(Compression Bases in Unital Groups

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Abstract

We introduce and launch a study of compression bases in unital groups. The family of all compressions on a compressible group and the family of all direct compressions on a unital group are examples of compression bases. In this article we show that the properties of the compatibility relation in a compressible group generalize to unital groups with compression bases.


Key Words and Phrases: normal sub-effect algebra, compatibility, unital group, compression, compressible group, compression base.

1. Normal Sub-Effect Algebras

If $E$ is an effect algebra [6], then a Mackey decomposition in $E$ of the ordered pair $(e, f) \in E \times E$ is an ordered triple $(e_1, f_1, d) \in E \times E \times E$ such that $e_1 \perp f_1$, $(e_1 \oplus f_1) \perp d$, $e = e_1 \oplus d$, and $f = f_1 \oplus d$. If there exists a Mackey decomposition in $E$ of $(e, f) \in E \times E$, then $e$ and $f$ are said to be Mackey compatible in $E$.

1.1 Definition Let $P$ be a sub-effect algebra of the effect algebra $E$ [6, Definition 2.6]. Then $P$ is a normal sub-effect algebra of $E$ iff, for all $e, f \in P$, if $(e_1, f_1, d) \in E \times E \times E$ is a Mackey decomposition in $E$ of $(e, f)$, then $d \in P$.

Suppose that $E$ is an effect algebra, $P$ is a sub-effect algebra of $E$, $e, f \in P$, and $(e_1, f_1, d) \in E \times E \times E$ is a Mackey decomposition of $(e, f)$ in $E$. Then
\[ e \text{ and } f \text{ are Mackey compatible in } E, \text{ but not necessarily in } P. \text{ However, if } P \text{ is a normal sub-effect algebra of } E, \text{ then } d \in P \text{ and, since } e_1 \oplus d = e, f_1 \oplus d = f, \text{ and } d, e, f \in P, \text{ it follows that } e_1, f_1 \in P, \text{ whence } (e_1, f_1, d) \in P \times P \times P \text{ is a Mackey decomposition in } P \text{ of } (e, f). \text{ Therefore, if } P \text{ is a normal sub-effect algebra of } E \text{ and } e, f \in P, \text{ then } e \text{ and } f \text{ are Mackey compatible in } E \text{ iff } e \text{ and } f \text{ are Mackey compatible in } P. \]

1.2 Example The center of an effect algebra \( E \) \(^7\) is a normal sub-effect algebra of \( E \).

Recall that \( G \) is a unital group with unit \( u \) and unit interval \( E \) iff \( G \) is a directed partially ordered abelian group \(^6\), such that \( u \in G^+ := \{ g \in G \mid 0 \leq g \}, \ E := \{ e \in G \mid 0 \leq e \leq u \}, \) and every element \( g \in G^+ \) can be written as \( g = \sum_{i=1}^{n} e_i \) with \( e_i \in E \) for \( i = 1, 2, ..., n \) \(^2\), p. 436. \( \) (The symbol := means “equals by definition.”)

Suppose that \( G \) is a unital group with unit \( u \) and unit interval \( E \). Then \( E \) is an effect algebra with unit \( u \) under the partially defined binary operation \( \oplus \) obtained by restriction of \( + \) on \( G \) to \( E \) \(^1\). \( \) We note that a sub-effect algebra \( P \) of \( E \) is a normal sub-effect algebra of \( E \) iff, for all \( e, f, d \in E \) with \( e + f + d \leq u \), we have \( e + d, f + d \in P \Rightarrow d \in P \).

1.3 Example Let \( \mathcal{H} \) be a Hilbert space. Then the additive abelian group \( G \) of all bounded self-adjoint operators on \( \mathcal{H} \), partially ordered in the usual way, is a unital group with unit \( 1 \). The unit interval \( \mathbb{E} \) in \( G \) is the standard effect algebra of all effect operators on \( \mathcal{H} \), and the orthomodular lattice \( \mathbb{P} \) of all projection operators on \( \mathcal{H} \) is a normal sub-effect algebra of \( \mathbb{E} \). \( \square \)

2. Retractions and Compressions

Let \( G \) be a unital group with unit \( u \) and unit interval \( E \). A retraction on \( G \) with focus \( p \) is defined to be an order-preserving group endomorphism \( J: G \rightarrow G \) with \( p = J(u) \in E \) such that, for all \( e \in E \), \( e \leq p \Rightarrow J(e) = e \). A retraction \( J \) on \( G \) with focus \( p \) is called a compression on \( G \) iff \( J(e) = 0 \Rightarrow e \leq u - p \) holds for all \( e \in E \) \(^3\).

