Large $N$ Scaling Behavior of the Lipkin-Meshkov-Glick Model

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We introduce a novel semiclassical approach to the Lipkin model. In this way the well-known phase transition arising at the critical value of the coupling is intuitively understood. New results — showing for strong couplings the existence of a threshold energy which separates deformed from undeformed states as well as the divergence of the density of states at the threshold energy — are explained straightforwardly and in quantitative terms by the appearance of a double well structure in a classical system corresponding to the Lipkin model. Previously unnoticed features of the eigenstates near the threshold energy are also predicted and found to hold.

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The Lipkin-Meshkov-Glick (LMG) model, originally introduced in nuclear physics has found applications in a broad range of other topics: statistical mechanics of quantum spin systems, Bose–Einstein condensates, as well as quantum entanglement, to name but a few. Quantum spin systems, Bose–Einstein condensates, and Wigner–Crystalline phases are some of the simplest to show a quantum transition in the regime of strong coupling.

This transition is by now well-understood: the ground state becomes degenerate and a macroscopic change in the ground state energy takes place. Furthermore, at the transition value of the coupling, the density of states diverges at the ground state energy as the number $N$ of interacting particles becomes large. The energy gap was recently found to vanish as $N^{-1/3}$. Furthermore, in this model a novel type of phase transition has recently been discovered for strong values of the coupling parameter. Indeed, in this regime the spectrum is divided by a critical energy $E_c$, where the behavior characteristic of strong coupling holds below $E_c$, while above $E_c$ the system reverts to the kind of behavior found below the phase transition. At the critical energy, the density of states is again found to diverge as the particle number $N$ goes to infinity. This divergence has been conjectured to be of the logarithmic type.

In recent papers different approaches have been used to explain these results. The continuous unitary transformation technique (flow equations) was applied to obtain the spectrum reliably for large $N$ values. An investigation of the singularities of the spectrum in the complex $\lambda$-plane (the exceptional points) sheds more light upon the complexity of the limit problem but a final answer about the limit attained has not been given. These singularities have been recognized as an essential mechanism to invoke the phase transition in a seminal paper by Lee and Yang and their significance for the partition function is discussed more recently again in. A different approach starts with the bosonization method using higher orders of powers in $1/N$ of the Holstein-Primakoff representation and then applies the flow equation technique. New results have been obtained in this way, in particular the correct analytic behavior of the level distance as a function of $N$ at the critical point. Earlier attempts have revealed different results such as the form of the wave functions beyond the phase transition.

In this Letter we introduce a novel semiclassical approach to the LMG model. It readily explains the above features, determines the precise value of the critical energy as a function of the coupling, proves the logarithmic divergence of the density of states near the critical energy as well as successfully predicts certain previously unnoticed behavior of the eigenstates near the critical energy. Finally, we obtain a qualitative understanding of all essentials of the model: the classical model we introduce has a double well structure above the phase transition, and the critical energy can then be identified with the separatrix energy. The approach given here also shows easily both the nature of the phase transition as a function of the coupling parameter as well as the scaling with $N$ of the vanishing gap at the critical coupling, which was previously shown to scale as $N^{-1/3}$.

We here recapitulate the basics of the model and discuss the essential properties for large values of $N$. It is given in terms of $2j+1 = N + 1$-dimensional representations of the SU(2) operators $J_k$, $k = x, y, z$ as follows

$$H(\lambda) = J_z + \frac{\lambda}{N}(J_x^2 - J_y^2).$$

Here the interaction is scaled by $N$ to ensure that $H$ is extensive. In this form the model has a phase transition just beyond $\lambda = 1$, the larger $N$ the closer the transition point at $\lambda = 1$. This has been discussed under various points of view in the literature, see e.g. and. The Hamiltonian allows reduction into two spaces: $m$ integer and $m$ half-integer, with $m$ the eigenvalues of $J_z$.
it corresponds to \( N \) even and odd respectively and is denoted as parity. For \( \lambda > 1 \), the even and odd levels are obviously separated and remain so for all \( \lambda < 1 \) while the levels become degenerate (up to terms vanishing exponentially fast in \( N \)) for \( \lambda > 1 \). The phase at \( \lambda < 1 \) is called the normal phase, while the symmetry (parity) breaking phase at \( \lambda > 1 \) is called the deformed phase. Recent calculations \[9\] apply non-perturbative flow equations allowing to obtain the spectrum for arbitrarily high yet finite values of \( N \). These have established \[8\] the existence of the phase transition in energy referred to above: the states having energy below a certain threshold behave as states of the deformed phase, whereas higher in the spectrum the states become undeformed again.

