First and Second Order Perturbations of Hypersurfaces

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Abstract

In this paper we find the first and second order perturbations of the induced metric and the extrinsic curvature of a non-degenerate hypersurface $\Sigma$ in a spacetime $(\mathcal{M}, g)$, when the metric $g$ is perturbed arbitrarily to second order and the hypersurface itself is allowed to change perturbatively (i.e. to move within spacetime) also to second order. The results are fully general and hold in arbitrary dimensions and signature. An application of these results for the perturbed matching theory between spacetimes is presented.

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1 Introduction

The aim of this paper is to analyze how the induced metric and extrinsic curvature of a hypersurface change when the spacetime metric is perturbed to second order and the hypersurface itself is deformed perturbatively to second order. The motivation to carry out this calculation is twofold. Firstly, to study the matching conditions between two spacetimes when the metric is perturbed arbitrarily to second order. Secondly, to study the dynamics of first and second order perturbations of an \((n-1)\)-brane and its backreaction on the bulk. These two problems involve in an essential way knowing the first and second order perturbations of the induced metric and extrinsic curvature of the hypersurface.

A natural question that should be addressed to start off is whether going to second order in perturbation theory is necessary and interesting. Taking for granted that perturbation theory is useful and powerful for a wealth problems, the point is why second order. First of all, our present understanding of linear perturbations (methods involved, subtleties, applications, etc.) has reached a level of maturity that allows us to go to second order as a natural next step. More importantly, there are many situations where linear theory is not accurate enough and second order non-linear effects have to be considered. One fundamental example is studying inhomogeneities in our Universe (see \[1\], \[2\], \[3\], \[4\], \[5\], and references therein) and particularly structure formation (e.g. \[6\], \[7\]). Another example is anisotropies in the cosmic microwave background, where the present and planned sensitivities of the detectors is high enough so that second order effects can already be detected (or ruled out) (see \[8\], \[9\], \[10\] for some recent references).

Besides cosmology, second order perturbations are also fundamental in slowly rotating stars. Since the seminal work of Hartle \[11\] we know that stationary and axially symmetric (rotating) perturbations of static stars have a linear component which sets the star into rotation without modifying the shape of its boundary. Its effect to second order is to modify the shape of the star as well as to modify the rest of the metric components. Dynamical (non-stationary) perturbations of stars (both static or slowly rotating) also require going to second order (see \[12\] and references therein). Second order perturbations have also been applied to back holes, specially to study the close limit in the collision of back holes (\[13\], \[14\] and therein references). When compared with numerical work, the results of these calculations show that the range of validity of the perturbative regime is much broader than expected.

In brane scenario, perturbations are also relevant. One important aspect has to do with stability of branes, for which a proper understanding of how the intrinsic geometry of the branes behaves under perturbations is required. Small (first order) perturbations of a worldsheet evolving in a fixed flat background were studied in \[15\]. The generalization to curved background (still fixed) was done in \[16\], \[17\]. Including the backreaction of the bulk (i.e. first order perturbations of a brane in a perturbed bulk) was analyzed in \[18\], where in particular the linear perturbation of the induced metric and second fundamental forms of an arbitrary (non-null) brane were calculated. The calculations were carried out using exclu-
sively spacetime tensors. This approach is very efficient because there is no need to use two

different sets of indices (spacetime indices and intrinsic indices on the brane). Linear per-

turbations of branes have also been studied in the context of cosmological perturbations on

the brane (see [19] for reviews). Different approaches have been considered: the most widely

used involves calculations in explicit coordinate systems (see [20] and references therein),

another approach uses a covariant formalism [21] and a third method is the doubly gauge in-

variant formalism developed by Mukohyama [22]. The latter is geometrically very clear and

has the advantage that the gauge freedom in the bulk and on the brane are neatly separated.

As far as I know [22] and [18] are the only papers where general first order perturbations of

an arbitrary non-null hypersurface in an arbitrary spacetime are studied.

Second order calculations have also been considered for some highly symmetric bulks

and branes [23], [24], [25], [26]. So far, the only completely general calculation to second order

can be found in [27] where the Lagrangian density of the Dirac-Goto-Nambu brane was

calculated when both the bulk and of the worldsheet were perturbed to second order. The

main motivation in [27] was to find a quadratic Lagrangian for the first order perturbation

equations. The results in the present paper generalize those in [27] and extend the calculation

of the induced volume form (i.e. the Lagrangian) to the full first and second fundamental

forms. However, only codimension one branes are considered here, unlike [27] where any

codimension was allowed.

Gauge invariance is a subtle point in any geometrical perturbation theory. In the case

of hypersurfaces the complexity increases due to the presence of a moving boundary, and

this has led to some confusion in the literature. Going to second order complicates matters

both conceptually and in the calculations. Thus, it becomes necessary to state clearly the

theoretical framework defining the perturbations. In this paper I will use a geometrical

method that, in some sense, combines the approaches of [18] and [22]. As in [22], the

hypersurface and its perturbations will be defined by embeddings of an abstract manifold

into the spacetime (thus splitting the intrinsic objects from the ambient spacetime objects

in a clear way). This allows us to separate the two gauge freedom sources neatly. For

the explicit calculation of perturbations, it is much more economic to use the spacetime

formalism, as in [18]. The result is finally written down in terms of intrinsic tensors on the

unperturbed hypersurface, where all objects are naturally defined. The calculations leading

to second order perturbations are difficult. If a correct approach is not taken, the expressions

easily become very large and unmanageable. Thus, I will spend some effort explaining how

the calculations are performed. A basic ingredient is a Lemma which relates perturbations of

intrinsic objects to a hypersurface and perturbations of suitable objects in the ambient space.

This Lemma will be used not only to calculate the perturbations within the hypersurface but

also to analyze the second order gauge freedom. This approach to the gauge transformations

is complementary to that in [28], [29] where the so-called knight diffeomorphisms were used.

The calculations presented in this paper have many potential applications. One which I

consider very relevant is the matching of spacetimes. Constructing spacetimes from the jun-

ction of two regions across their boundary has been a very useful tool in gravitational theory.
The set of conditions ensuring that two spacetime regions can be joined are well understood (see e.g. [30] for a detailed account). The role of the discontinuity of the second fundamental form and its relationship with distributional parts in the energy-momentum tensor were clarified by Israel and have become known as Israel matching conditions [31]. Since perturbation theory has been useful for many problems, it is not surprising that perturbing spacetimes constructed from matching two regions is also of interest. Obvious examples are perturbed stars, voids in the universe (as in the Einstein-Straus cheese model [32]), shells of matter, impulsive waves, etc. In this paper I will present, as a by product, the first and second order perturbed matching conditions for non-null hypersurfaces.

The paper is intended to be self-contained, so that all the subtleties and difficulties of second order calculations on hypersurfaces can be properly understood. I will, of course, give credit to previous results whenever necessary. The paper is organized as follows. In section 2, a brief summary of perturbation theory is given. This fixes our framework and notation. In section 3, the Lemma mentioned above will be stated and proven. This result will be used throughout the paper. A consequence of this Lemma is that second order perturbations of hypersurfaces can be described by two vector fields defined on the unperturbed hypersurface. Their explicit form in an arbitrary coordinate system is discussed in Section 4. Section 5 deals with first order perturbations of the fundamental forms. Here the results of [18] and [22] mentioned above will be recovered. In this section some useful Lemmas to carry out the second order calculations are presented. The result of the second order perturbations is stated in Section 6, leaving the details of the proof to Appendices A and B. Section 7 discusses first and second order gauge transformations. For hypersurfaces there are two types of gauge freedom, namely the one coming from the ambient spacetime and another one intrinsic to the hypersurface. Both are discussed in this section. Section 8 applies the previous results to the perturbed matching conditions between two spacetimes. A theorem giving the necessary and sufficient conditions for second order perturbations to match across a matching hypersurface in the background is presented. This theorem can be potentially applied to many situations. One case that has already been investigated involves first and second order stationary and axially symmetric perturbations of spherical stars [33]. The two Appendices contain the main steps in the calculations of the second order perturbations. The reader who is not interested in detailed calculations may skip the Appendixes and concentrate on the main text.

2 Summary of perturbation theory

Perturbation theory deals with one parameter families of spacetimes\(^1\) \((\mathcal{M}_\epsilon, \hat{g}_\epsilon)\) and their first and higher order variations around one element of the family, say \((\mathcal{M}_0, g_0)\). In order

\(^1\)A \(C^m\) spacetime is a Hausdorff, connected \(C^{m+1}\) manifold of dimension \(n\) endowed with a Lorentzian metric of class \(C^m\). If we are considering a manifold-with-boundary then the boundary \(\partial \mathcal{M}\) is also assumed to be \(C^{m+1}\). Our signature and sign conventions for the Riemann and Ricci tensors follow [34].
to take derivatives with respect to $\epsilon$ we need some fixed set of points, i.e. we need all $M_\epsilon$ to be diffeomorphic to $M_0$. Through these diffeomorphisms we can define a one-parameter family of metrics $g_\epsilon$ on $M_0$ associated to $\hat{g}_\epsilon$. We denote $M_0$ simply as $M$. Since we want to take second variations of extrinsic curvatures we take $M$ to be $C^4$, as a manifold. We thus consider a differentiable family of Lorentzian metrics on $M$, i.e. a $C^2$ map

$$T : I \rightarrow \mathcal{G}(M)$$

$$\epsilon \rightarrow g_\epsilon$$

where $\mathcal{G}(M)$ denotes the set of $C^3$ symmetric, non-degenerate, two-index tensor fields in $M$ ($\mathcal{G}(M)$ can be endowed with a natural differential structure, $T$ is $C^2$ with respect to this structure).

We denote by $\nabla^\epsilon$ the Levi-Civita covariant derivative of the metric $g_\epsilon$. Given two arbitrary metrics $g$ and $\overline{g}$, the corresponding covariant derivatives $\nabla$ and $\overline{\nabla}$ are well-known [34] to be related by

$$\nabla^\epsilon_{\mu} T^\alpha_{\beta} = \nabla_{\mu} T^\alpha_{\beta} + C^\nu_{\beta \mu} T^\alpha_{\nu} - C^\alpha_{\nu \mu} T^\nu_{\beta},$$

(1)

where $C^\alpha_{\beta \mu} = (1/2)\overline{g}^{\nu \mu}(\nabla_{\beta} \overline{g}_{\mu \gamma} + \nabla_{\gamma} \overline{g}_{\mu \beta} - \nabla_{\mu} \overline{g}_{\beta \gamma})$. Similar expressions hold for tensors with any number of indices. Substituting $g$ by $g_\epsilon = 0$ and $\overline{g}$ by $g_\epsilon$, we can take $\epsilon$-derivatives in (1) at $\epsilon = 0$ to get

$$\left. \frac{(d\nabla^\epsilon_{\mu} T^\alpha_{\beta})}{d\epsilon} \right|_{\epsilon=0} = S^\nu_{\beta \mu} T^\alpha_{\nu} - S^\alpha_{\nu \mu} T^\nu_{\beta},$$

(2)

where we have defined

$$K'_{\alpha \beta} \equiv \frac{dg_{\alpha \beta}}{d\epsilon} \bigg|_{\epsilon=0}, \quad S^\alpha_{\beta \gamma} \equiv \frac{1}{2} \left( \nabla_{\beta} K'^{\alpha}_{\gamma} + \nabla_{\gamma} K'^{\alpha}_{\beta} - \nabla^{\alpha} K'_{\beta \gamma} \right).$$

(3)

Here and in the following we set $g_0 \to g$ and $\nabla^0 \to \nabla$. The tensor $K'$ is the first order perturbation of $g$ along the family $g_\epsilon$. We will simply call it first order perturbed metric. Similar expressions as (2) hold for tensors with any number of indices. The second derivatives of (1) gives

$$\left. \frac{(d^2\nabla^\epsilon_{\mu} T^\alpha_{\beta})}{d\epsilon^2} \right|_{\epsilon=0} = \left( S^{\nu \mu}_{\beta \rho} + 2K'^{\nu \rho}_{\beta \rho} S^\delta_{\beta \rho} \right) T^\alpha_{\nu} - \left( S^{\mu \alpha}_{\nu \mu} + 2K'^{\mu \alpha}_{\nu \mu} S^\delta_{\nu \mu} \right) T^\nu_{\beta},$$

where $K''_{\alpha \beta} \equiv \frac{d^2g_{\alpha \beta}}{d\epsilon^2} \bigg|_{\epsilon=0}$ is the second order perturbed metric and

$$S^{\mu \alpha}_{\beta \gamma} \equiv \frac{1}{2} \left( \nabla_{\beta} K''^{\mu \alpha}_{\gamma} + \nabla_{\gamma} K''^{\mu \alpha}_{\beta} - \nabla^{\alpha} K''_{\beta \gamma} \right).$$

(4)
3 A useful Lemma

We want to calculate how the first and second fundamental forms of a hypersurface change when the ambient metric and the hypersurface are perturbed. Thus, we consider a one-parameter family of hypersurfaces $\Sigma_\epsilon$ of $\mathcal{M}$. As before, in order to define variations we need a fixed set of points, so we assume all $\Sigma_\epsilon$ to be diffeomorphic to each other. We allow the hypersurfaces to change both as a subset of points of $\mathcal{M}$ and also in the way we coordinate them for different $\epsilon$. Since the fundamental forms are pull-backs of tensors on $\mathcal{M}$, their dependence on $\epsilon$ arise because of three facts: (i) because the ambient metric depends on $\epsilon$, (ii) because the hypersurface $\Sigma_\epsilon$ considered as a subset of $\mathcal{M}$ changes with $\epsilon$ and (iii) because the way in which we coordinate $\Sigma_\epsilon$ is allowed to depend on $\epsilon$ (even if the hypersurface as a set of points remains unchanged). Points (ii) and (iii) can be treated together by viewing the hypersurfaces $\Sigma_\epsilon$ as embedded hypersurfaces, i.e. as the images of a family of embeddings $\Phi_\epsilon : \Sigma \rightarrow \mathcal{M}$, where $\Sigma$ is a copy of any of the $\Sigma_\epsilon$, say $\Sigma_0$. It is useful to view $\Sigma$ as an abstract manifold detached from the spacetime so that one knows clearly where the different objects are defined. Thus, we shall distinguish between $\Sigma$ as an $(n-1)$-dimensional manifold and $\Sigma_0 = \Phi_0(\Sigma)$, which is the hypersurface in $\mathcal{M}$.