The unital group \( G \) always admits at least two compressions, namely the zero mapping, \( g \mapsto 0 \) for all \( g \in G \) and the identity mapping \( g \mapsto g \) for all \( g \in G \). Conversely, the only retraction on \( G \) with focus \( 0 \) is the zero mapping, and the only retraction on \( G \) with focus \( u \) is the identity mapping. Suppose \( J \) is a retraction with focus \( p \) on \( G \). Then, \( J \) is idempotent \( (\text{i.e., } J \circ J = J) \)
and \( J(p) = p \). Also, for all \( e \in E \), \( e \leq u - p \Rightarrow J(e) = 0 \) and, if \( J \) is a compression, then \( e \leq u - p \Leftrightarrow J(e) = 0 \) [3].

2.1 Lemma Let \( G \) be a unital group with unit \( u \) and unit interval \( E \). Suppose that \( J \) is a compression on \( G \) with focus \( p \), and \( J' \) is a retraction on \( G \) with focus \( u - p \). Then, for all \( g \in G^+ \), \( J(g) = 0 \Leftrightarrow J'(g) = g \).

Proof Let \( e \in E \). As \( 0 \leq e \leq u \), we have \( 0 \leq J'(e) \leq J'(u) = u - p \), whence \( J(J'(e)) = 0 \). Since \( E \) generates \( G \) as a group and \( J \circ J' \) is an endomorphism on \( G \), we have \( J(J'(g)) = 0 \) for all \( g \in G \). As \( J \) is a compression with focus \( p \), it follows that \( J(e) = 0 \Rightarrow e \leq u - p \Rightarrow J'(e) = e \). Now let \( g \in G^+ \) and write \( g = \sum_{i=1}^{n} e_i \) with \( e_i \in E \) for \( i = 1, 2, ..., n \). If \( J(g) = 0 \), then \( \sum_{i=1}^{n} J(e_i) = 0 \) and, since \( 0 \leq J(e_i) \) for \( i = 1, 2, ..., n \), it follows that \( J(e_i) = 0 \) for \( i = 1, 2, ..., n \), whence \( J'(e_i) = e_i \) for \( i = 1, 2, ..., n \), and therefore \( J'(g) = g \). Conversely, if \( J'(g) = g \), then \( J(g) = J(J'(g)) = 0 \). \( \square \)

A compressible group is defined to be a unital group \( G \) such that (1) every retraction on \( G \) is uniquely determined by its focus, and (2) if \( J \) is a retraction on \( G \), there exists a retraction \( J' \) on \( G \) such that, for all \( g \in G^+ \), \( J(g) = 0 \Leftrightarrow J'(g) = g \) and \( J'(g) = 0 \Leftrightarrow J(g) = g \) [3]. Definition 3.3]. If \( G \) is a compressible group, then an element \( p \in G \) is called a projection if it is the focus of a retraction on \( G \). Suppose that \( G \) is a compressible group and \( P \) is the set of all projections in \( G \). Then every retraction on \( G \) is a compression, and if \( p \in P \), then the unique retraction (hence compression) on \( G \) with focus \( p \) is denoted by \( J_p \). The set \( P \) is a sub-effect algebra of \( E \) and, in its own right, it forms an orthomodular poset (OMP) [2] Corollary 5.2 (iii)].

2.2 Example Let \( A \) be a unital C\(^*\)-algebra and let \( G \) be the additive group of all self-adjoint elements in \( A \). Then \( G \) is a unital group with unit 1 and positive cone \( G^+ = \{ aa^* \mid a \in A \} \). The unital group \( G \) is a compressible group with \( P = \{ p \in G \mid p = p^2 \} \) and, if \( p \in P \), then \( J_p(g) = pgp \) for all \( g \in G \) [3]. \( \square \)

2.3 Theorem Let \( G \) be a compressible group with unit \( u \) and unit interval \( E \). Then: (i) \( P \) is a normal sub-effect algebra of \( E \). (ii) If \( p, q, r \in P \) with \( p + q + r \leq u \), then \( J_{p+r} \circ J_{q+r} = J_r \).