Since the commutator \([H(\lambda), (\hat{J})^2]\) vanishes, we confine ourselves to a fixed value of \( j = N/2 \). For large \( N \) we consider the Hamiltonian \[11\] on the sphere of radius \( j = N/2 \). In other words, we rewrite the Hamilton operator \[11\] as a classical Hamilton function

\[
H = \frac{N}{2} \left( -\sin \theta \cos \phi - \frac{\lambda}{2} (\cos^2 \theta - \sin^2 \theta \sin^2 \phi) \right)
\]  

having introduced the polar angles as \[17\]

\[
J_z = -\frac{N}{2} \sin \theta \cos \phi
J_x = \frac{N}{2} \cos \theta
J_y = \frac{N}{2} \sin \theta \sin \phi.
\]

Note that the transition to a classical Hamiltonian has also been achieved in a different way using the coherent state approach developed in \[18\]. At this point we notice that, with \( \mu = \cos \theta \), the Poisson bracket

\[
\{\mu, \phi\} = \frac{2}{N}
\]  

suggests how to quantize the Hamilton function of the single particle problem in the two canonical conjugate coordinates \( \phi \) and \( \mu \). It can be written as

\[
K \equiv \frac{2H}{N} = -\sqrt{1 - \mu^2} \cos \phi - \frac{\lambda}{2} (\mu^2 - (1 - \mu^2) \sin^2 \phi).
\]  

To obtain information about the ground state and the low lying states we expand \( 2H/N \) around its minimum which is found at

\[
\sin \phi_0 = 0
\]

\[
\mu_0 = \begin{cases} 
0 & (\lambda \leq 1) \\
\pm \sqrt{1 - \lambda^{-2}} & (\lambda \geq 1)
\end{cases}
\]  

and the corresponding minimum values of \( H \) at

\[
H = \begin{cases} 
-\frac{N}{2} \lambda & (\lambda \leq 1) \\
-\frac{N}{2} \lambda^{-1} & (\lambda \geq 1)
\end{cases}
\]  

Expanding around the minimum reveals in fact all essential features. Keeping to lowest order terms we obtain around \( \mu = \phi = 0 \)

\[
K = -1 + \frac{1 + \lambda}{2} \phi^2 + \frac{1 - \lambda}{2} \mu^2 + \frac{\mu^4}{8} + \ldots
\]  

In this form, quantization is straightforward. Based on \[9\] we identify \( \phi \) and \( \mu \) with momentum and position, respectively, and \( 2/N \) with \( \hbar \), i.e. we use the usual canonical commutation relations for \( \mu \) and \( \phi \). With this identification the Hamiltonian \[8\] represents a quartic oscillator which behaves for \( \lambda < 1 \) basically like a traditional oscillator with a harmonic spectrum (for the lower states) \( E_k \sim k \sqrt{1 - \lambda^2} \), \( k = 1, 2, \ldots \). For \( \lambda > 1 \), the lower states must be determined at the minimum in \( \mu \) around the values \( \mu_0 \) as given in \[10\]. With \( \epsilon = \mu - \mu_0 \) the expansion of the Hamiltonian \( K \) yields, up to the constant given in \[10\]

\[
K = \frac{\lambda^2 - 1}{2} \lambda^2 + \frac{\phi^2}{\lambda^2}.
\]

For low-lying levels, the harmonic spectrum is again obtained from \[9\] with frequency \( \sqrt{2(\lambda^2 - 1)} \) well-known from previous work \[3, 19\].

To evaluate the average energy density, the features of which have been the object of recent work \[8\], we exploit, as we are in the semi-classical regime, the WKB relation

\[
S(2E_k/N) = 2\pi \left(k + \frac{1}{2}\right) \hbar.
\]

Here \( S(E) \) denotes the action corresponding to the Hamiltonian \( K \). By differentiation we obtain

\[
\Delta E = E_{k+1} - E_k = \frac{\pi \hbar N}{T(2E/N)} = \frac{2\pi}{T(E)}
\]  

where \( T(E) \) is the period of the orbit with respect to \( K \) as a function of energy and \( E \) is \((E_{k+1} + E_k)/2\). Use is made of \( T(E) = dS/dE \).

The last relation \[11\] contains virtually all basic information of the Lipkin model. Firstly, for \( \lambda < 1 \) nothing dramatic happens: the density of states merely changes smoothly as the period varies. An important fact should still be noted: the bosonization approach predicts, a constant \( \Delta E \) in this case. We see here that this does not hold over the entire energy range: the period varies smoothly over large energy scales, and so does the average energy spacing; it increases with the energy.

Secondly, for \( \lambda > 1 \) there is an energy, in fact the separatrix, being situated for \( K \) at the value \(-1\), where the period \( T(E) \) diverges. To estimate the average energy spacing in this region, we consider trajectories near the classical separatrix where they spend a long time near the unstable equilibrium, that is, in a region in which \( \phi \)
As the third major result, (11) provides the leading analytic behavior of the spectrum at the transition point \( \lambda = 1 \). Since the Hamiltonian is quartic for (classically) low energies, one can evaluate \( \Delta E \) using the formula for \( T(E) \) for a quartic oscillator valid for energies high up in the spectrum but still small with respect to \( N \). It is given by \( T(E) = \text{const.} \cdot E^{-1/4} \), from which
\[
\Delta E = \text{const.} \cdot (E/N)^{1/4}
\] (14)
follows, and it entails
\[
E_k \sim k^{4/3}/N^{1/3}.
\] (15)
The \( N^{-1/3} \) behavior has been obtained recently \(^7\) and the \( k^{4/3} \) behavior is confirmed numerically (see Fig.2). We stress the non-uniform nature of the limit played by the critical point \( \lambda = 1 \) when comparing (14) and (15).