The fact that the fundamental forms on $\Sigma$ depend on $\epsilon$ from several sources and that we want to do the calculation up to second order makes it very important to use a method as covariant as possible and use coordinates only when absolutely necessary. Moreover, it is very convenient to work on the ambient manifold as much as possible and perform the pull-back only at the very end (in agreement with [18]). The alternative of calculating the derivative directly on $\Sigma$ is, of course, possible but much more difficult. In this section I present a Lemma which shows how derivatives of geometric tensors on an embedded submanifold (or arbitrary codimension, including codimension 0) can be calculated from derivatives performed purely on the ambient manifold. This result will be crucial for the calculations in the following section. Throughout this section all differentiable objects are $C^3$ unless otherwise specified.

Thus, let $\mathcal{N}$ and $\mathcal{M}$ be two differentiable manifolds and let $\chi_\epsilon : \mathcal{N} \rightarrow \mathcal{M}$ be a family of differentiable maps. Let us consider a $C^2$ family of covariant tensor fields $T_\epsilon$ on $\mathcal{M}$. We can pull-back this family to $\mathcal{N}$ and define a one-parameter family of tensors $T_\epsilon$ on $\mathcal{N}$. We are interested in determining the first and second derivatives of $T_\epsilon$ with respect to $\epsilon$. Using directly the definition of derivative,

$$\frac{dT_\epsilon}{d\epsilon} = \lim_{h \to 0} \frac{\chi_\epsilon+h(T_\epsilon+h) - \chi_\epsilon(T_\epsilon)}{h} = \lim_{h \to 0} \frac{\chi_\epsilon+h(T_\epsilon+h) - \chi_\epsilon(T_\epsilon+h)}{h} + \chi_\epsilon \left( \lim_{h \to 0} \frac{T_\epsilon+h - T_\epsilon}{h} \right),$$  

(5)

where we have added and subtracted $\chi_\epsilon(T_\epsilon+h)$ in the numerator and we have used the linearity of $\chi_\epsilon$. The second term is the pull-back of the derivative of $T_\epsilon$. The first term cannot be written directly in a simple form because there is no a priori relationship between $\chi_\epsilon+h$ and $\chi_\epsilon$ (contrarily to what happens for instance in a one-parameter group of diffeomorphisms). Assume now that $\chi_\epsilon$ are embeddings. Then, there exists a set of diffeomorphisms $\Psi_\epsilon : \mathcal{M} \rightarrow \mathcal{N}$.
\( \mathcal{M} \) of the ambient space such that the diagram

\[
\begin{array}{c}
\mathcal{N} \\
\downarrow \chi_{\epsilon} \downarrow \chi_{h+\epsilon} \\
\mathcal{M} \xrightarrow{\Psi_{h}^{\epsilon}} \mathcal{M}
\end{array}
\]

is commutative. We moreover fix \( \Psi_{0}^{\epsilon} = \mathbb{I}_{\mathcal{M}} \) (the identity on \( \mathcal{M} \)) for all \( \epsilon \). Geometrically, \( \Psi_{h}^{\epsilon} \) transforms the point \( \chi_{\epsilon}(p) \) into the point \( \chi_{\epsilon+h}(p) \) for all \( p \in \mathcal{N} \). Notice that \( \Psi_{h}^{\epsilon} \) is fixed only on \( \chi_{\epsilon}(\mathcal{N}) \) and therefore it is non-unique in general. However, when \( \mathcal{N} \) has the same dimension as \( \mathcal{M} \) and \( \chi_{\epsilon} \) are diffeomorphisms, then \( \Psi_{h}^{\epsilon} \) is unique and given by \( \chi_{\epsilon+h} \circ \chi_{\epsilon}^{-1} \).

The following Lemma gives explicit expressions for the first and second \( \epsilon \)-derivatives of \( T_{\epsilon} \) in terms of objects defined in the ambient space \( \mathcal{M} \).

**Lemma 1** Let \( T_{\epsilon} \) be a \( C^{2} \) one-parameter family of covariant tensor fields on \( \mathcal{M} \), \( \chi_{\epsilon} : \mathcal{N} \rightarrow \mathcal{M} \) a \( C^{2} \) family of embeddings and define \( T_{\epsilon} = \chi_{\epsilon}^{\ast}(T_{\epsilon}) \). Then

\[
\frac{dT_{\epsilon}}{d\epsilon} = \chi_{\epsilon}^{\ast} \left( \mathcal{L}_{\bar{Z}_{\epsilon}} T_{\epsilon} + \frac{dT_{\epsilon}}{d\epsilon} \right),
\]

and

\[
\frac{d^{2}T_{\epsilon}}{d\epsilon^{2}} = \chi_{\epsilon}^{\ast} \left( \mathcal{L}_{\bar{Z}_{\epsilon}} T_{\epsilon} + \mathcal{L}_{\bar{Z}_{\epsilon}} T_{\epsilon} + 2 \mathcal{L}_{\bar{Z}_{\epsilon}} \left( \frac{dT_{\epsilon}}{d\epsilon} + \frac{d^{2}T_{\epsilon}}{d\epsilon^{2}} \right) \right),
\]

where

\[
\bar{Z}_{\epsilon} = \frac{\partial \Psi_{h}^{\epsilon}}{\partial h} \bigg|_{h=0}, \quad \bar{W}_{\epsilon} = \frac{d\bar{Z}_{\epsilon}}{d\epsilon}
\]

and \( \Psi_{h}^{\epsilon} \) is any set of diffeomorphisms of \( \mathcal{M} \) which makes the diagram \( (\mathcal{A}) \) commutative.

**Proof:** The commutativity of the diagram \( (\mathcal{A}) \) implies that the first term in \( (\mathcal{A}) \) can be written as

\[
\lim_{h \to 0} \frac{\chi_{\epsilon+h}^{\ast}(T_{\epsilon+h}) - \chi_{\epsilon}^{\ast}(T_{\epsilon+h})}{h} = \chi_{\epsilon}^{\ast} \left( \lim_{h \to 0} \frac{\Psi_{h}^{\epsilon\ast}(T_{\epsilon+h}) - T_{\epsilon+h}}{h} \right) = \chi_{\epsilon}^{\ast} \left( \mathcal{L}_{\bar{Z}_{\epsilon}} T_{\epsilon} \right),
\]

where the definition of Lie derivative has been used. This proves \( (7) \). For the second derivative we apply this expression twice

\[
\frac{d^{2}T_{\epsilon}}{d\epsilon^{2}} = \chi_{\epsilon}^{\ast} \left( \left( \mathcal{L}_{\bar{Z}_{\epsilon}} + \frac{d}{d\epsilon} \right) \left( \mathcal{L}_{\bar{Z}_{\epsilon}} + \frac{d}{d\epsilon} \right) T_{\epsilon} \right) =
\]

\[
\chi_{\epsilon}^{\ast} \left( \mathcal{L}_{\bar{Z}_{\epsilon}} \mathcal{L}_{\bar{Z}_{\epsilon}} T_{\epsilon} + \mathcal{L}_{\bar{Z}_{\epsilon}} \left( \frac{dT_{\epsilon}}{d\epsilon} \right) + \frac{d}{d\epsilon} \left( \mathcal{L}_{\bar{Z}_{\epsilon}} T_{\epsilon} \right) + \frac{d^{2}T_{\epsilon}}{d\epsilon^{2}} \right) \quad \text{(10)}
\]
Only the third term needs to be elaborated. Using again the definition of derivative and adding and subtracting a suitable term we get
\[
\frac{d}{d\epsilon}(\mathcal{L}_{\mathbf{Z}} T_\epsilon) = \lim_{h \to 0} \frac{\mathcal{L}_{\mathbf{Z}_{\epsilon+h}} T_{\epsilon+h} - \mathcal{L}_{\mathbf{Z}_\epsilon} T_{\epsilon+h}}{h} + \lim_{h \to 0} \frac{\mathcal{L}_{\mathbf{Z}_\epsilon} T_{\epsilon+h} - \mathcal{L}_{\mathbf{Z}_\epsilon} T_{\epsilon}}{h},
\]
which, using the linearity of the Lie derivative, becomes
\[
\frac{d}{d\epsilon}(\mathcal{L}_{\mathbf{Z}} T_\epsilon) = \mathcal{L}_{\mathcal{Z}} \frac{d}{d\epsilon} T_\epsilon + \mathcal{L}_{\mathbf{Z}_\epsilon} \left( \frac{dT_\epsilon}{d\epsilon} \right).
\]
Inserting this into (10) and using the definition \(\mathbf{W}_\epsilon = \frac{d\mathbf{Z}_\epsilon}{d\epsilon}\) the lemma follows. □

**Remark.** Notice that when \(\mathcal{N}\) is a hypersurface \(\Sigma\), all the information regarding the first and second variation of the hypersurface \(\Sigma_\epsilon\) around \(\Sigma_0\) is encoded in the two vector fields \(\mathbf{Z}_1 \equiv \mathbf{Z}_{\epsilon=0}\) and \(\mathbf{W} \equiv W_{\epsilon=0}\). These vectors are, by construction, defined everywhere on \(\mathcal{M}\) (because they are defined in terms of the diffeomorphisms \(\Psi^\epsilon_h\)). However, only their values on \(\Sigma_0\) should matter. Geometrically, they define how the hypersurface is deformed to first and second order (as we shall see, the second order variation is best defined by using a suitable combination of \(\mathbf{W}\) and \(\mathbf{Z}_1\)). These vectors have tangential and normal components. The normal part determines how the hypersurface moves in spacetime as a set of points, while the tangential part encodes the information on how the different \(\Sigma_\epsilon\) are coordinated. The fact that these vectors have been extended off \(\Sigma_0\), and that this extension is essentially arbitrary (due to the large freedom in defining \(\Psi^\epsilon_h\)) provides a powerful check for the validity of the final results, namely that they must be independent of the extension of these vectors outside \(\Sigma_0\).

### 4 First and second order perturbations vectors of \(\Sigma\)

Let us now concentrate in the case where \(\mathcal{N}\) is a hypersurface \(\Sigma\). We replace \(\chi_\epsilon \to \Phi_\epsilon\) in all the expressions above. Our aim in this section is to find explicit expressions for the vectors \(Z_1 \equiv \left. \frac{\partial \psi^i}{\partial h} \right|_{\epsilon=h=0}\) and \(\mathbf{W} \equiv \left. \frac{d\mathbf{Z}_\epsilon}{d\epsilon} \right|_{\epsilon=0}\), so that they can be determined in explicit examples. For the second variation we use \(\mathbf{Z}_2 \equiv \mathbf{W} + \nabla_{\mathbf{Z}_1} \mathbf{Z}_1\) instead of \(\mathbf{W}\). The reason will become clear later on. We call \(\mathbf{Z}_1\) and \(\mathbf{Z}_2\) respectively as *first and second order perturbation vectors of \(\Sigma\)*. Let us choose local coordinate systems on \(\Sigma\) and on \(\mathcal{M}\) so that \(\Phi_\epsilon\) are written
\[
\Phi_\epsilon : \quad \Sigma \quad \longrightarrow \quad \mathcal{M},
\]
\[
y^i \quad \longrightarrow \quad x^\alpha = \Phi^\alpha(y^i, \epsilon).
\]
Proposition 1 The first and second order perturbation vectors $Z_1^\alpha(y)$ and $Z_2^\alpha(y)$ of the hypersurface $\Sigma$ read

\[
Z_1^\alpha(y^i) = \frac{\partial \Phi^\alpha(y^i, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0},
\]

\[
Z_2^\alpha(y^i) = \frac{\partial^2 \Phi^\alpha(y^i, \epsilon)}{\partial \epsilon^2} \bigg|_{\epsilon=0} + \Gamma^\alpha_{\beta\gamma}(x_0(y^i))Z_1^\beta(y^i)Z_1^\gamma(y^i).
\]

where $x_0(y^i)$ is the local form of the unperturbed embedding $\Phi_0$.

