high density toward lower density. For example, the region near the neutron star surface is very high density, corresponding to high electron degeneracy. This will tend to suppress in-medium neutrino mixing. A hydrodynamic flow away from the neutron star surface will carry a fluid element into regions of lower temperature and density and net neutrino fluxes. At large enough radius the neutrino-electron potential will scale like \( A \sim r^{-3} \), while the flavor-diagonal and off-diagonal neutrino-neutrino potentials will scale as \( r^{-4} \). As a result, there may be some region where the neutrino-neutrino potentials dominate.

The neutrino resonance energy experienced in this fluid element at radius \( r \) will be \( E_{\text{res}} = \delta m^2 \cos 2\theta / 2(A + B) \). Near the neutron star surface \( E_{\text{res}} \) will be extremely small. Further out, in an adiabatic and roughly hydrostatic envelope (notation as in the last section), the resonance energy at radius \( r \) will be

\[
E_{\text{res}} \approx \frac{45}{4\sqrt{2\pi}^2} \left( \frac{m_{\nu_l}^2}{M_{\odot} m_p} \right)^3 \frac{\delta m^2 \cos 2\theta S_N^4 \nu^3}{G_F (Y_e + Y_{\nu_{\text{eff}}})} \approx (2.85 \times 10^{-3} \text{ MeV}) \left( \frac{1.4 M_{\odot}}{M_{\text{NS}}} \right)^3 \left( \frac{\delta m^2 \cos 2\theta}{3 \times 10^{-3} \text{ eV}^2} \right) \frac{S_{100}^4 \nu^3}{(Y_e + Y_{\nu_{\text{eff}}})}. \quad (84)
\]

where we define the effective net number of neutrinos per baryon through \( B = \sqrt{2}G_F n_b Y_{\nu_{\text{eff}}} \). Because \( \delta m^2 \) is small, the resonance energy also tends to be small at distances where neutrino fluxes are appreciable.

However, if neutrinos transform their flavors via a strict MSW evolution then \( B \) (and \( Y_{\nu_{\text{eff}}} \)) will drop with the radius of the fluid element and eventually will be driven *negative* [3]. To see this consider first an example (hierarchical) energy spectrum for \( \nu_e \) and \( \nu_e \) neutrinos as they leave the neutrino sphere. In Fig. \( 5 \) we show Fermi-Dirac-type energy spectra for these species, taking the neutrino degeneracy parameter for both to be \( \eta_{\nu_e} = 3 \), and taking average energies \( \langle E_{\nu_e} \rangle = 10 \text{ MeV} \) and \( \langle E_{\nu_e} \rangle = 27 \text{ MeV} \). The actual supernova neutrino energy spectra may differ significantly from these, but they serve to illustrate general trends. Note that for our chosen spectral parameters, the \( \nu_e \) population at lower energies is larger than the \( \nu_{\tau} \) population for comparable luminosity in the two neutrino species.

As our example fluid element moves out to larger \( r \),
From Eq. (78) it is clear that this term could be substantially smaller than $B_{\text{BDS}}^\text{res}$ if $E_\nu$ is a typical neutrino energy. Even if this is not true for $E_\nu = E_\text{res}$ at very high density where $E_\text{res}$ is small, higher energy neutrinos and antineutrinos may experience significant in-medium mixing angles over a broad range of energy. Though not strictly our BDS, this may nevertheless approximate it.

Previous numerical simulation work on neutrino flavor evolution in the supernova environment may offer only limited guidance here. The simulation in Ref. [6] made the same $2 \times 2$, and one-dimensional approximations as we make here. (By “one-dimensional” we mean that flavor histories on neutrino trajectories of any polar angle are taken to be the same as a radially directed path for the same lapse of Affine parameter along these trajectories.) Additionally, the work in Ref. [6] employed the density profiles and neutrino fluxes of the Mayle & Wilson late time supernova models and it adopted a range for $\delta m^2$ which is now known to be un-physically large for active-active neutrino evolution. Both of these features combined to produce only minimal effects from rather small values of $B_{\nu \tau}$. 