Proof (i) By [2] Corollary 5.2 (ii)], \( P \) is a sub-effect algebra of \( E \). Suppose \( e, f, d \in E \), \( e + f + d \leq u \), \( e + d \in P \), \( f + d \in P \), and define \( J := J_{e+d} \circ J_{f+d} \). Then \( J : G \rightarrow G \) is an order-preserving endomorphism and \( J(u) = \)
\[ J_{e+d}(J_{f+d}(u)) = J_{e+d}(f + d) = J_{e+d}(f) + J_{e+d}(d). \] But, \( e + f + d \leq u \), so \( f \leq u - (e + d) \), and \( d \leq e + d \), whence \( J(u) = 0 + d = d \). Suppose \( h \in E \) with \( h \leq d \). Then \( h \leq e + d, f + d \), and it follows that \( J(h) = J_{e+d}(J_{f+d}(h)) = J_{e+d}(h) = h \). Therefore \( J \) is a retraction with focus \( d \), so \( d \in P \).

(ii) If \( p, q, r \in P \) and \( p + q + r \leq u \), then by the proof of (i) above with \( e \) replaced by \( p \), \( f \) replaced by \( q \), and \( d \) replaced by \( r \), we have \( J_{p+r} \circ J_{q+r} = J_r \).

3. Compression Bases

By Theorem 2.3, the notion of a “compression base,” as per the following definition, generalizes the family \((J_p)_{p \in P}\) of compressions in a compressible group.

3.1 Definition Let \( G \) be a unital group with unit interval \( E \). A family \((J_p)_{p \in P}\) of compressions on \( G \), indexed by a normal sub-effect algebra \( P \) of \( E \), is called a compression base for \( G \) iff (i) each \( p \in P \) is the focus of the corresponding compression \( J_p \), and (ii) if \( p, q, r \in P \) and \( p + q + r \leq u \), then \( J_{p+r} \circ J_{q+r} = J_r \).

The conditions for a unital group to be a compressible group are quite strong and they rule out many otherwise interesting unital groups. On the other hand, the notion of a unital group \( G \) with a specified compression base \((J_p)_{p \in P}\) is very general, yet most of the salient properties of projections and compressions for a compressible group generalize, mutatis mutandis, to the elements \( p \in P \) and to the compressions \( J_p \) in the compression base for \( G \).

3.2 Example A retraction \( J \) on the unital group \( G \) is direct iff \( J(g) \leq g \) for all \( g \in G^+ \) [Definition 2.6]. For instance, the zero mapping \( g \mapsto 0 \) and the identity mapping \( g \mapsto g \) for all \( g \in G \) are direct compressions on \( G \). Let \( P \) be the set of all foci of direct retractions on \( G \). Then \( P \) is a sub-effect algebra of the center of \( E \). Also, if \( p \in P \), there is a unique retraction \( J_p \) on \( G \) with focus \( p \), and \( J_p \) is a compression. Furthermore, the family \((J_p)_{p \in P}\) is a compression base for \( G \).

3.3 Standing Assumption In the sequel, we assume that \( G \) is a unital group with unit \( u \) and unit interval \( E \) and that \((J_p)_{p \in P}\) is a compression base for \( G \).

3.4 Theorem \( P \) is an orthomodular poset and, if \( p \in P \) and \( g \in G^+ \), then \( J_p(g) = 0 \Leftrightarrow J_{u-p}(g) = g \).
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ibility in a compressible group [2, Definition 4.1] carries over, as follows, to

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le to a unital group with a compression base.

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4.3 Theorem Let \( p, q \in P \). Then the following conditions are mutually equivalent: (i) \( J_p \circ J_q = J_q \circ J_p \). (ii) \( J_p(q) = J_q(p) \). (iii) \( J_p(q) \leq q \). (iv) \( p \) is Mackey compatible with \( q \) in \( E \). (v) \( p \) is Mackey compatible with \( q \) in \( P \). (vi) \( \exists r \in P, J_p \circ J_q = J_r \). (vii) \( J_p(q) \in P \). (viii) \( qCp \)

Proof (i) \( \Rightarrow \) (ii). If (i) holds, then \( J_p(q) = J_p(J_q(u)) = J_q(J_p(u)) = J_q(p) \).

(ii) \( \Rightarrow \) (iii). If (ii) holds, then \( J_p(q) = J_q(p) \leq q \).