Proof: $\Psi^\epsilon : M \to M$ satisfies $\Phi_{\epsilon+h} = \Psi^\epsilon \circ \Phi_\epsilon$. Let $\Psi^\alpha(x^\alpha, h, \epsilon)$ be the local coordinate form of $\Psi^\epsilon$. By construction, $\Phi^\alpha$ and $\Psi^\alpha$ satisfy

\[
\Phi^\alpha(y^i, \epsilon + h) = \Psi^\alpha(\Phi^\beta(y^i), h, \epsilon),
\]

which has as immediate consequence

\[
\frac{\partial \Psi^\alpha}{\partial h} \bigg|_{(\Phi(y^i, \epsilon), h, \epsilon)} = \frac{\partial \Psi^\alpha}{\partial x^\beta} \bigg|_{(\Phi(y^i, \epsilon), h, \epsilon)} \frac{\partial \Phi^\beta}{\partial \epsilon} \bigg|_{(y^i, \epsilon)} + \frac{\partial \Psi^\alpha}{\partial \epsilon} \bigg|_{(\Phi(y^i, \epsilon), h, \epsilon)}.
\]

Evaluating at $\epsilon = h = 0$ and using the fact that $\Psi^\epsilon_0$ is the identity on $M$ for all $\epsilon$ we obtain,

\[
\frac{\partial \Psi^\alpha}{\partial h} \bigg|_{(x^\alpha=x_0^\alpha(y^i), h=0, \epsilon=0)} = \frac{\partial \Phi^\alpha(y^i, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0}.
\]

According to its definition, $\tilde{Z}_\epsilon$ has components $Z_{\epsilon}^\alpha(x) = \frac{\partial \Psi^\alpha(x^\beta, h, \epsilon)}{\partial h} \bigg|_{h=0}$. Expression (11) follows directly from (13). For the second order perturbation vector $\tilde{Z}_2$, let us first find the coordinate form of $\partial_\epsilon \tilde{Z}_\epsilon + \nabla_{\tilde{Z}_\epsilon} \tilde{Z}_\epsilon$. Directly from its definition one finds

\[
\partial_\epsilon Z_{\epsilon}^\alpha(x) + \nabla_{\tilde{Z}_\epsilon} Z_{\epsilon}^\alpha(x) \bigg|_{\epsilon=0} = \frac{\partial^2 \Psi^\alpha}{\partial h \partial \epsilon} \bigg|_{(x,0,0)} + \frac{\partial \Phi^\beta}{\partial h} \bigg|_{(x,0,0)} \frac{\partial^2 \Psi^\alpha}{\partial x^\beta \partial h} \bigg|_{(x,0,0)} + \Gamma^\beta_{\beta\gamma}(x) \frac{\partial \Psi^\beta}{\partial h} \bigg|_{(x,0,0)} \frac{\partial \Psi^\gamma}{\partial h} \bigg|_{(x,0,0)},
\]

which contains no second derivatives with respect to $h$. Performing the second $\epsilon, h$ derivative of (13) and evaluating at $\epsilon = h = 0$ gives

\[
\frac{\partial^2 \Phi^\alpha(y^i, \epsilon + h)}{\partial \epsilon \partial h} \bigg|_{\epsilon=h=0} = \frac{\partial^2 \Psi^\alpha}{\partial x^\beta \partial h} \bigg|_{(x=x_0(y^i), 0, 0)} \frac{\partial \Phi^\beta(y^i, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} + \frac{\partial^2 \Psi^\alpha}{\partial h \partial \epsilon} \bigg|_{(x=x_0(y^i), 0, 0)}
\]

Taking into account $\partial_\epsilon \partial_\epsilon \Phi^\alpha(y^i, \epsilon + h) \big|_{(\epsilon=0, h=0)} = \partial^2 \Phi^\alpha(y^i, \epsilon + h) \big|_{(\epsilon=0, h=0)}$, and noticing that $\partial_\epsilon^2 \Phi^\alpha(y^i, \epsilon + h) \big|_{\epsilon=h=0} = \partial_\epsilon \partial_\epsilon \Phi^\alpha(y^i, \epsilon) \big|_{\epsilon=0}$, the vector field $\partial_\epsilon Z_{\epsilon}^\alpha + \nabla_{\tilde{Z}_\epsilon} Z_{\epsilon}^\alpha$ evaluated on $\Sigma$ and at $\epsilon = 0$ becomes

\[
\partial_\epsilon Z_{\epsilon}^\alpha + \nabla_{\tilde{Z}_\epsilon} Z_{\epsilon}^\alpha \bigg|_{(x=x_0(y^i), \epsilon=0)} = \frac{\partial^2 \Phi^\alpha(y^i, \epsilon)}{\partial \epsilon^2} \bigg|_{\epsilon=0} + \Gamma^\alpha_{\beta\gamma}(x_0(y^i))Z_1^\beta(y^i)Z_1^\gamma(y^i).
\]
and (12) follows directly from its definition $Z_2^\alpha(y^i) = \partial_\epsilon Z_\epsilon^\alpha \big|_{(x=x_0(y^i), \epsilon=0)}$.

**Remark:** Both the first and second order perturbation vectors $\vec{Z}_1$ and $\vec{Z}_2$ depend only on the family of embeddings $\Phi_\epsilon$ (i.e. the hypersurfaces $\Sigma_\epsilon$) and not on the specific choice of $\Psi_\epsilon^h$. Moreover, it is clear from (12) that $Z_2^\alpha(y^i)$ corresponds to the covariant acceleration, evaluated on $\Sigma_0$, of the curve defined by the motion of a fixed point of $\Sigma$ when the hypersurface moves, i.e. by the curve $\Phi^\alpha(y^i, \epsilon)$ with $y^i$ fixed. Notice also that, had we chosen $\vec{W} = \partial_\epsilon Z_\epsilon \big|_{\epsilon=0}$ as our second order perturbation vector, we would have obtained a vector field which depends on $\Psi_\epsilon^h$, i.e. it would not be defined solely in terms of the one parameter family of embeddings. This is why $\vec{Z}_2$ is preferable to $\vec{W}$.

## 5 First order perturbation of the hypersurface

We can now start the calculations of the first and second order variations of the fundamental forms of $\Sigma_0$. We shall assume that this hypersurface contains no null points, i.e. that its induced first fundamental form $h = \Phi_0^\alpha(g)$ defines a metric. For small enough $\epsilon$ the same will be true for $\Sigma_\epsilon$ at least on compact subsets. Since we are only interested in derivatives at $\epsilon = 0$ we can assume without loss of generality that all $\Sigma_\epsilon$ are non-degenerate. Let us denote by $h_\epsilon = \Phi_\epsilon^*(g)$ the one-parameter family of induced metrics. Notice that all of them are defined on the same manifold $\Sigma$. Let also $\vec{n}_\epsilon$ be the unit normal to $\Sigma_\epsilon$ with respect to $g_\epsilon$. Its orientation is taken arbitrarily on $\Sigma_0$ and extended to all $\epsilon$ by continuity. Let us extend $\vec{n}_\epsilon$ to an open neighbourhood $\mathcal{U}$ of $\Sigma_\epsilon$. By working locally near one point we can, without loss of generality, choose $\mathcal{U}$ to be independent of $\epsilon$. We keep $\vec{n}_\epsilon$ unit everywhere on $\mathcal{U}$, i.e. $g_{\alpha\beta}n^\alpha_\epsilon n^\beta_\epsilon|\mathcal{U} = \sigma$, where $\sigma = +1$ for timelike hypersurfaces and $\sigma = -1$ for spacelike ones. Hence, we have at hand a one-parameter family of one-forms $\vec{n}_\epsilon$ defined everywhere on $\mathcal{U}$ and both covariant derivatives at constant $\epsilon$ and $\epsilon$-derivatives at fixed spacetime point $x \in \mathcal{U}$ can be performed. $\Sigma$ inherits a one-parameter family of second fundamental forms $\kappa_\epsilon = \Phi_\epsilon^*(\nabla^* \vec{n}_\epsilon)$. We drop the subindex 0 for any background object, thus $\Sigma_0$ is endowed with a metric $h$, covariant derivative $D$, second fundamental form $\kappa$ and has unit normal $\vec{n}$. We also write the background embedding simply as $\Phi$. All spacetime indices are lowered and raised with the background metric $g_{\alpha\beta}$ and it inverse. Similarly all hypersurface indices are lowered and raised with $h_{ij}$ and its inverse. Tensors on $\Sigma$ will carry Latin indices.

The first and second order perturbations of the induced metric and second fundamental forms are obviously

First order perturbations:

$$h' = \left. \frac{\partial h_\epsilon}{\partial \epsilon} \right|_{\epsilon=0}, \quad \kappa' = \left. \frac{\partial \kappa_\epsilon}{\partial \epsilon} \right|_{\epsilon=0},$$

Second order perturbations:

$$h'' = \left. \frac{\partial^2 h_\epsilon}{\partial \epsilon^2} \right|_{\epsilon=0}, \quad \kappa'' = \left. \frac{\partial^2 \kappa_\epsilon}{\partial \epsilon^2} \right|_{\epsilon=0},$$

---

2In this paper boldface letters are used to denote one-forms.
These derivatives are taken at fixed point \( p \) in the abstract manifold \( \Sigma \). In this section we shall obtain the explicit expressions for the first order perturbed quantities. The second order quantities are considered in the next section.

Let us start by introducing some notation. It is well known that covariant tensors on the background hypersurface \( \Sigma \) are in one to one correspondence with spacetime tensors defined on \( \Sigma_0 \) and which are totally tangent to \( \Sigma_0 \), i.e. tensors \( C_{\mu_1 \cdots \mu_m} \) satisfying \( C_{\mu_1 \cdots \mu_m} n^\mu_a = 0 \) \((a = 1 \cdots m)\) on \( \Sigma_0 \). The one-form \( n \) is obviously hypersurface orthogonal on \( \Sigma_0 \). We can assume without loss of generality that its extension off \( \Sigma_0 \) is chosen so that this property is kept. We could also choose \( \vec{n} \) so that it defines a geodesic affinely parametrized congruence. These two conditions would fix \( \vec{n} \) uniquely and would simplify the calculations below. However we prefer to leave the acceleration \( \vec{a} = \nabla \vec{n} \vec{n} \) completely free. The increase in complexity is compensated by the fact that the result has to be independent of \( \vec{a} \). This provides a non-trivial check for the validity of the result. Since, as we shall see, the calculation is quite involved, it is convenient to keep non-trivial checks at hand.

Being \( n \) hypersurface orthogonal, its covariant derivative reads
\[
\nabla_\alpha n_\beta = \sigma n_\alpha a_\beta + \kappa_{\alpha \beta},
\]
where \( \kappa_{\alpha \beta} \) is symmetric and completely orthogonal to \( \vec{n} \). This tensor is obviously the counterpart on \( M \) of the second fundamental form \( \kappa_{ij} \). From (18) it follows
\[
\bar{\mathcal{L}}_{\vec{n}} g_{\alpha \beta} = \sigma n_\alpha a_\beta + \sigma n_\beta a_\alpha + 2 \kappa_{\alpha \beta}.
\]

Covariant derivatives of a tensor \( C_{i_1 \cdots i_m} \) within the hypersurface can be calculated by considering its counterpart \( C_{\mu_1 \cdots \mu_m} \) on spacetime. Indeed, if we extend this tensor to a neighbourhood of \( \Sigma_0 \) in such a way that it remains orthogonal to \( \vec{n} \) (and otherwise arbitrarily), the three dimensional covariant derivative can be calculated as
\[
D_\alpha C_{\mu_1 \cdots \mu_m} \equiv h_\alpha^\nu h_{\mu_1}^\beta \cdots h_{\mu_m}^\beta \nabla_\nu C_{\beta_1 \cdots \beta_m},
\]
where \( h_\alpha^\beta \equiv \delta_\alpha^\beta - \sigma n^\alpha n_\beta \) is the projector to the hypersurface. More concretely, this means that \( D_\beta C_{i_1 \cdots i_m} \) has \( D_\alpha C_{\mu_1 \cdots \mu_m} \) as its spacetime counterpart. A simple integration by parts shows that covariant derivatives and three-dimensional derivatives are related by
\[
\nabla_\alpha C_{\mu_1 \cdots \mu_m} = D_\alpha C_{\mu_1 \cdots \mu_m} + \sigma n_\alpha n^\rho \nabla_\rho C_{\mu_1 \cdots \mu_m} - \sigma \sum_{i=1}^n C_{\mu_1 \cdots \rho \cdots \mu_m} \kappa^\rho_{\alpha} n_\mu_i.
\]

For later use let us notice some useful expressions. The first one is obvious: for any covariant tensor \( A_{\alpha_1 \cdots \alpha_m} \) and any function \( F \)
\[
(\mathcal{L}_F \bar{n} A)_{\alpha_1 \cdots \alpha_m} = F (\mathcal{L}_n A)_{\alpha_1 \cdots \alpha_m} + \sum_{i=1}^n A_{\alpha_1 \cdots \mu \cdots \alpha_m} n^\mu \nabla_{\alpha_i} F.
\]

Less immediate, but still easy, are the following three Lemmas. The first one is well-known
Lemma 2 Let $\Sigma$ be an embedded submanifold of $\mathcal{M}$ with embedding $\Phi : \Sigma \rightarrow \mathcal{M}$ and let $\vec{V}$ be a vector field on a neighbourhood $\mathcal{U}$ of $\Phi(\Sigma)$ tangent to this hypersurface (i.e. $\vec{V}|_{\Phi(\Sigma)} = \Phi_*(\vec{V}_\Sigma)$ for some vector $\vec{V}_\Sigma$ on $\Sigma$). Then, for any covariant tensor $A$ on $\mathcal{U}$

$$\Phi^*(\mathcal{L}_{\vec{V}} A) = \mathcal{L}_{\vec{V}_\Sigma}(\Phi^* A).$$

Remark. For simplicity we will use the same symbol to denote $\vec{V}_\Sigma$ and $\vec{V}$. The precise meaning will be clear from the context.

A consequence of this Lemma is that, for any vector field $\vec{X} = R\vec{n} + \vec{V}$, with $\vec{V}$ orthogonal to $\vec{n}$,

$$\Phi^*(\mathcal{L}_{\vec{X}} g) = \mathcal{L}_{\vec{V}} h + 2R \kappa,$$

(21)

Lemma 3 Let $B_{\alpha \beta}$ be any symmetric tensor and $\vec{X}$ any vector field. Defining $S(B)^\mu_{\alpha \beta} \equiv \frac{1}{2}(\nabla_\alpha B^\mu_{\beta} + \nabla_\beta B^\mu_{\alpha} - \nabla^\mu B_{\alpha \beta})$ and $H^\alpha = B^\alpha \mu X_\mu$. The following identity holds

$$(\mathcal{L}_{\vec{X}} B)_{\alpha \beta} + 2X_\mu S(B)^\mu_{\alpha \beta} = (\mathcal{L}_{\vec{V}} g)_{\alpha \beta}.$$ 

The next Lemma tells us how to perform second Lie derivatives. The Riemann tensor of $(\mathcal{M}, g)$ is denoted by $R_{\alpha \beta \gamma \delta}$.