Likewise, the numerical simulation of Ref. [10] considered one-dimensional, $2 \times 2$ neutrino flavor evolution with un-physically large mass-squared difference. The conclusions in this work regarding real supernovae are suspect because: (1) the large $\delta m^2$ used would in reality demand the incorporation of sterile neutrinos which mix significantly with actives and this was left out; and (2) the feedback of neutrino flavor conversion on $Y_\tau$ was not correctly modeled since the threshold was neglected in the rate for $\bar{\nu}_\tau + p \rightarrow n + e^+$ and the weak magnetism corrections were also neglected. However, this numerical simulation was the first to follow neutrino phases in detail in this environment. Synchronization of large amplitude neutrino flavor oscillations was seen. This behavior is at least qualitatively like some aspects of the BDS, especially as regards significant in-medium mixing.

Though the conditions for establishment of the BDS are manifest in many regions of the post-shock supernova environment, it has not been seen unambiguously in simulations to date. However, there is considerable room for improvement in the sophistication of these simulations. Flavor evolution histories on different neutrino trajectories needs to be followed in detail, including all coupling. The role of density fluctuations in getting some neutrino and antineutrinos must be followed. Finally, the effects of neutrino mixing on neutrino transport in the neutron star core may be important and recent formulations of this problem represent significant progress.
C. The Ephemeral Nature of the BDS

Changing neutrino luminosities and fluxes and changing matter density will quickly lead to the development of complex amplitudes in the unitary transformation between the neutrino mass/energy and flavor bases which, in turn, will lead to complex potentials. This will signal the end of the strict validity of our particular BDS discussed above. However, it may not signal the immediate end of appreciable in-medium mixing among the flavors of neutrinos and antineutrinos.

If we ride along with a fluid element being driven from the neutron star’s surface by heating we will see a local fall off in matter density and neutrino fluxes and so a decrease in neutrino-electron and neutrino-neutrino forward scattering-induced potentials in this Lagrangian frame. What is the effect of this time dependence on in-medium flavor mixing? Using the flavor basis evolution equation (Eq. (50) and ignoring the term proportional to the identity we can find a second order equation for, e.g., $a_{ee}$, the amplitude for a neutrino of initial flavor $\alpha$ to be a $\nu_e$:

$$\ddot{a}_{ee} + \omega^2 a_{ee} = \frac{\dot{B}_{\tau\tau}}{B_{\tau\tau}} a_{ee}. \quad (87)$$

Here the dots over quantities denote time derivatives and

$$\omega^2 = \frac{1}{4} \left[ |B_{\tau\tau}|^2 + \delta^2 + 2i \dot{\delta} - 2i \frac{\dot{B}_{\tau\tau}}{B_{\tau\tau}} \right], \quad (88)$$

with $\delta \equiv A + B - \Delta \cos 2\theta$ and $\dot{\delta} = \dot{A} + \dot{B}$.

In our BDS all time derivatives vanish, $\omega \approx |B_{\tau\tau}|/2$, and we can solve Eq. (87), use unitarity $(|a_{ee}|^2 + |a_{\tau\tau}|^2 = 1)$ and take $a_{ee} = a_{\tau\tau} = a_{\tau\tau} = \pm \exp(\pm i\omega t)/\sqrt{2} \; \text{and} \; a_{\tau\tau} = \mp \exp(\pm i\omega t)/\sqrt{2}$, and likewise, $a_{ee} = a_{\tau\tau} = \pm \exp(\pm i\omega t)/\sqrt{2} \; \text{and} \; a_{\tau\tau} = \mp \exp(\pm i\omega t)/\sqrt{2}$. If we employ these solutions in the general flavor-basis form for the off-diagonal potential,

$$B_{\tau\tau} = \sqrt{2G_F} \sum_\alpha \int (1 - \cos \theta_{pq}) [dn_{\nu_\alpha} a_{ee} a_{\tau\alpha}^* - dn_{\bar{\nu}_\alpha} \bar{a}_{ee} \bar{a}_{\tau\alpha}^*], \quad (89)$$

we will recover the BDS form for this [cf., Eq. (14)] discussed above:

$$B_{\tau\tau}^{\text{BDS}} \approx \sqrt{2G_F} \int (1 - \cos \theta_{pq}) [(dn_{\nu_e} - dn_{\nu_\alpha}) - (dn_{\bar{\nu}_e} - dn_{\bar{\nu}_\alpha})], \quad (90)$$