(iii) \( \Rightarrow \) (iv). Let \( r := J_p(q) \) and assume that \( r \leq q \). Then \( 0 \leq r \leq p, q \), whence \( e := p - r \in E \) and \( f := q - r \in E \) with \( e + r = p \) and \( f + r = q \). As \( J_p(f) = J_p(q-r) = r-r = 0 \), we have \( f \leq u-p \), whence \( e+f+r = f+p \leq u \), and it follows the \( p \) is Mackey compatible with \( q \) in \( E \).

(iv) \( \Rightarrow \) (v). As \( P \) is a normal sub-effect algebra of \( E \), we have (iv) \( \Rightarrow \) (v).

(v) \( \Rightarrow \) (vi). If (v) holds, there exist \( e, f, r \in P \) with \( e+f+r \leq u, p = e+r \) and \( q = f + r \). Therefore, by Definition 3.1 (ii), \( J_p \circ J_q = J_{e+r} \circ J_{f+r} = J_r \).

(vi) \( \Rightarrow \) (vii). Suppose that \( r \in P \) and \( J_p \circ J_q = J_r \). Then \( J_p(q) = J_p(J_q(u)) = J_r(u) = r \in P \).

(vii) \( \Rightarrow \) (viii). Assume (vii) and let \( r := J_p(q) \). Then \( J_r(q) \leq r \leq p \), so \( 0 \leq r - J_r(q) \). Thus, by Lemma 3.5, \( r - J_r(q) = r - (J_r \circ J_p)(q) = r - J_r(J_p(q)) = r - J_r(r) = r-r = 0 \), i.e., \( r = J_r(q) \). Therefore, \( J_r(u-q) = r-r = 0 \), so \( u-q \leq u-r \), i.e., \( r \leq q \), and it follows from Lemma 4.2 that \( pCq \).

(viii) \( \Rightarrow \) (i). Assume that \( qCp \). Then, by Lemma 4.2, \( J_p(q) \leq q \), so (iii) holds. We have already shown that (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v), so there exist \( e, f, r \in P \) with \( e+f+r \leq u, p = e+r \), and \( q = f + r \). Therefore, by Definition 3.1 (ii), \( J_p \circ J_q = J_{e+r} \circ J_{f+r} = J_r \), and \( J_p \circ J_q = J_p \).

Because conditions (i), (ii), (iv), and (v) in Theorem 4.3 are symmetric in \( p \) and \( q \), so are conditions (iii), (vi), (vii), and (viii). In particular, for \( p, q \in P \), we have \( pCq \Leftrightarrow qCp \).

4.4 Corollary Let \( p, q \in P \) and suppose that \( pCq \). Then \( J_q(p) = J_p(q) = p \wedge q \) is the greatest lower bound of \( p \) and \( q \) both in \( E \) and in \( P \), and \( J_p \circ J_q = J_q \circ J_p = J_{p \wedge q} \).

Proof Suppose that \( p, q \in P \) and \( pCq \). By Theorem 4.3, there exists \( r \in P \) with \( J_p \circ J_q = J_q \circ J_p = J_r \). Thus, \( r = J_p(J_q(u)) = J_q(q) = J_q(p) \leq p, q \).

If \( e \in E \) with \( e \leq p, q \), then \( e = J_p(J_q(e)) = J_r(e) \leq r \), so \( r \) is the greatest lower bound of \( p \) and \( q \) in \( E \), hence also in \( P \).

4.5 Theorem Let \( v \in P \) and define \( H := J_v(G) \), \( E_H := \{ e \in E \mid e \leq v \} \), and \( P_H := \{ q \in P \mid q \leq v \} \). For each \( q \in P_H \), let \( J_q^H \) be the restriction of \( J_q \)
to $H$. Then: (i) With the induced partial order, $H = \{ h \in G \mid h = J_v(h) \}$ is a unital group with unit $v$ and unit interval $H \cap E = E_H$. (ii) $H \cap P = P_H$, and if $q \in P_H$, then $J^H_q$ is a compression on $H$. (iii) $P_H$ is a normal sub-effect algebra of $E_H$. (iv) $(J^H_q)_{q \in P_H}$ is a compression base for $H$.

Proof (i) By [2, Lemma 2.4], $H$ is a unital group with unit $v$ and unit interval $H \cap E$. As $J_v$ is idempotent, $H = \{ h \in G \mid h = J_v(h) \}$. Thus, for $e \in E$, $e \leq v \iff e = J_v(e) \iff e \in H$, whence $H \cap E = \{ e \in E \mid e \leq v \}$.