Lemma 4 Let $\vec{X}$ be an arbitrary vector field and $B_{\alpha \beta}$ any symmetric tensor. Then

$$(\mathcal{L}_{\vec{X}} \mathcal{L}_{\vec{X}} B)_{\alpha \beta} = \left(\mathcal{L}_{\nabla_X \vec{X}} B\right)_{\alpha \beta} + X^\mu X^\nu \nabla_\mu \nabla_\nu B_{\alpha \beta} - B_{\alpha \nu} X^\mu X^\gamma R^\nu_{\gamma \beta \mu} - B_{\beta \nu} X^\mu X^\gamma R^\nu_{\gamma \alpha \mu} + 2(X^\mu \nabla_\mu B_{\alpha \nu}) \nabla_\beta X^\nu + 2(X^\mu \nabla_\mu B_{\beta \nu}) \nabla_\alpha X^\nu + 2B_{\mu \nu} (\nabla_\alpha X^\mu) (\nabla_\beta X^\nu),$$

Proof: Expand the first term and use the Ricci identity applied to $\vec{X}$. □

A particular case of this Lemma is obtained for $B_{\alpha \beta} = g_{\alpha \beta}$:

$$(\mathcal{L}_{\vec{X}} \mathcal{L}_{\vec{X}} g)_{\alpha \beta} = (\mathcal{L}_{\nabla_X \vec{X}} g)_{\alpha \beta} - 2X^\mu X^\nu R_{\alpha \mu \beta \nu} + 2(\nabla_\alpha X^\mu) (\nabla_\beta X_\mu).$$

(22)

Combining this with the general expression (20) it follows easily that, for any pair of functions $F_1$ and $F_2$,

$$\mathcal{L}_{F_1 \vec{n}} \mathcal{L}_{F_2 \vec{n}} g_{\alpha \beta} = \mathcal{L}_{F_1 F_2 \vec{n}} g_{\alpha \beta} + 2F_1 F_2 \left(-n^\mu n^\nu R_{\alpha \mu \beta \nu} + \kappa_{\alpha \mu} \kappa_{\beta \nu}^\mu \right) + \sigma(\nabla_\alpha F_1 \nabla_\beta F_2 + \nabla_\alpha F_2 \nabla_\beta F_1) + 2\kappa_{\alpha \beta} F_1 \vec{n}(F_2) + n_\alpha G_\beta + n_\beta G_\alpha,$$

(23)

where $G_\alpha = \sigma a_\alpha F_1 \vec{n}(F_2) + \mathcal{L}_{F_1 \vec{n}}(\mathcal{L}_{\alpha} F_2) + 2\sigma F_1 F_2 a^\mu \kappa_{\mu \alpha} + F_1 F_2 n_\alpha a^\mu a_\mu$. With $F_1 \rightarrow Q_1$ and $F_2 \rightarrow 1$ we get

$$\mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{n}} g_{\alpha \beta} = \mathcal{L}_{Q_1 \vec{n}} g_{\alpha \beta} + 2Q_1 \left(-n^\mu n^\nu R_{\alpha \mu \beta \nu} + \kappa_{\alpha \mu} \kappa_{\beta \nu}^\mu + n_\alpha n_\beta a^\mu a_\mu + \sigma n_\alpha a^\mu \kappa_{\beta \mu} + \sigma n_\beta a^\mu \kappa_{\alpha \mu}\right) (24)$$

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It is convenient to decompose the first perturbed metric into tangential and normal components with respect to \( \vec{n} \). Explicitly

\[
K'_{\alpha\beta} = Y' n_\alpha n_\beta + \sigma n_\alpha \tau'_{\beta} + \sigma n_\beta \tau'_{\alpha} + K'^t_{\alpha\beta},
\]

where obviously \( \tau'_{\alpha} \) and \( K'^t_{\alpha\beta} \) are orthogonal to \( n^\alpha \). We can now find the first order perturbations of the first and second fundamental forms of \( \Sigma_0 \).

**Proposition 2 (Battye & Carter, 1995)** Let \((\mathcal{M}, g)\) be a \( C^3 \) spacetime of any dimension and \( \Sigma_0 \) an arbitrary non-degenerate hypersurface defined by an embedding \( \Phi : \Sigma \to \mathcal{M} \). Let \( h \) be the induced metric, \( \kappa \) the extrinsic curvature and \( \vec{n} \) the unit normal to the hypersurface.

If the metric \( g \) is perturbed to first order with a vector field \( \vec{Z} \) and \( \vec{n} \) be the induced metric, \( \kappa \) the extrinsic curvature and \( \vec{n} \) the unit normal to the hypersurface.

\[
\Phi : \Sigma_0 \to M
\]

For its normal part, it is convenient to use the normal on \( \vec{e} \)-form,\( \Phi^* \)

\[
\Phi^*(\vec{n}) = \Phi^* n_\alpha \vec{e}^\alpha
\]

where \( Y' = K'_{\alpha\beta} n^\alpha n^\beta \), \( S' \) is given in (3) and \( e^i = \Phi_*(\partial_i) \) are tangent vectors to \( \Sigma_0 \).

**Proof:** From Lemma 1

\[
h' = \partial_\epsilon h|_{\epsilon=0} = \Phi^*(\mathcal{L}_{\vec{Z}}g) + \Phi^*(K')
\]

and (20) follows directly from Lemma 2 and (21). For \( \kappa' \), we notice that \( 2\kappa' = 2\Phi^*_\epsilon(\nabla' n_\epsilon) = \Phi^*_\epsilon(\mathcal{L}_{\vec{n}}g_\epsilon) \). Applying Lemma 1

\[
2\partial_\epsilon \kappa|_{\epsilon=0} = \Phi^*(\mathcal{L}_{\vec{Z}}\nabla_\epsilon g + \mathcal{L}_{\vec{n}}g + \mathcal{L}_{\vec{n}}K')
\]

where \( \vec{n}_1 \equiv \partial_\epsilon \vec{n}|_{\epsilon=0} \). Let us identify this vector: for its normal component we use that \( \vec{n}_\epsilon \) is unit for all \( \epsilon \), i.e. \( (\vec{n}_\epsilon, \vec{n}_\epsilon)_g = \sigma \). The derivative at \( \epsilon = 0 \) gives \( n^\alpha_1 n_\alpha = -\frac{1}{2} Y' \). For its tangential part, it is convenient to use the normal one-form, \( m_1 \equiv \partial_\epsilon n_\epsilon|_{\epsilon=0} \). From \( \Phi^*_\epsilon(n_\epsilon) = 0 \), Lemma 1 gives \( \Phi^*(\mathcal{L}_{\vec{Z}} n + m_1) = 0 \). Using \( \mathcal{L}_\alpha(n)_\alpha = a_\alpha \) and (20) with \( A = n \), \( F = Q_1 \) gives

\[
\Phi^*(m_1)_i = -(Q_1 a_i + \sigma D_i Q_1).
\]

The identity \( \partial_i g^\alpha\beta = -g^\alpha\mu g^\nu\beta \partial_i g_{\mu\nu} \) implies that \( \vec{n}_1 \) and \( \vec{n}_1 \) are related by \( n^\alpha_1 = -K'^{\alpha\beta} n_\beta + m_1^\alpha \). Hence \( m_1^\alpha n^\alpha = \frac{1}{2} Y' \) which, together with (29), gives

\[
m_1^\alpha = \frac{\sigma}{2} Y' n^\alpha - (Q_1 a^\alpha + \sigma D^\alpha Q_1),
\]

\[
n_1^\alpha = -\frac{\sigma}{2} Y' n^\alpha - (\tau^\alpha + Q_1 a^\alpha + \sigma D^\alpha Q_1),
\]

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where the decomposition (25) has been used. Inserting (31) and (24) into (28) yields
\[ 2\partial_{\xi} \kappa_{ij}|_{\xi=0} = 2\mathcal{L}_{\vec{T}_1} \kappa_{ij} + \mathcal{L}_{Q_1} h_{ij} + 2Q_1 \left( -n^{\mu} n^{\nu} R_{\alpha\mu\beta\nu} e_i^\alpha e_2^\beta + \kappa_{ij} \right) - \sigma Y'. \kappa_{ij} \\
- \mathcal{L}_{\vec{\tau} + \sigma Q_1} h_{ij} + \Phi^* (\mathcal{L}_{\vec{\varpi}} K') , \tag{32} \]
where \((\text{grad} \ Q_1)^i = D^i Q_1\). For the last term we apply Lemma 3 with \(B_{\alpha\beta} = K'_{\alpha\beta}\) and \(\vec{\chi} = \vec{n}, \) i.e.
\[ (\mathcal{L}_{\vec{\varpi}} K')_{\alpha\beta} = -2n_{\mu} S'_{\alpha\beta} + \mathcal{L}_{\vec{\tau} + \sigma Y' \vec{n}} g_{\alpha\beta} \tag{33} \]
and the Lemma follows. \(\square\)

**Remark.** This result was also derived by Mukohyama \cite{22} in his study of first order perturbed matching conditions between spacetimes. The proof presented here is based on Lemma 1 which makes the calculations very efficient. This will allow us to push the calculation to second order.

### 6 Second order perturbation of the hypersurface

Lemma 4 gives an expression for second order Lie derivatives. The next Lemma gives an alternative expression adapted to decompositions into tangential and normal components to the hypersurface.

**Lemma 5** Let \(\Sigma_0\) be an arbitrary non-generate hypersurface in a spacetime \((\mathcal{M}, g)\). Let \(\vec{n}\) be a hypersurface orthogonal unit vector in a neighbourhood of \(\Sigma_0\) which is orthogonal to \(\Sigma_0\). Let \(B\) be any tensor and \(\vec{Z}_1 = Q_1 \vec{n} + \vec{T}_1\), with \(\vec{T}_1\) orthogonal to \(\vec{n}\). Then
\[ \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{Z}_1} B = \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{Z}_1} B + \mathcal{L}_{\vec{T}_1} \mathcal{L}_{\vec{T}_1} B + 2\mathcal{L}_{\vec{T}_1} \mathcal{L}_{Q_1} \vec{n} B - \mathcal{L}_{C_1} \vec{n} + D_1 B - \mathcal{L}_{Q_1} \vec{n} \mathcal{L}_{Q_1} \vec{n} B , \tag{34} \]
where
\[ C_1 \equiv Q_1 \vec{n} (Q_1) + 2\vec{T}_1 (Q_1) - \sigma T_1^\alpha T_1^\beta \kappa_{\alpha\beta}, \quad D_1^\mu = 2Q_1 T_1^\alpha \kappa_{\alpha\mu} + T_1^\alpha D_\alpha T_1^\mu . \tag{35} \]

**Proof:** Applying \ref{19} to \(T_1^\mu\) in the decomposition \(\vec{Z}_1 = Q_1 \vec{n} + \vec{T}_1\), we get
\[ \nabla_\alpha Z_1^\mu = n_\alpha \left( Q_1 a_{\mu} + n^\rho \nabla_\rho T_1^\mu \right) + n_\mu \left( \nabla_\alpha Q_1 - \sigma T_1^\rho \kappa_{\rho\alpha} + Q_1 \kappa_{\alpha\mu} + D_\alpha T_1^\mu \right) . \tag{36} \]
Furthermore, linearity and the general property \(\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = [X, Y]\) imply
\[ \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{Z}_1} B = \mathcal{L}_{\vec{T}_1} \mathcal{L}_{\vec{T}_1} B + 2\mathcal{L}_{\vec{T}_1} \mathcal{L}_{Q_1} \vec{n} B + \mathcal{L}_{\left[ Q_1, \vec{T}_1 \right]} \vec{n} \mathcal{L}_{Q_1} \vec{n} B . \tag{37} \]
We want to introduce a term \(\nabla_{\vec{Z}_1} \vec{Z}_1\) in the right hand side. To that aim we rewrite \([Q_1, \vec{T}_1]\) in the following way
\[ [Q_1, \vec{T}_1]^\mu = [\vec{Z}_1, \vec{T}_1]^\mu = \nabla_{\vec{Z}_1} T_1^\mu - \nabla_{\vec{T}_1} Z_1^\mu = \nabla_{\vec{Z}_1} Z_1^\mu - \nabla_{\vec{Z}_1} (Q_1 n^\mu) - \nabla_{\vec{T}_1} Z_1^\mu = \nabla_{\vec{Z}_1} Z_1^\mu - Q_1^2 a^\mu - 2Q_1 T_1^\alpha \kappa_{\alpha^\mu} - T_1^\alpha D_\alpha T_1^\mu + n^\mu \left( -Q_1 \vec{n} (Q_1) - 2\vec{T}_1 (Q_1) + \sigma T_1^\alpha T_1^\beta \kappa_{\alpha\beta} \right) . \tag{38} \]
where the first three equalities are immediate and the last one follows directly from (18) and (30). Combining (37) and (38) the Lemma follows.

As before, let us decompose the second order perturbed metric into tangential and normal components

\[ K''_{\alpha\beta} = Y'' n_\alpha n_\beta + \sigma n_\alpha \tau'' + \sigma n_\beta \tau'' + K''_{\alpha\beta}. \]

We can now write down our main result of this section. The proof is given in Appendix A.