However, once we allow the potentials to change in time, amplitudes will quickly acquire a non-sinusoidal time dependence which will lead to the development of potentials with imaginary components. With complex potentials we will lose a key assumption used in obtaining the BDS of Eq. (14). Flavor evolution from that point on will be complicated, but there is nothing in the evolution equations that demands an immediate return to medium-suppressed flavor mixing for most neutrino energies.

IV. THE BDS IN LEPTON-DEGENERATE COSMOLOGIES

Coherent active-active neutrino flavor evolution in the early universe also can be dominated by the flavor off-diagonal potential whenever significant net lepton numbers reside in the neutrino seas. Collision-associated decoherence dominates neutrino flavor conversion in the early universe at temperatures above Weak Decoupling, $T > 1 \text{ MeV}$. Neutrino inelastic scattering rates are large compared to the expansion rate in that regime. By con-
neutrino flavor evolution below this scale is largely coherent. We will concentrate on this episode. For illustrative purposes we also will confine our discussion to cosmologies which have identical lepton numbers and interactions for both mu and tau neutrinos. With this condition we can reduce the flavor evolution problem to the same 2 × 2 channel we dealt with for the supernova environment.

Coherent medium-enhanced flavor transformation in the channel $\nu_\alpha \leftrightarrow \nu_\beta$ (where $\alpha, \beta = e, \mu, \tau$ and $\alpha \neq \beta$) in the early universe is governed by the flavor-basis Hamiltonian

$$H_{EU} = \frac{1}{2} \left( V - \Delta \cos 2\theta \Delta \sin 2\theta + B_{\tau e} \Delta \cos 2\theta - V \right), \quad (91)$$

where $V = A + B + \delta C$, we take $\alpha = e$ and $\beta = \tau$, and we ignore the components of the coherent Hamiltonian proportional to the identity. The time evolution of the flavor and mass/energy amplitudes can be handled by a mean field Schrödinger-like equation in complete analogy to Eqs. (33) & (34).

The thermal contribution to the flavor-diagonal potential is $\delta C$. The high entropy of the universe ($S \approx 10^{10}$) dictates a sometimes appreciable contribution to neutrino effective mass stemming from forward scattering on thermal fluctuations (cf. Ref. [28]). The thermal term relative to zero potential is $C \approx r_e G_f^2 T^5$, where $r_e \approx 79.34$ and $r_\tau \approx 22.22$ at the epoch of interest, $T < 2$ MeV. The difference of these contributions is

$$\delta C \approx \delta r e G_f^2 T^5.$$ 

This term in the Hamiltonian is negligible in the post Weak Decoupling epoch environment whenever the lepton numbers are significantly larger than the baryon-to-photon ratio, $|\ell_{\nu_e}| \gg \eta$. We therefore will neglect $\delta C$ here, as we consider the large lepton number case only. Similarly, we also neglect thermal contributions to the flavor off-diagonal potential.

Let us consider the case where $\ell_{\nu_e} > \ell_{\nu_\tau}$. We impose the BDS for this case by: (1) assuming completely adiabatic neutrino flavor evolution; and (2) employing the maximal mixing angles of Eqs. (63), (64), (65), (66).

The first of these conditions allows the use of the adiabatic forms of $B$ and $B_{\tau e}$, Eqs. (55) and (56), respectively. Employing the second assumption in these potential forms immediately implies that $B \approx 0$.

We will have $|a_{1\tau}|^2 = 0$, $|a_{1e}|^2 = 1$, $|a_{1e}|^2 = 1$, and $|a_{1\tau}|^2 = 0$ in the adiabatic limit when $\ell_{\nu_e} > \ell_{\nu_\tau} > 0$.