It is obvious that the spectra of the two Hamiltonians \(^1\) and \(^2\) differ. This is due to the issue of ordering: in order to make sense of \(^2\), we must specify in which way we order \( \mu \) and \( \phi \) to obtain a self-adjoint operator. There is no unique prescription for this so that an unknown difference exists between the two Hamiltonians. It is, however, known that if care is taken, these errors are of order \( \hbar^2 \), that is, of order \( N^{-2} \). From this follows that we also expect the singularities to be different. We recall that they are associated with the critical point and the transitions for \( \lambda > 1 \). As the semi-classical treatment preserves these basic features it is expected that at least the qualitative pattern of the exceptional points remains. In fact, there is (i) the special feature at \( \lambda = 1 \) (an accumulation point for \( N \rightarrow \infty \) \(^8\)), (ii) a high density of EPs near to the separatrix \(^9\), that is for energies around \( E_c(\lambda) \), (iii) the absence of singularities near to real values for \( \lambda < 1 \), and for \( \lambda > 1 \) for energies sufficiently distant from \( E_c(\lambda) \).

Of interest in the semi-classical treatment is the behavior of the wave function at the phase transition for \( \lambda > 1 \). In accordance with the long dwelling time classically at the saddle point, there is the phenomenon of super-scarring for the wave function. In fact, this has been shown generally \(^2\) for the occurrence of such double wells. It arises for the specific values of \( k \) where the separatrix itself satisfies the WKB condition \(^10\). The wave function then shows a dramatic concentration at the saddle-point of the Hamiltonian (or at the maximum in configuration space): in particular, it can be shown that there is an interval, the length of which goes to zero with \( \hbar \), that is with \( 1/N \), in which the whole wave function is asymptotically concentrated. This is reflected in the eigenvectors of \(^1\) associated with the eigenvalues at the minimal gap: in the basis of \( J_z \) they become relatively concentrated in the first few components, that is around \( \mu = 0 \); while the number of substantial components increases with \( N \), the first twenty components exhaust the norm by about 50\% irrespective of \( N \). This has been tested for \( \lambda = 1.1, 1.5 \) and 2.0 where the transition occurs at \( k \approx N/120, N/16 \) and \( N/8 \), respectively.
The effect is rather significant – the more so the nearer \( \lambda \) to unity – in that the first twenty components of the neighboring wave functions contribute appreciably less to the total norm. In principle, wave functions can be directly determined semi-classically. As this is not the major focus of this paper we only outline the procedure.

Switching to the usual polar coordinates defined around the \( z \)-axis but keeping the definition of \( \mu \) as before, the renormalized Hamiltonian \( K \) reads

\[
K = \mu + \frac{\lambda}{2} (1 - \mu^2) \cos 2\phi. \tag{16}
\]

In these variables, the usual spherical harmonics are eigenfunctions of the operator \( \hat{\mu} = (2i/N) \partial / \partial \phi \) with an eigenvalue proportional to that of \( J_z \). The semi-classical eigenfunctions of \( \hat{\mu} \) in the eigenbasis of \( \hat{\mu} \) are then expressed as a function of the energy by means of standard WKB formulae for the one-dimensional Hamiltonian \( \hat{\mu} \). The expressions fail, of course, in the usual manner near the turning points of \( \hat{\mu} \).

To summarize: using the semi-classical version of the original model \( \hat{\mu} \) leads to \( \hat{\mu} \). The WKB approximation then yields the expansion \( \hat{\mu} \). Most information can then be extracted from \( \hat{\mu} \) being based on \( \hat{\mu} \). There is (i) the qualitative result about the phase transition occurring for \( \lambda > 1 \) as discussed in \( \hat{\mu} \). This includes the deviation from the strict equidistant level sequence for large \( N \) and for \( \lambda < 1 \) as well as the exponential (in \( N \)) separation of the degenerate levels for \( \lambda > 1 \). There is (ii) the expression \( \hat{\mu} \) for the level distance at the transition point for \( \lambda > 1 \); there is (iii) the finding \( \hat{\mu} \) at \( \lambda = 1 \). The apparent contrast of results (ii) and (iii) underlines once again the nonuniform nature of the large \( N \) limit at \( \lambda = 1 \). Note that all these results have been confirmed numerically. Additional results referring to the semi-classical wave function, in particular the super-scarring at the saddle point in phase space are presented. The qualitative behavior of the singularities of the spectrum (the exceptional points) seems to be preserved in the semi-classical approach.

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[17] The unusual convention used by this choice avoids the coincidence of the physically important South pole (see below) with the singular point of the polar coordinates.