**Proposition 3** With the same assumptions and notation as in Proposition 2, if the metric is perturbed to second order with \( K'' \) and the hypersurface is perturbed to second order with \( \tilde{Z}_2 = Q_2 \tilde{n} + \tilde{T}_2 \) (with \( \tilde{T}_2 \) orthogonal to \( \tilde{n} \)) then the induced metric and extrinsic curvature are perturbed to second order as

\[
\begin{aligned}
    h''_{ij} &= \mathcal{L}_{\tilde{T}_2} h_{ij} + 2Q_2 \kappa_{ij} + K''_{\alpha\beta} e_i^\alpha e_j^\beta + 2 \mathcal{L}_{\tilde{T}_1} h''_{ij} - \mathcal{L}_{\tilde{T}_1} \mathcal{L}_{\tilde{T}_1} h_{ij} + \\
    &+ \mathcal{L}_{2Q_1 \gamma - 2Q_1 \kappa (\tilde{n})} \tilde{T}_1 h_{ij} + 2 \left( \sigma T_1^l T_1^* \kappa_{ls} - 2 \tilde{T}_1^l (Q_1) + 2 \sigma Q_1 Y' \right) \kappa_{ij} + \\
    &+ 2Q_1^2 \left( -n'' n'' R_{\alpha \beta \theta \nu} e_i^\alpha e_j^\beta + \kappa_{il} \kappa_{lj} \right) + 2 \sigma D_1 Q_1 D_1 Q_1 - 4Q_1 n^\mu S''_{\alpha \beta} e_i^\alpha e_j^\beta, \\
    \end{aligned}
\]

where \( S'' \) is given in (4) and, for any tangent vector \( \tilde{V} \), \( \kappa (\tilde{V}) \) is the vector with components \( \kappa_i^j \).

**Remark:** Propositions 2 and 3 are still true for metrics \( g \) of arbitrary signature.

The expressions in this Proposition are rather involved, and the calculations leading to them are not easy. Thus, it is useful to have ways of checking whether the expressions are indeed correct. We have already mentioned two such tests, namely that the result must be completely independent of the extension of the normal \( n \) and of the perturbations vectors \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) outside \( \Sigma_0 \). This is obvious from the expressions above because no term containing
the acceleration $\ddot{a}$ is present. Moreover, there are no normal derivatives of objects defined intrinsically on $\Sigma_0$. Another important test is that the expressions must transform correctly under gauge transformations. In the next section we study gauge transformations when there is a hypersurface present.

7 First and second order gauge transformations

The definition of the perturbation of the metric and of the hypersurface is based on an a priori identification of $\mathcal{M}_\epsilon$ with $\mathcal{M}_0 = \mathcal{M}$ and of $\Sigma_\epsilon$ with the abstract manifold $\Sigma$. Obviously this identification is not unique. The consequence of this non-uniqueness if the gauge freedom inherent to any geometric perturbation theory. In our case we have two different sources of gauge freedom, namely the identification of the ambient space and the identification of the hypersurfaces. Let us describe them in detail.

The freedom in the identification of $\mathcal{M}_\epsilon$ is described by the possibility of performing an arbitrary diffeomorphism of $\mathcal{M}_\epsilon$ (which, of course, will in general depend on $\epsilon$) before doing the identification with $\mathcal{M} = \mathcal{M}_0$. An equivalent way of stating this fact is that for any one-parameter family of metrics $g_\epsilon$ on $\mathcal{M}$ we can define an infinite number of equivalent families by performing an $\epsilon$-diffeomorphism of $g_\epsilon$. The equivalence is shown as follows.

Each $\mathcal{M}_\epsilon$ has a metric that we denote by $\hat{g}_\epsilon$. These metrics cannot be compared with each other because they are defined on different spaces. Thus, we need a one-parameter family of diffeomorphisms $A_\epsilon: \mathcal{M} \to \mathcal{M}_\epsilon$ in order to be able to relate them, and this defines a one-parameter family of metrics on $\mathcal{M}$ by $g_\epsilon = A_\epsilon^*(\hat{g}_\epsilon)$. If we now perform a diffeomorphism $\Omega_\epsilon$ on $\mathcal{M}$ before identifying with $\mathcal{M}_\epsilon$, it is clear that $A_\epsilon$ becomes $A_\epsilon^{(g)} = A_\epsilon \circ \Omega_\epsilon$ (the superscript $(g)$ stands for “gauge transformed”), and the new parameter family of metrics is $g_\epsilon^{(g)} = A_\epsilon^{(g)*}(\hat{g}_\epsilon) = \Omega_\epsilon^*(g_\epsilon)$, as claimed. All these families are geometrically equivalent. So, although the first and second order perturbation metrics obtained from $g_\epsilon^{(g)}$ and $g_\epsilon$ are indeed different, they are intrinsically the same (i.e. gauge equivalent). In this section we use Lemma 1 to find the gauge transformation law for $K'_\alpha\beta$ and $K''_{\alpha\beta}$ (c.f. [28] and [29] where the so-called knight diffeomorphism are used to describe gauge transformations of any order).

Let us then apply Lemma 1 with $\mathcal{N} = \mathcal{M}$ and $\chi_\epsilon = \Omega_\epsilon$. In this case the diffeomorphisms $\Psi_\epsilon$ which make the diagram (6) commutative are uniquely given by $\Psi_\epsilon = \Omega_{\epsilon+h} \circ \Omega_{\epsilon}^{-1}$. Choose $T_\epsilon = g_\epsilon$ so that $T_\epsilon = g_\epsilon^{(g)}$. We write, as usual, $K' = \frac{dg_\epsilon}{d\epsilon} \big|_{\epsilon=0}$, $K'' = \frac{d^2g_\epsilon}{d\epsilon^2} \big|_{\epsilon=0}$ and we define their gauge transformed tensors as

$$K'_g \equiv \frac{dg_\epsilon^{(g)}}{d\epsilon} \bigg|_{\epsilon=0}, \quad K''_g \equiv \frac{d^2g_\epsilon^{(g)}}{d\epsilon^2} \bigg|_{\epsilon=0}.$$
Let us also define
\[ \vec{S}_1 = \frac{\partial \Omega_{\epsilon}}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad \vec{S}_\epsilon = \frac{\partial (\Omega_{h+\epsilon} \circ \Omega^{-1})}{\partial h} \bigg|_{h=0}, \quad \vec{V} = \frac{\partial \vec{S}_\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} \]
and call \( \vec{S}_1 \) and \( \vec{S}_2 \) the first and second order gauge vectors.

**Proposition 4 (Bruni et. al., 1997)** Under a gauge transformation defined by the vectors \( \vec{S}_1 \) and \( \vec{S}_2 \), the first and second order perturbed metric transform as
\[ K'_{g \alpha \beta} = K'_{\alpha \beta} + \mathcal{L}_{\vec{S}_1} g_{\alpha \beta}, \]
\[ K''_{g \alpha \beta} = K''_{\alpha \beta} + \mathcal{L}_{\vec{S}_2} g_{\alpha \beta} + 2 \mathcal{L}_{\vec{S}_1} K'_{\alpha \beta} - 2 S_1^\mu S_1^\nu R_{\alpha \mu \beta \nu} + 2 \nabla_\alpha S_1^\mu \nabla_\beta S_1^\nu. \]

**Proof:** The first expression is a direct consequence of Lemma 1. For the second order perturbation, the same Lemma gives
\[ K'' = \mathcal{L}_V g + \mathcal{L}_{\vec{S}_1} \mathcal{L}_{\vec{S}_1} g + 2 \mathcal{L}_{\vec{S}_1} K' + K''. \]
The Proposition follows directly from Lemma 1 with \( X \to \vec{S}_1 \).

**Remark:** In [28] the vector \( \vec{V} \) was used instead of \( \vec{S}_2 \) to define second order gauge transformations. Both vectors are equally suited in this case (unlike for hypersurfaces, where \( \vec{Z}_2 \) is intrinsic to the perturbation \( \Sigma \) while \( \vec{W} \) is not, see Remark after Proposition 1). Second order gauge transformation were analyzed also in [27] and \( \vec{S}_2 \) was used there.

When performing a gauge transformation not only the perturbed metrics change but also the vectors \( \vec{Z}_1 \) and \( \vec{Z}_2 \) of the hypersurface are modified. Geometrically, this is clear because changing the way how the different manifolds \( M_\epsilon \) are identified to each other affects how the abstract manifold \( \Sigma \) is embedded into \( \mathcal{M} \) at different \( \epsilon \), and this obviously changes \( \vec{Z}_1 \) and \( \vec{Z}_2 \). It is clear, for instance, that in suitably chosen gauges one can always make these two vectors identically zero. Let us therefore determine the behaviour of \( \vec{Z}_1 \) and \( \vec{Z}_2 \) under general gauge transformations. We denote \( \vec{Z}_1^{(g)} \) and \( \vec{Z}_2^{(g)} \) the corresponding gauge transformed vectors.

**Proposition 5** Under a gauge transformation defined by \( \vec{S}_1 \) and \( \vec{S}_2 \), the first and second order perturbation vectors of \( \Sigma \) transform as
\[ \vec{Z}_1^{(g)} = \vec{Z}_1 - \vec{S}_1, \]
\[ \vec{Z}_2^{(g)} = \vec{Z}_2 - \vec{S}_2 - 2 \nabla_{\vec{S}_1} \vec{S}_1 + 2 \nabla_{\vec{S}_1} \vec{S}_1. \]
independent, i.e. $A^{(g)}_\epsilon \circ \Phi^{(g)}_\epsilon = A_\epsilon \circ \Phi_\epsilon$. Under a gauge transformation $A_\epsilon$ transforms to $A^{(g)}_\epsilon = A_\epsilon \circ \Omega_\epsilon$, therefore the transformation law for the embeddings is

$$\Phi^{(g)}_\epsilon = \Omega^{-1}_\epsilon \circ \Phi_\epsilon.$$ 

Hence, the gauge transformed diffeomorphisms which make the diagram commutative (6) (i.e. such that $\Phi^{(g)}_{\epsilon+h} = \Phi^{(g)}_\epsilon \circ \Phi^{(g)}_h$) can be chosen to be $\Psi^{(g)}_{\epsilon+h} = \Omega^{-1}_\epsilon \circ \Psi^{(g)}_\epsilon \circ \Omega_\epsilon$, which is more conveniently written as $\Psi^{(g)}_h \circ \Omega_\epsilon = \Omega_{\epsilon+h} \circ \Psi^{(g)}_\epsilon$. Taking the first derivative with respect to $h$ at $h = 0$ we find

$$\vec{Z}_\epsilon(\Omega_\epsilon(x)) = \vec{R}_\epsilon(x) + \frac{\partial \Omega_\epsilon(x)}{\partial x^\alpha} \vec{Z}^{(g)}_\alpha(x),$$

(44)

where $\vec{R}_\epsilon(x) \equiv \frac{\partial \Omega_\epsilon(x)}{\partial \epsilon}$.

Putting $\epsilon = 0$ in (44) and recalling that $\Omega_0 = I_M$ the first order transformation $\vec{Z}^{(g)}_1 = \vec{Z} - \vec{S}_1$ follows. For the second order, we take an $\epsilon$-derivative of (44) at $\epsilon = 0$. Applying directly the definitions we find

$$W^\alpha(x) + S_1^\beta(x) \partial_\beta \vec{Z}^{(g)}_1(x) = \frac{\partial R^\alpha_\epsilon(x)}{\partial \epsilon} \bigg|_{\epsilon=0} + \vec{Z}^{(g)}_1(x) \partial_\beta S_1^\alpha(x) + W^{(g)}_\alpha(x).$$

(46)

Now, the derivative of (45) at $\epsilon = 0$ gives

$$\frac{\partial R^\alpha_\epsilon(x)}{\partial \epsilon} \bigg|_{\epsilon=0} \equiv \frac{\partial^2 \Omega^\alpha_\epsilon}{\partial \epsilon^2} \bigg|_{\epsilon=0} = V^\alpha(x) + S_1^\beta(x) \partial_\beta S_1^\alpha(x).$$

(47)

Inserting this into (46) and recalling the definition of $\vec{S}_2$ and $\vec{Z}_2$ the transformation follows.

**Remark:** From (47) a useful expression for $\vec{S}_2$ directly in terms of derivatives of $\Omega_\epsilon$ follows (c.f. Proposition 3)

$$S_2^\alpha(x) = \frac{\partial^2 \Omega^\alpha_\epsilon(x)}{\partial \epsilon^2} \bigg|_{\epsilon=0} + \Gamma^\alpha_{\beta\gamma}(x) S_1^\beta(x) S_1^\gamma(x).$$

Under spacetime gauge transformations, the tensors $h'$, $h''$, $\kappa'$, and $\kappa''$ must be gauge invariant because they are defined intrinsically on $\Sigma$. This provides a very strong potential check for the validity of the expressions given in Proposition 3. The calculations required to perform the check are however very involved and have not been done analytically. Nevertheless, with the aid of a computer algebra program written in Reduce, I have checked
gauge invariance for an important number of non-trivial examples, with positive results in all cases.

As already mentioned perturbed hypersurfaces in a perturbed spacetime have two independent gauge transformations. The first one has already been analyzed. The second gauge freedom comes from the fact that the hypersurfaces \( \Sigma_\epsilon \) embedded into \( \mathcal{M}_\epsilon \) must be identified with an abstract copy of \( \Sigma \). This identification entails the freedom of performing an \( \epsilon \)-dependent diffeomorphism \( \chi_\epsilon \) of \( \Sigma \) before embedding this manifold into \( \mathcal{M} \). Thus, the gauge freedom is given by the transformation \( \Phi_\epsilon^{(g)} = \Phi_\epsilon \circ \chi_\epsilon \) (recall that \( \Phi_\epsilon : \Sigma \to \mathcal{M} \) defines the embedded hypersurface \( \Sigma_\epsilon \)). In terms of coordinates, this gauge transformation corresponds to \( \epsilon \)-dependent coordinate changes \( \hat{y}^i(y^j, \epsilon) \) of the intrinsic coordinates in the hypersurface. It is obvious that this gauge transformation does not affect the first and second order perturbed metrics \( K', K'' \). However it does affect how points with fixed coordinates \( y^i \) move on spacetime and therefore it affects the vectors \( \vec{Z}_1 \) and \( \vec{Z}_2 \). In order to find their gauge transformations, we need to define suitable first and second gauge vectors \( \vec{u}_1 \) and \( \vec{u}_2 \).