(There is a level crossing for the antineutrinos in this case but not for the neutrinos.) Employing these amplitudes in Eq. (55) and noting the isotropic nature of the neutrino distribution functions in the early universe, we find

$$B_{\tau e} \approx \sqrt{2G_f} (\ell_{\nu_e} - \ell_{\nu_\tau} - (\ell_{\nu_e} - \ell_{\nu_\tau})) \quad (92)$$

and

$$\approx 2\sqrt{2}\zeta(3) G_f T^3 (\ell_{\nu_e} - \ell_{\nu_\tau}). \quad (93)$$

For this solution to be self consistent, we must have $|B_{\tau e}| \gg A$ and adiabatic neutrino flavor evolution. The first condition will be true long as

$$(\ell_{\nu_e} - \ell_{\nu_\tau}) \gg Y_e \eta \sim 3 \times 10^{-10}. \quad (94)$$

This condition follows on noting that $A = \sqrt{2}G_f Y_e n_b$ and the baryon density is $n_b = \eta n_\gamma$.

Adiabatic flavor evolution in this case is all but guaranteed on account of the slow expansion rate at this epoch and the known values for the atmospheric and solar mass-squared difference scales (see the discussion of adiabaticity in Ref. [29]). The effective density scale height for weak charge at an epoch with temperature $T$ in the early universe is roughly a third of the causal horizon length or, more correctly,

$$\mathcal{H} \equiv \left| \frac{1}{V} \frac{dV}{dt} \right|^{-1} \approx \frac{1}{3} H^{-1} \left| 1 + \frac{\dot{\gamma}}{g} - \frac{\Delta \dot{\ell}/\Delta \ell}{3H} \right|^{-1}. \quad (95)$$

The statistical weight in relativistic particles at this epoch is $g$ and the difference in lepton numbers here is $\Delta \ell \equiv \ell_{\nu_e} - \ell_{\nu_\tau}$. The expansion rate (Hubble parameter) is $H \approx (8\pi^3/90)^{1/2} g^{1/2} T^2 / m_\Pi$ and the horizon length in $d_H(t) = 2t = H^{-1}$. Neglecting the rate of change of $g$ and the rate of change of lepton difference, $\Delta \dot{\ell}$, relative to the expansion rate, the adiabaticity parameter for the BDS [cf. Eq. (82)] is in this case

$$\gamma \approx \frac{\sqrt{10} \zeta(3) G_f m_\Pi}{\pi g_{7/2}} \frac{T (\Delta \ell)^2}{g^{1/2} Y_e \eta} \eta \approx \left( 5 \times 10^{18} \right) \left( \frac{10.75}{g} \right)^{1/2} \Delta \ell^2 (T_{\text{MeV}}). \quad (96)$$

In the second approximation we have used the measured baryon-to-photon ratio $\eta \approx 6 \times 10^{-10}$. For an epoch where $T \approx 1$ MeV, we will have $g \approx 10.75$ and $Y_e \approx 0.5$ to 0.16, so that we have adiabatic neutrino flavor evolution for $\Delta \ell \gg \eta$. Since this is the same condition as that required for $|B_{\tau e}| \gg A$ to obtain, we see that $B_{\tau e}^{\text{BDS}}$ indeed can be a self consistent solution in the early universe.

If we had a lepton number degenerate cosmology with $\ell_{\nu_e} > \ell_{\nu_\tau}$ as described above, then we would expect a rapid “evening up” of the electron and tau lepton numbers if the maximal mixing BDS obtained. In fact, numerical simulations which include the neutrino background terms and follow neutrino amplitudes with phases find just this end result [12]. They also find that neutrino oscillations are synchronized in phase in time/space. To what extent is this result related to the BDS?

In contrast to the supernova case, it may be much easier to evolve into the BDS in a large lepton number early universe scenario. First, with modest lepton numbers, $|\ell_{\nu_e}| < 10^{-4}$, there could be significant transformation of neutrino flavors associated with collisions and de-coherence in the epoch where temperatures are above the weak decoupling scale. The extent of this conversion depends on vacuum mixing angle $\theta$ and, hence, possibly
on $\theta_{13}$. Significant flavor conversion at this epoch would imply that there may already be a significant $|B_{e\tau}|$ by the time neutrino flavor evolution is dominated by coherent forward scattering in the epoch after weak decoupling.