(ii) As $P \subseteq E$, we have $H \cap P = P_H$. If $h \in H$ and $q \in P_H$, then by Lemma 3.5, $J_q(h) = J_v(J_q(h)) \in H$. Therefore $J^H_q : H \to H$ is an order-preserving group endomorphism, and by Lemma 3.5 again, $J^H_q(v) = J_q(v) = q$. Also, if $e \in E_H$ with $e \leq q$, then $J^H_q(e) = J_q(e) = e$, so $J^H_q$ is a retraction on $H$. Suppose $e \in E_H$ and $J^H_q(e) = 0$. Then $e \leq u - q$, so $e + q \leq u$. By [3, Lemma 2.3 (iv)], $v$ is a principal element of $E$, hence, since $0 \leq e, q \leq v$, it follows that $e + q \leq v$, i.e., $e \leq v - q$. Therefore, $J^H_q$ is a compression on $H$.

(iii) Suppose $e, f, d \in E_H$, $e + f + d \leq v$, and $e + d, f + d \in P_H$. Then $e, f, d \in E$, $e + f + d \leq v \leq u$, and $e + d, f + d \in P$. As $P$ is a normal sub-effect algebra of $E$, it follows that $d \in P$. But $d \leq v$, so $d \in P_H$.

(iv) Suppose $s, t, r \in P_H$ with $s + t + r \leq u$. Then $s, t, r \in P$ with $s + t + r \leq u$, whence $J_s \circ J_t \circ J_r = J_{s + t + r} = J_v$, and it follows that $J^H_{s + t + r} = J^H_v$.

4.6 Theorem Let $v \in P$ and define $C := C(v)$. For each $s \in C \cap P$, let $J^C_s$ be the restriction of $J_s$ to $C$. Then: (i) With the induced partial order, $C$ is a unital group with unit $v$ and unit interval $C \cap E = \{ e + f \mid e, f \in E, e \leq v, f \leq u - v \}$. (ii) If $s \in C \cap P$, then $J^C_s$ is a compression on $C$. (iii) $C \cap P$ is a normal sub-effect algebra of $C \cap E$. (iv) $(J^C_s)_{s \in C \cap P}$ is a compression base for $C$.

Proof Part (i) follows from [2, Lemma 4.2 (iv)], part (iii) is obvious, and part (iv) is easily confirmed once part (ii) is proved. To prove part (ii), assume that $g \in C = C(v)$ and $s \in P \cap C$. Then, by Lemma 3.5, $J^C_s(g) = J_s(J_v(g) + J_u-v(g)) = J_s(J_v(g)) + J_s(J_u-v(g)) = J_v(J_s(g)) + J_u-v(J_s(g))$, so $J^C_s(g) = J_s(g) \in C(v) = C$. Therefore $J^C_s : C \to C$ is an order-preserving group endomorphism, hence it is obviously a compression on $C$.

4.7 Definition If $C$ and $W$ are unital groups with units $u$ and $w$, respectively, and if $(J^C_q)_{q \in Q}$ and $(J^W_t)_{t \in T}$ are compression bases in $C$ and $W$, respectively, then an order-preserving group homomorphism $\phi : C \to W$ is called a morphism of unital groups with compression bases if $\phi(u) = w, \phi(Q) \subseteq T$, and $J^W_{\phi(q)} \circ \phi = \phi \circ J^C_q$ for all $q \in Q$. 

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We omit the straightforward proof of the following theorem.

4.7 Theorem Suppose \( v \in P \) and define \( H := J_v(G) \), \( K := J_{u-v}(G) \), and \( C := C(v) \). Organize \( H \), \( K \), and \( C \) into unital groups with compression bases \( (J^H_q)_{q \in P_H} \), \( (J^K_r)_{r \in P_K} \), and \( (J^C_s)_{s \in C \cap P} \), respectively, as in Theorems 4.5 and 4.6. Let \( \eta \) be the restriction to \( C \) of \( J_v \) and let \( \kappa \) be the restriction to \( C \) of \( J_{u-v} \). Then \( \eta: C \to H \) and \( \kappa: C \to K \) are surjective morphisms of unital groups with compression bases and, in the category of unital groups with compression bases, \( \eta \) and \( \kappa \) provide a representation of \( C \) as a direct product of \( H \) and \( K \).

In subsequent papers we shall prove that all of the major results in [2, 3, 4, 5] can be generalized to unital groups with compression bases.

References