It is clear from our previous discussion that they can be defined as

\[
\vec{u}_1 = \frac{\partial \chi_\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad \vec{u}_2 = \frac{\partial^2 (\chi_{\epsilon+h} \circ \chi_\epsilon^{-1})}{\partial h \partial \epsilon} \bigg|_{\epsilon=h=0} + D_{\vec{u}_1} \vec{u}_1.
\]

Being defined on \( \Sigma \) they can be pushed-forward to spacetimes vectors defined on \( \Sigma_0 \). As before, we shall use the same symbol for both objects.

**Proposition 6** Under a gauge transformation on \( \Sigma \) defined by gauge vectors \( \vec{u}_1 \) and \( \vec{u}_2 \), \( \vec{Z}_1(p) \) and \( \vec{Z}_2(p) \) at any point \( p \in \Sigma_0 \) transform as

\[
\vec{Z}_1^{(g)}(p) = \vec{Z}_1(p) + \vec{u}_1(p), \quad (48)
\]

\[
\vec{Z}_2^{(g)}(p) = \left( \vec{Z}_2 + \vec{u}_2 + 2 \nabla_{\vec{u}_1} \vec{Z}_1 - \sigma (\kappa_{ij} u_1^i u_1^j) \vec{n} \right) \bigg|_p, \quad (49)
\]

where \( \vec{n} \) is a unit normal, \( (\vec{n}, \vec{n}) = \sigma \), and \( \kappa_{ij} \) is the second fundamental form of \( \Sigma_0 \).

**Remark:** The covariant derivative in the second expression is a *spacetime* covariant derivative, not a covariant derivative on \( \Sigma \) push-forwarded to \( \Sigma_0 \).

**Proof:** A similar calculation as the one leading to (47) shows that

\[
u_2^i(y) = \frac{\partial^2 \chi_\epsilon^i(y)}{\partial \epsilon^2} \bigg|_{\epsilon=0} + \Gamma^{(3)ij}_l(y) u_1^j(y) u_1^l(y), \quad (50)
\]

where \( \Gamma^{(3)ij}_l \) are the Christoffel symbols of \( (\Sigma, h) \). Writing (44) with the substitutions \( \Phi_\epsilon \to \Phi_\epsilon^{(g)} = \Phi_\epsilon \circ \chi_\epsilon \), \( \vec{Z}_1 \to \vec{Z}_1^{(g)} \) immediately implies

\[
Z_1^{(g)\alpha}(y) = \frac{\partial \Phi_\epsilon^{\alpha}(y)}{\partial \epsilon} \bigg|_{\epsilon=0} + e_\alpha^i(y) \frac{\partial \chi_\epsilon^i(y)}{\partial \epsilon} \bigg|_{\epsilon=0},
\]

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where \( e_1^\alpha(y) = \frac{\partial \Phi^\alpha(y)}{\partial y^l} \) are tangent vectors to \( \Sigma_0 \) and we have used that \( \chi_{\epsilon=0} = I_{\Sigma} \). This proves (48). For the second derivative we find, after gauge transforming (12),

\[
Z_2^{(g)\alpha}(y) = Z_2^\alpha(y) + \left. \frac{\partial^2 \chi^i(y)}{\partial \epsilon^2} \right|_{\epsilon=0} e_i^\alpha(y) + 2u_i^j \left( e_i^\beta \nabla_\beta Z_1^\alpha \right) + \frac{\partial^2 \Phi^\alpha(y)}{\partial y^i \partial y^j} + \Gamma_\beta^\alpha(x_0(y)) e_\beta^\gamma e_\gamma^j,
\]

(51)

The term in parenthesis can be rewritten as \( e_i^\beta \nabla_\beta e_j^\alpha \). Decomposing this vector into tangent and normal components we have \( e_i^\beta \nabla_\beta e_j^\alpha = -\sigma_{ij} n^\alpha + \nabla_\gamma (\Gamma_i^\alpha_{\beta\gamma}) e_j^\alpha \), where we have used the fact that the projection on \( \Sigma_0 \) of the spacetime covariant derivative is precisely the three-dimensional covariant derivative. Using this and (50) into (51) yields the result. □

**Remark:** The gauge transformation on \( \Sigma \) only affects the way how points on \( \Sigma \) are identified before the \( \epsilon \)-derivative is performed. One could think that the only effect of this should be transforming the vectors \( \vec{Z}_1 \) and \( \vec{Z}_2 \) with tangential components. While this is clearly so for the first order vector, it is not true for the second variation (i.e. \( \vec{Z}_2^{(g)} - \vec{Z}_2 \) contains in general normal components). The reason is that the vector \( \vec{Z}_2 \) measures essentially spacetime accelerations, and curves fully contained within \( \Sigma_0 \) in general have a non-zero normal acceleration unless the hypersurface is totally geodesic, i.e. \( \kappa_{ij} = 0 \). Notice that in this case the transformation (49) does not have normal components (provided \( \vec{Z}_1 \) is tangent to \( \Sigma_0 \), of course).

## 8 First and second order linearized matching conditions

In this section we apply the previous results to perturbed matching theory. More specifically, we will consider two spacetimes \( (\mathcal{M}^+, g^+) \) and \( (\mathcal{M}^-, g^-) \) joined across a common hypersurface \( \Sigma_0 \), i.e. such that the so-called matching conditions are satisfied on \( \Sigma_0 \). On both regions we perturb the metric to first and second order with \( K_1 \pm \) and \( K_2 \mp \) respectively. Our aim is to obtain necessary and sufficient conditions on \( \Sigma_0 \) such that the matching conditions are also satisfied in a perturbed sense (i.e. to first and second order).

Let us start with a brief discussion of the matching conditions. Matching theory between spacetimes deals with two \( C^2 \) spacetimes \( (\mathcal{M}^+, g^+) \) with boundary. The respective boundaries are \( C^3 \) hypersurfaces of \( \mathcal{M}^\pm \) which are called matching hypersurfaces. We will denote them by \( \Sigma^\pm \). Although a fully successful theory can be developed for boundary of an arbitrary causal character [30], for simplicity we will concentrate here in the case where both \( \Sigma^\pm \) are either timelike or spacelike everywhere. The matching theory asserts that a \( C^0 \) spacetime \( (\mathcal{M}^0, g^0) \) can be constructed by joining \( (\mathcal{M}^+, g^+) \) and \( (\mathcal{M}^-, g^-) \) if and only if there exists a diffeomorphism \( \varphi : \Sigma^+ \to \Sigma^- \) which is an isometry with respect to the induced metrics
on each boundary. This is equivalent to introducing an abstract \((n - 1)\)-dimensional \(C^3\) manifold \(\Sigma\), and demanding the existence of two embeddings \(\Phi^\pm : \Sigma \rightarrow \mathcal{M}^\pm\) such that (i) \(\Phi^\pm(\Sigma) = \Sigma^\pm \subset \mathcal{M}^\pm\) and (ii) that the two induced metrics \(h^\pm \equiv \Phi^\pm^*(g^\pm)\) on \(\Sigma\) coincide, i.e. \(h^+ = h^-\). Condition (ii) is called first set of matching conditions. Furthermore, the Riemann tensor in the joined spacetime \((\mathcal{M}^t, g^t)\) is free of Dirac delta distributions if and only if the second fundamental forms on \(\Sigma\) coincide, i.e.

\[
\Phi^+^* (\nabla^+ n^+) = \Phi^-^* (\nabla^- n^-),
\]

where \(\pm\) always means that the objects are calculated from the \(\mathcal{M}^\pm\) side. The unit normal \(n^\pm\) to \(\Sigma^\pm\) are chosen to have the same relative orientation after the matching, i.e. either \(\vec{n}^+\) points outwards and \(\vec{n}^-\) inwards or vice versa (“inwards” and “outwards” are well-defined concepts for vector fields on the boundary of a manifold-with-boundary). Notice that the matching conditions are fully covariant both with respect to the spacetimes \(\mathcal{M}^\pm\) and with respect to the matching hypersurface \(\Sigma\). This means that any local coordinate system in \(\mathcal{M}^+\) and any coordinate system in \(\mathcal{M}^-\) are equally valid to impose the matching conditions. Moreover any coordinate system can be used in the abstract manifold \(\Sigma\).

We want to perturb the spacetimes with boundary \((\mathcal{M}^\pm, g^\pm)\) and analyze under which conditions the matching conditions are satisfied perturbatively (assuming the background spacetimes do match across their boundary). First of all, we need to consider how to define perturbations for manifolds with boundary. As mentioned above, in important ingredient in perturbation theory is the need to identify the different manifolds \(\mathcal{M}_\epsilon\) with each other. Now, our manifolds \(\mathcal{M}^\pm_\epsilon\) have boundary (we concentrate on the “+” side; similar considerations hold for \(\mathcal{M}^-\)). If we imagine them as subsets of larger manifolds without boundary \(\tilde{\mathcal{M}}_\epsilon\) and identify these, it is clear that, generically, the identification will not transform the boundaries \(\Sigma^\pm_\epsilon\) among themselves. We could insist that the identification preserves the boundaries, but only at the expense of restricting strongly the gauge freedom (at least near the boundary). This may be quite inconvenient for other purposes. Thus, it is better to let the boundary “move” freely in the identification. From the point of view of the family of manifolds with boundary this means that, strictly speaking, we are not taking diffeomorphisms between them. They are diffeomorphisms except in some closed neighbourhood of the boundary \(\Sigma^\pm_\epsilon\), for each \(\epsilon\). Thus, in strict terms, we cannot talk of a background manifold \(\mathcal{M}^+_0\) with boundary \(\Sigma^+_0\) and a family of metrics \(g_\epsilon\) defined on it. Nevertheless, in perturbation theory we only care about derivatives at \(\epsilon = 0\) of the \(\epsilon\)-family and this can be consistently defined on points at the boundary \(\Sigma^+_0\) by taking one-sided derivatives (i.e. restricting the variations to positive or to negative \(\epsilon\) depending on the point of the boundary we are considering). This allows one to define, similarly as in the case without boundary, a background manifold with boundary \(\mathcal{M}^+\) and two symmetric tensors \(K^+\) and \(K''^+\) defined everywhere up to and including the boundary, so that we have a proper perturbation theory up to second order (or higher order if desired). Having this in mind, we will abuse notation and still talk about diffeomorphisms between different \(\mathcal{M}^\pm_\epsilon\). This also allows us to talk about how the hypersurfaces \(\Sigma_\epsilon\) move on the background and therefore introduce vectors \(Z_1^+\) and \(Z_2^+\) on
the unperturbed boundary, exactly as we did in Section 4. With this particularity in mind, it now easy to write down the perturbed matching conditions. Indeed, for each $\epsilon$ the matching conditions demand the equality of the first and second fundamental forms on each side, i.e. $h^+_{ij} = h^-_{ij}$, $\kappa^+_{ij} = \kappa^-_{ij}$. This tensors are all defined on the abstract hypersurface $\Sigma$ and therefore can be compared with each other (and differentiated with respect to $\epsilon$). It is also clear that the matching conditions will be satisfied in a perturbed sense if and only if the first and second derivatives of the first and second fundamental forms coincide from both sides. Using the explicit form of these derivatives found in previous sections we have a practical method of determining whether perturbations of a spacetime constructed by joining two regions across $\Sigma_0$ can be matched across this hypersurface. We state this result in the form of a theorem

**Theorem 1** Let $(M, g)$ be a spacetime constructed by joining two spacetimes with boundary $(M^+, g^+)$ and $(M^-, g^-)$ across their corresponding boundaries $\Sigma^\pm$. Let $\Sigma$ be an abstract copy of $\Sigma^+$ and $\Phi^\pm : \Sigma \to M^\pm$ be the embeddings defining the background matching. Let also $K'^\pm$ and $K''^\pm$ be first and second order metric perturbations in $M^\pm$.

The first order perturbed (i.e. linearized) matching conditions are fulfilled if and only if there exist two scalars $Q^1_\pm$ and two vectors $\vec{T}^1_\pm$ on $\Sigma$ for which

$$h'^+_\ ij = h'^-_\ ij, \quad \kappa'^+_\ ij = \kappa'^-_\ ij,$$

holds, where $h'^\pm_{ij}$, $\kappa'^\pm_{ij}$ are given in Proposition 2 after the substitution $Q_1 \to Q^1_\pm$, $\vec{T}_1 \to \vec{T}^1_\pm$, $g \to g^\pm$, $K' \to K'^\pm$ and $e_\alpha^i \to e_\alpha^i \pm$.

The second order perturbed matching conditions are satisfied if and only if there exist two scalars $Q^2_\pm$ and two vector fields $\vec{T}^2_\pm$ on $\Sigma$ such that

$$h''^+_\ ij = h''^-_\ ij, \quad \kappa''^+_\ ij = \kappa''^-_\ ij,$$

where these objects are obtained from (40)-(41) after similar substitutions.

**Remark:** It is important to stress the fact that satisfying the perturbed matching conditions require the existence of vector fields $\vec{Z}^1_\pm$ and $\vec{Z}^2_\pm$ such the equations above are satisfied. These vector fields are not known a priori. Moreover they need not be the same vector on both sides. This is obvious from the fact that these vectors are gauge dependent and the gauge may be chosen differently in the different regions $M^\pm$ (actually one can not even compare the two gauges, in general). The gauge can always be chosen so that these vectors coincide but this may not be the most convenient choice. Linearized matching conditions have often been analyzed by using specific gauges where the vectors $\vec{Z}_1$ and $\vec{Z}_2$ take simpler forms. A common choice is to use Gauss coordinates adapted to the matching hypersurfaces for all $\epsilon$ (this obviously makes $\vec{Z}_1^+ = \vec{Z}_1^- = 0$) and then transform to the desired gauge. This is the approach taken in [36] for instance. In spherical symmetry, the linearized matching conditions in arbitrary gauge was first studied in [37], [38] and completed in [39]. The
general linearized matching conditions in an arbitrary background were first presented by Mukohyama \cite{22}. In this paper a vector field \( \vec{Z} \) was introduced describing the perturbation of the matching hypersurface to first order. However, this vector was assumed to be the same in both sides \( \mathcal{M}^+ \) and \( \mathcal{M}^- \). As I have already stressed this need not be case and the fully general perturbed matching conditions require the use of two vectors \( \vec{Z}_1^+ \) and \( \vec{Z}_1^- \) (and two more vectors to second order). Notice also that the gauge freedom within \( \Sigma \) (see Proposition \ref{prop:6}) allows us to choose the tangential components of \( \vec{Z}_1 \) and \( \vec{Z}_2 \) in any way we want, but only on one of the sides, i.e. either on \( \mathcal{M}^+ \) or on \( \mathcal{M}^- \). Once a choice on one side has been made, the other side must be left free and determined by the matching conditions (if they happen to be consistent). This is similar to the fact that when solving a matching problem one not only looks for a pair of matching hypersurfaces with suitable properties, but also for a specific pair of embeddings on each side, i.e. a way of identifying the two hypersurfaces pointwise.