Also, depending on $\Delta \ell$, neutrino resonance energies could be appreciable at this epoch. The resonance condition implies that the resonant scaled energy is

$$\epsilon_{\text{res}} \approx \frac{\pi^2}{4\sqrt{2}} \frac{\delta m^2 \cos 2\theta}{\langle 3 \rangle_{\text{GF}}} \frac{1}{(\Delta \ell + \eta \gamma_e) T^4}. \quad (98)$$

$$E_{\text{res}} \approx (6.2 \times 10^5 \text{MeV}) \left( \frac{\delta m^2 \cos 2\theta}{3 \times 10^{-3} \text{eV}^2} \right) \left( \frac{\text{MeV}}{T} \right)^3 \frac{1}{\gamma_e + \Delta \ell/\eta}. \quad (99)$$

In the BDS only the $A$ term determines resonance energies. This case would correspond to $\Delta \ell = 0$ in Eq. (98). This would give a large resonance energy and would likely lead to a significant $|B_{e\tau}|$ and appreciable mixing across a broad range of neutrino energies.

As outlined above, the numerical simulations of Ref.s \cite{15} show synchronization of large amplitude neutrino and/or antineutrino flavor oscillations in large lepton number early universe scenarios. As for the supernova case, this looks at least qualitatively similar to the BDS. This is especially true while $\Delta \ell$ is still large in these models. As this quantity is decreased by neutrino flavor mixing, the flavor off-diagonal potential is decreased and neutrino flavor inter-conversion will be less efficient and the BDS conditions may no longer apply.

\section{V. CONCLUSION}

We have investigated the complicated problem of $2 \times 2$ coherent neutrino flavor evolution in the limit of large flavor off-diagonal neutrino-neutrino forward scattering potential both in the early universe and in the post-shock supernova environment. We have identified a simple solution/limit in this problem. This solution (BDS) is governed by a dominant off-diagonal potential. This constitutes a viable solution only under a number of restrictive assumptions, but it is evident that even a rough facsimile to this solution will retain key principal features of the BDS. These include maximal or near maximal in-medium mixing angles for both neutrinos and antineutrinos over broad ranges of neutrino/antineutrino energies. These features are very different from conventional neutrino flavor amplitude evolution with the MSW effect.

Indeed, it has been generally thought that the small values of neutrino mass-squared difference among the active neutrinos preclude significant effects from medium-enhanced neutrino flavor conversion in the dense environment above the neutron star in supernova models. This is largely the case for conventional MSW neutrino evolution. It would not take much coherent flavor transformation to produce a prodigious $B_{e\tau}$, at least on the scale of the vacuum term in Eq. (98). This term can be especially small here on account of the large resonance energies. For example, using Eq. (98), the resonance energy $= T\epsilon_{\text{res}}$ is

The stakes may be high. If neutrino energy spectra or fluxes for the different neutrino flavors are appreciably different at any point in the some 20 s time frame following core bounce, then neutrino and antineutrino mixing could affect shock re-heating physics, neutrino heating on the neutrino-driven wind, and slow outflow r-process scenarios. Of course, the neutrino signal could be affected by any kind of neutrino/antineutrino flavor mixing. The effect we point out here, if it is ever realized deep in the supernova envelope, could appreciably alter the emergent neutrino energy spectra and fluxes over those calculated via conventional MSW evolution alone.

Finally, our considerations extend to any environment where neutrino fluxes are appreciable and where neutrino flavor mixing may have important consequences for the neutron-to-proton ratio and/or energetics and dynamics. Gamma-ray burst fireball models sited in the vicinity of a hot or collapsed compact object are a case in point. In this environment, the supernova problem, and in the early universe we are hard pressed to follow computationally the evolution of the neutrino/antineutrino component. It is unsatisfactory that this remains true
even in the face of the tremendous strides in experimental neutrino physics which have given us the neutrino mass-squared differences and most of the vacuum mixing parameters.

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