In this theorem, only non-null hypersurfaces are considered. This is because the classical Darmois matching conditions (discussed above) are not adequate for hypersurfaces with null points. In that case the continuity of the second fundamental form does not ensure the absence of distributional parts in the Riemann tensor. The matching conditions for null hypersurfaces where first discussed by Clarke and Dray \cite{35} and later extended to hypersurfaces of arbitrary causal character (including a changing one) in \cite{30}. They involve the continuity of a tensor that generalizes the second fundamental form. In order to find the perturbed matching conditions in this case we would need to find how this new tensor is perturbed to second order. This issue is of interest and should be studied. The methods described in the present paper are useful to find perturbations of any geometric tensor defined on a hypersurface and therefore are applicable to this situation too. The calculations for hypersurfaces with null points are probably more difficult but still manageable.

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9 Appendix A: Proof of Proposition \ref{prop:3}

Proof: Let us start with \( h'' \). From Lemma \ref{lem:1} we find

\[
\partial_i \partial_i h_e \big|_{e=0} = \Phi^* \left( K'' + 2 \mathcal{L}_{\vec{Z}_1} K' + \mathcal{L}_{\vec{W}} g + \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{Z}_1} g \right). \tag{53}
\]
Applying (34) with \( B \to g \) and using (28) with \( F_1 = F_2 = Q_1 \) we readily obtain

\[
\mathcal{L}_{\vec{Z}_i} \mathcal{L}_{\vec{Z}_i} g_{\alpha \beta} = \mathcal{L}_{\nabla \vec{Z}_i} \vec{Z}_i g_{\alpha \beta} + 2 \mathcal{L}_{\vec{T}_1} \mathcal{L}_{\vec{Z}_i} g_{\alpha \beta} - \mathcal{L}_{\vec{T}_1} \mathcal{L}_{\vec{T}_1} g_{\alpha \beta} - \mathcal{L}_{\vec{D}_1} g_{\alpha \beta} + 2 \left( \sigma T_1^{\mu} T_1^{\nu} \kappa_{\mu \nu} \right) \\
- 2 T_1^{\nu} (Q_1) \kappa_{\alpha \beta} + 2 Q_1^2 \left( -n^\mu n^\nu R_{\alpha \mu \beta \nu} + \kappa_{\alpha \mu} \kappa_\beta^\mu \right) + 2 \sigma D_\alpha Q_1 D_\beta Q_1 + n_\alpha P_\beta + n_\beta P_\alpha.
\]

Here and in the following \( P_\alpha, P_{\alpha \beta}, \cdots \) stands for expressions whose explicit form does not concern us. Notice that its meaning may be different even in different parts of the same formula.

Substituting into (28) we observe that a term \( 2 \mathcal{L}_{\vec{T}_1} (K' + \mathcal{L}_{\vec{Z}_i} g) \) appears. From Proposition 2 the pull-back of this term is just \( 2 \mathcal{L}_{\vec{T}_1} h' \). Recalling that \( \vec{Z}_2 \equiv \vec{W} + \nabla \vec{Z}_1 \vec{Z}_1 \) and the decomposition \( \vec{Z}_2 = Q_2 \vec{n} + \vec{T}_2 \) yields the first two terms in (40). Only \( 2 \mathcal{L}_{Q_1 \vec{n}} K'_{\alpha \beta} \) remains to be analyzed. This is dealt with using Lemma 3 with \( \vec{X} \to Q_1 \vec{n} \) which gives

\[
2 \mathcal{L}_{Q_1 \vec{n}} K'_{\alpha \beta} = -4 Q_1 n_\mu S^{\mu}_{\alpha \beta} + 2 \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{n}} g_{\alpha \beta} + 4 \sigma Q_1 Y' \kappa_{\alpha \beta} + n_\alpha P_\beta + n_\beta P_\alpha,
\]

and expression (40) follows directly. Let us next consider \( \kappa'' \), which involves the longest and most difficult calculation. Applying Lemma 1 to \( \kappa_e \) we find

\[
2 \partial_e \partial_e \kappa_e \big|_{e=0} = \Phi^* \left( \mathcal{L}_{\vec{n}_2} g + 2 \mathcal{L}_{\vec{n}_1} K' + \mathcal{L}_{\vec{n}_1} K'' + 2 \mathcal{L}_{\vec{Z}_i} \mathcal{L}_{\vec{n}_1} g + 2 \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{n}_1} K' + \right. \\
+ \mathcal{L}_{\vec{W}} \mathcal{L}_{\vec{n}_2} g + \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{n}_2} g + \mathcal{L}_{\vec{T}_1} \mathcal{L}_{\vec{n}_2} g + \mathcal{L}_{\vec{T}_1} M \big).
\]

In Proposition 2 we evaluated \( \partial_e \kappa_e \big|_{e=0} \), which required calculating the pull-back on \( \Sigma \) of \( M \equiv \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{n}} g + \mathcal{L}_{\vec{n}} g + \mathcal{L}_{\vec{n}} K' \) (see 28). We want to identify in (54) terms giving the Lie derivative of \( M \) along \( \vec{T}_1 \). Adding and subtracting \( 2 \mathcal{L}_{\vec{T}_1} \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{n}} g \) we can write

\[
2 \partial_e \partial_e \kappa_e \big|_{e=0} = \Phi^* \left( \mathcal{L}_{\vec{n}_2} g + 2 \mathcal{L}_{\vec{n}_1} K' + \mathcal{L}_{\vec{n}_1} K'' + 2 \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{n}_1} g + 2 \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{n}_1} K' + \right. \\
+ \mathcal{L}_{\vec{W}} \mathcal{L}_{\vec{n}_2} g + \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{n}_2} g - 2 \mathcal{L}_{\vec{T}_1} \mathcal{L}_{\vec{Z}_1} \mathcal{L}_{\vec{n}_2} g + 2 \mathcal{L}_{\vec{T}_1} M \big).
\]

Let us deal with the different terms in this expression, starting with \( \mathcal{L}_{\vec{n}_1} K' \). A direct application of (20) and the explicit expression for \( \vec{n}_1 \) (31) yield, after using (33),

\[
\mathcal{L}_{\vec{n}_1} K'_{\alpha \beta} = -Y'^{\beta} \kappa_{\alpha \beta} - \frac{\sigma}{2} Y' \mathcal{L}_{\vec{\tau}} g_{\alpha \beta} + \sigma Y' n_\mu S^{\mu}_{\alpha \beta} - \frac{\sigma}{2} (\tau'_\alpha D_\beta Y' + \tau'_\beta D_\alpha Y') - \\
- \mathcal{L}_{\vec{\tau} + Q_1 \vec{a} + \sigma \text{grad} Q_1} K'_{\alpha \beta} + n_\alpha P_\beta + n_\beta P_\alpha.
\]

Next, we analyze the combination \( 2 \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{n}_1} g + 2 \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{n}} K' \). Using expression (33) and the first equality in (33) we get

\[
2 \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{n}_1} g_{\alpha \beta} + 2 \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{n}} K'_{\alpha \beta} = \sigma \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{\vec{Y}'} \vec{n} g_{\alpha \beta} - 2 \mathcal{L}_{Q_1 \vec{n}} \mathcal{L}_{Q_1 \vec{a} + \sigma \text{grad} Q} g_{\alpha \beta} - 4 \mathcal{L}_{Q_1 \vec{n}} (n_\mu S^{\mu}_{\alpha \beta}).
\]
Using now identity (23) with $F_1 = Q_1$, $F_2 = Y'$ and writing the Lie derivative of the last term explicitly using covariant derivatives yields

\[
2 \mathcal{L}_{Q_1} \bar{n} \mathcal{L}_{Q_1} g_{\alpha \beta} + 2 \mathcal{L}_{Q_1} \bar{n} \mathcal{L}_{\kappa} K'_{\alpha \beta} = \sigma \mathcal{L}_{Q_1} Y' \bar{g}_{\alpha \beta} \\
+ 2 \sigma Q_1 Y' (-n^\nu n^\rho R_{\alpha \mu \beta \nu} + \kappa_{\alpha \mu} \kappa_{\beta}^\mu) + (\nabla_\alpha Q_1 \nabla_\beta Y' + \nabla_\alpha Y' \nabla_\beta Q_1) + 2 \mathcal{L}_{Q_1} \bar{n} \mathcal{L}_{Q_2} \bar{a} g_{\alpha \beta} - 4 Q_1 a_\mu S^\mu_{\alpha \beta} - 4 Q_1 n_\mu n^\nu \nabla_\nu S^\mu_{\alpha \beta} - 4 n_\mu n^\nu S^\mu_{\alpha \beta} + 4 n_\mu S^\mu_{\alpha \beta} \kappa_{\beta}^\nu - 4 n_\mu S^\mu_{\alpha \beta} \kappa_{\alpha}^\nu + n_\alpha P_\beta + n_\beta P_\alpha.
\]

(57)

Next we analyze the terms involving third derivatives in (58). Lemma 3 implies

\[
\mathcal{L}_{\bar{Z}_1} \mathcal{L}_{Z_1} \bar{L}_\kappa g - \mathcal{L}_{\bar{L}} \mathcal{L}_1 \bar{L}_1 \bar{L}_\kappa g = \mathcal{L}_{\bar{Z}_2} \mathcal{L}_{Z_2} \bar{L}_\kappa g - \mathcal{L}_{\bar{L}_1} \mathcal{L}_1 \mathcal{L}_1 \bar{L}_\kappa g - \mathcal{L}_{\bar{L}_2} \mathcal{L}_1 \mathcal{L}_1 \bar{L}_\kappa g + \mathcal{L}_{\bar{L}_1} \mathcal{L}_1 \bar{L}_1 \mathcal{L}_1 \bar{L}_\kappa g.
\]

Except for the last term $\mathcal{L}_{Q_1} \mathcal{L}_{Q_1} \mathcal{L}_{Q_1} \bar{L}_\kappa g$, the pull-back of the right-hand side is easily obtained from Lemma 2 and (23). For the last term we have, directly from (23),

\[
\mathcal{L}_{Q_1} \mathcal{L}_{Q_1} \mathcal{L}_{Q_1} \bar{L}_\kappa g = \mathcal{L}_{Q_1} \mathcal{L}_{Q_1} \mathcal{L}_{Q_1} \bar{L}_\kappa g + 2 Q_1 \bar{n}(Q_1) (-n^\nu n^\rho R_{\alpha \mu \beta \nu} + \kappa_{\alpha \mu} \kappa_{\beta}^\mu) + 2 Q_1 \bar{n}(Q_1) (\kappa_{\alpha \mu} \kappa_{\beta}^\mu - n^\mu n^\nu R_{\alpha \mu \beta \nu}) + 2 \sigma Q_1 \left( a^\mu \kappa_{\alpha \mu} (Q_1 a_\beta + \sigma D_\beta Q_1) + a^\nu \kappa_{\beta}^\nu (Q_1 a_\alpha + \sigma D_\alpha Q_1) \right) + n_\alpha P_\beta + n_\beta P_\alpha.
\]

(58)

In order to elaborate this expression further we expand the Lie derivative in the third summand in terms of covariant derivatives. We get

\[
n^\mu n^\nu \mathcal{L}_{Q_1} \mathcal{L}_{Q_1} \mathcal{L}_{Q_1} R_{\alpha \mu \beta \nu} = Q_1 n^\mu n^\nu \bar{n}^\delta \nabla_\delta R_{\alpha \mu \beta \nu} + 2 \bar{n}(Q_1) n^\mu n^\nu R_{\alpha \mu \beta \nu} + Q_1 \left( a^\mu n^\nu R_{\alpha \beta \mu \nu} + n^\mu a^\nu R_{\alpha \beta \mu \nu} + \kappa_{\delta}^\rho R_{\alpha \mu \beta \nu} n^\mu n^\nu + \kappa_{\beta}^\rho R_{\delta \mu \rho \nu} n^\mu n^\nu \right) + n_\alpha P_\beta + n_\beta P_\alpha.
\]

(59)

For the Lie derivative of $\kappa_{\alpha \beta}$ along $Q_1 \bar{n}$ we have

\[
\mathcal{L}_{Q_1} \bar{n} \kappa_{\alpha \beta} = -\sigma Q_1 a_\alpha a_\beta + Q_1 (-n^\mu n^\nu R_{\alpha \mu \beta \nu} + \kappa_{\alpha \mu} \kappa_{\beta}^\mu) + \frac{Q_1}{\alpha} h^\mu_{\alpha} h^\nu_{\beta} \bar{L}_\kappa g_{\mu \nu}
\]

(60)

which follows from $\mathcal{L}_{Q_1} \bar{n} h_\alpha^\mu = -\sigma (Q_1 \delta_\beta + \sigma D_\beta Q_1) n^\alpha$ and $\kappa_{\alpha \beta} = \frac{1}{2} h^\mu_{\alpha} h^\nu_{\beta} \mathcal{L}_{Q_1} \bar{n} g_{\mu \nu}$ after applying (23). Inserting (59) and (60) into (58) the following expression is found

\[
\mathcal{L}_{Q_1} \bar{n} \mathcal{L}_{Q_1} \bar{n} \mathcal{L}_{Q_1} \bar{n} g_{\alpha \beta} = \mathcal{L}_{Q_1} \bar{n} \mathcal{L}_{Q_1} \bar{n} g_{\alpha \beta} + 2 Q_1 \bar{n}(Q_1) (-n^\mu n^\nu R_{\alpha \mu \beta \nu} + \kappa_{\alpha \mu} \kappa_{\beta}^\mu) - 2 \bar{n}^2 (n^\mu n^\nu \bar{n}^\delta R_{\alpha \mu \beta \nu} + a^\mu n^\nu R_{\alpha \mu \beta \nu} + n^\mu a^\nu R_{\alpha \mu \beta \nu} + \kappa_{\delta}^\rho R_{\delta \mu \rho \nu} n^\mu n^\nu + \kappa_{\beta}^\rho R_{\delta \mu \rho \nu} n^\mu n^\nu) + 2 Q_1 a^\mu \kappa_{\mu} D_\alpha Q_1 + 2 Q_1 a^\mu \kappa_{\mu} D_\beta Q_1 + Q_1^2 (\kappa_{\beta}^\rho \bar{L}_\kappa g_{\alpha \mu} + \kappa_{\beta}^\rho \bar{L}_\kappa g_{\beta \mu}) + n_\alpha P_\beta + n_\beta P_\alpha
\]

(61)

It only remains to calculate the first term in (58), i.e. to find the vector $\bar{n}_\beta = \partial_\beta \bar{n}_\kappa |_{\epsilon = 0}$. The calculation is somewhat long and will be given in Appendix B. The result is

\[
n_\alpha^2 = n^\alpha \left( -\frac{1}{2} \sigma Y^{\nu} + \frac{3}{4} Y'^{\nu} + \sigma \tau^{\nu} \tau^\mu \right) - \tau^{\nu} \left( (Q_2 a^\alpha + \sigma D^\alpha Q_2) + \sigma Y^\nu \tau^\alpha \right) + (C_1 a^\alpha + \sigma D^\alpha C_1) - (\bar{n}(Q_1) + \sigma Y')(Q_1 a^\alpha + \sigma D^\alpha Q_1) + 2 K'^{\alpha \beta} \tau_{\beta} + Q_1 a_\alpha + \sigma D_\alpha Q_1) + [Q_1 \bar{n}, Q_1 \bar{n} + \sigma \text{grad} Q_1]^\alpha + 2 Q_1^2 a^\mu \kappa_{\mu}^\alpha + 2 \sigma Q_1 \kappa_{\mu}^\alpha D^\mu Q_1
\]

(62)
Our interest is to calculate \( \mathcal{L}_{\bar{n}^2} g_{\mu\nu} \). Only the term involving \( 2K^{t\alpha\beta} (\tau' + Q_1 \alpha \beta + \sigma D_\beta Q_1) \) requires further analysis. Applying Lemma 6 and the fact that \( S'^\mu_{\alpha\beta} = S(K^{t\alpha\beta} \mu + \sigma T^\mu_{\alpha\beta} + n^\mu P_{\alpha\beta} + n^\alpha P_{\beta\mu} + n^\beta P_{\alpha\mu} \), which follows directly from the definition of \( S' \) and the decomposition (23), we get

\[
\mathcal{L}_{2K^{t\mu\nu}(\tau' + Q_1 a_{\mu} + \sigma D_\nu Q_1)} g_{\alpha\beta} = 2 \mathcal{L}_{\tau' + Q_1 a_{\mu} + \sigma D_\nu Q_1} K^{t\alpha\beta} + 4 (\tau' + Q_1 a_{\mu} + \sigma D_{\mu} Q_1) (S'^\mu_{\alpha\beta} - \sigma \mu_{\alpha\beta}) + n^\alpha P_{\beta} + n^\beta P_{\alpha}.
\]

We are now in a position where all terms in (55) can be collected. Several obvious cancellations happen which will not be described in any detail. More subtle is the use of the following identity

\[
- \mathcal{L}_{Q_1 \bar{n}} \mathcal{L}_{Q_1 \bar{n}} g_{\alpha\beta} - \mathcal{L}_{Q_1 \bar{n}} \mathcal{L}_{Q_1 \bar{n}} g_{\alpha\beta} + Q_1^2 \kappa_{\alpha} \mathcal{L}_{\bar{n}} g_{\beta\mu} + Q_1^2 \kappa_{\beta} \mathcal{L}_{\bar{n}} g_{\alpha\mu} + 2Q_1 (a^\mu \kappa_{\beta} D_\alpha Q_1 + a^\mu \kappa_{\alpha} D_\beta Q_1) - 2Q_1^2 (n^\mu a^\nu R_{\alpha\mu\beta\nu} + a^\mu n^\nu R_{\alpha\mu\beta\nu}) + 2Q_1 \bar{n}(Q_1) \kappa_{\alpha\beta} + 2 \mathcal{L}_{Q_1 \bar{n}^\nu \kappa_{\beta} g_{\alpha\beta}} = n^\alpha P_{\beta} + n^\beta P_{\alpha}.
\]

This expression follows by direct calculation using the Codazzi identity written in the space-time form \( D_{\alpha} \kappa_{\mu\nu} - D_{\mu} \kappa_{\alpha\nu} = n^\delta R_{\sigma\delta\beta\rho} h^\rho_{\alpha} h^\delta_{\mu} \) and the fact that \( D_{\alpha} a_\beta - D_{\beta} a_\alpha = 0 \), which is a direct consequence of the definition of acceleration in our hypersurface orthogonal case.

Finally, in order to arrive at the final expression, two more ingredients are required. The first one is

\[
[Q_1 \bar{n}, \text{grad } Q_1]^\alpha = Q_1 D^\alpha (\bar{n}(Q_1)) - \sigma Q_1 \bar{n}(Q_1) a^\alpha - 2Q_1 \kappa_{\mu}^\alpha D^\mu Q_1 - (D_{\mu} Q_1 D^\mu Q_1 + \sigma Q_1 \bar{a}(Q_1)) n^\alpha,
\]

which is checked directly. The second one is \( \bar{n}(Y') = 2\tau'_\alpha a^\alpha + 2n_\mu n^\delta S'^\mu_{\delta\beta} \), which is immediate. Using also the explicit expression for \( C_1 \) and \( \bar{D}_1 \) in (55) and collecting all terms we find (11).

### 10 Appendix B: Calculation of \( \bar{n}_2 \)

In this appendix we find an explicit expression for \( \partial_i \bar{n}_i \).

**Lemma 6** With the same notation and conventions as in Proposition 5 we have

\[
n_2^\alpha = n^\alpha \left( \frac{1}{2} Y'' + \frac{3}{4} \tau'^2 + \sigma \tau' \tau'' \right) - \tau''^\alpha - (Q_2 a^\alpha + \sigma D^\alpha Q_2) + \sigma Y' \tau''^\alpha + (C_1 a^\alpha + \sigma D^\alpha C_1) - (\bar{n}(Q_1) + n^\alpha Y') (Q_1 a^\alpha + \sigma D^\alpha Q_1) + 2K^{t\alpha\beta} (\tau' + Q_1 a_\beta + \sigma D_\beta Q_1)
\]

\[
+ \left[ Q_1 \bar{n}, Q_1 \bar{a} + \text{grad } Q_1 \right]^\alpha + 2Q_1^2 a^\mu \kappa_{\mu}^\alpha + 2\sigma Q_1 \kappa_{\mu}^\alpha D^\mu Q_1. \quad (63)
\]
Proof: We proceed as we did for \( \vec{n}_1 \), i.e. we first determine the normal component of \( \vec{n}_2 \) and then its tangential part. Taking the second \( \epsilon \) derivative of \((\vec{n}_\epsilon, \vec{n}_\epsilon)_{g_\epsilon} = \sigma \) and evaluating at \( \epsilon = 0 \) we find

\[
2n_2^\mu n_\mu + 2n_1^\mu n_1^\mu + 4n_1^\mu n_1^\nu K_\mu^\nu + n^\mu n^\nu K_\mu^\nu = 0,
\]

which after substitution of the expression for \( \vec{n}_1 \) yields

\[
n_2^\nu n_\nu = -\frac{1}{2}Y'' + \frac{3}{4}Y'^2 + (\tau'_\mu - Q_1 a_\mu - \sigma D_\mu Q_1) (\tau'' + Q_1 a^\mu + \sigma D^\mu Q_1).
\] (64)

For the tangential components, it is convenient to use the second variation of the normal one-form, i.e \( m_2 \equiv \partial_\nu n_\epsilon |_{\epsilon = 0} \). The relationship with \( \vec{n}_2 \) is immediate from the second \( \epsilon \)-derivative of \( g_\epsilon (n_\epsilon, \cdot) = n_\epsilon \), i.e.

\[
n_2^\alpha = -K^\alpha_\beta n_\beta + \sigma Y^{\epsilon}_{\epsilon} K^{\epsilon}_{\beta} n_\beta + 2K^{\epsilon}_{\alpha \beta} (\tau'_\beta + Q_1 a_\beta + \sigma D_\beta Q_1) + m_2^\alpha.
\]

Decomposing this into tangential and normal components and using (64) one finds

\[
n_2^\alpha = n^\alpha \left[ -\frac{1}{2}Y'' + \frac{3}{4}Y'^2 + \sigma (\tau'_\mu - Q_1 a_\mu - \sigma D_\mu Q_1) (\tau'' + Q_1 a^\mu + \sigma D^\mu Q_1) \right]
\]

\[
-\tau''^\alpha + \sigma Y''_\tau^{\epsilon} + 2K'' t^\alpha_\beta (\tau'_\beta + Q_1 a_\beta + \sigma D_\beta Q_1) + h^\alpha_\beta m_2^\beta.
\] (65)

It only remains to find \( h^\alpha_\beta m_2^\beta \). Applying Lemma 4 to \( \Phi^\epsilon (n_\epsilon) = 0, \forall \epsilon \) yields

\[
\Phi^\ast \left( L_{\vec{W}} n + L_{\vec{Z}_1} L_{\vec{Z}_2} n + 2L_{\vec{Z}_1} m_1 + m_2 \right) = 0.
\] (66)

Moreover, Lemma 5 applied to \( n \) leads to

\[
L_{\vec{W}} n + L_{\vec{Z}_1} L_{\vec{Z}_1} n = L_{\vec{Z}_2} n + L_{\vec{T}_1} L_{\vec{T}_1} n + 2L_{\vec{T}_1} L_{Q_1 \vec{n}} n - L_{\vec{C}_1 \vec{n}} + \vec{D}_1 n - L_{Q_1 \vec{n}} L_{Q_1 \vec{n}} n.
\]

Now, for any pair of functions \( F_1 \) and \( F_2 \) we have \( L_{F_1 \vec{n}} (F_2 n) = \vec{n} (F_1 F_2) n_\alpha + F_2 (F_1 a_\alpha + \sigma D_\alpha F_1) \). Using also the fact that \( \Phi^\ast (L_{\vec{V}} n) = 0 \) for any vector field \( \vec{V} \) tangent to \( \Sigma \), i.e. \( L_{\vec{V}} n \propto n \) one finds

\[
L_{\vec{W}} n_\alpha + L_{\vec{Z}_1} L_{\vec{Z}_1} n_\alpha = Q_2 a_\alpha + \sigma D_\alpha Q_2 + 2L_{\vec{T}_1} (Q_1 a_\alpha + \sigma D_\alpha Q_1) - (C_1 a_\alpha + \sigma D_\alpha C_1) + \vec{n} (Q_1) (Q_1 a_\alpha + \sigma D_\alpha Q_1) + L_{Q_1 \vec{n}} (Q_1 a_\alpha + \sigma D_\alpha Q_1) + Pn_\alpha.
\]

Moreover, from (60)

\[
L_{\vec{Z}_1} m_1 a = -L_{\vec{T}_1} (Q_1 a_\alpha + \sigma D_\alpha Q_1) + \frac{\sigma}{2} Y'' (Q_1 a_\alpha + \sigma D_\alpha Q_1) - L_{Q_1 \vec{n}} (Q_1 a_\alpha + \sigma D_\alpha Q_1) + Pn_\alpha.
\]

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Then (66) implies
\[
h^\beta_\alpha m_{2\beta} = - (Q_2 a_\alpha + \sigma D_\alpha Q_2) + (C_1 a_\alpha + \sigma D_\alpha C_1) - (\tilde{n}(Q_1) + \sigma Y')(Q_1 a_\alpha + \sigma D_\alpha Q_1) + 
+ h^\beta_\alpha \mathcal{L}_{Q_1} \tilde{n}(Q_1 a_\beta + \sigma D_\beta Q_1). \tag{67}
\]

Let us elaborate the last term. Using \( \mathcal{L}_{Q_1} \tilde{n} h^{\alpha\beta} = -2Q_1 \kappa^{\alpha\beta} - \sigma n^{\alpha}(Q_1 a^\beta + \sigma D^\beta Q_1) - \sigma n^\beta(Q_1 a^\alpha + \sigma D^\alpha Q_1) \), and integrating by parts we find
\[
h^{\alpha\beta} \mathcal{L}_{Q_1} \tilde{n}(Q_1 a_\beta + \sigma D_\beta Q_1) = [Q_1 \tilde{n}, Q_1 \tilde{a} + \sigma \text{grad} Q_1]^{\alpha} + 2Q_1^2 a^\mu \kappa_\mu^\alpha + 2\sigma Q_1 \kappa_\mu^\alpha D^\mu Q_1 
+ \sigma n^{\alpha}(Q_1 a_\mu + \sigma D_\mu Q_1)(Q_1 a^\mu + \sigma D^\mu Q_1).
\]

Plugging this into (67) and the resulting expression in (65), the Lemma follows. \( \Box \).

References


