The Schwinger Representation of a Group: Concept and Applications

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The concept of the Schwinger Representation of a finite or compact simple Lie group is set up as a multiplicity-free direct sum of all the unitary irreducible representations of the group. This is abstracted from the properties of the Schwinger oscillator construction for SU(2), and its relevance in several quantum mechanical contexts is highlighted. The Schwinger representations for SU(2), SO(3) and SU(n) for all n are constructed via specific carrier spaces and group actions. In the SU(2) case connections to the oscillator construction and to Majorana's theorem on pure states for any spin are worked out. The role of the Schwinger Representation in setting up the Wigner-Weyl isomorphism for quantum mechanics on a compact simple Lie group is brought out.

I. INTRODUCTION

The Schwinger construction of the Lie algebra of SU(2) in terms of the annihilation and creation operators of two independent quantum mechanical harmonic oscillators has been used in a wide variety of contexts [1]. These include the physics of strongly correlated systems [2], quantum optics of two mode radiation fields [3], analysis of partially coherent classical Gaussian Schell model beams [4], extension to all three-dimensional Lie algebras and analysis of both classical and q-deformed versions [5], applications in the context of quantum computing [6], and a new approach to the spin-statistics theorem [7], to mention only a few. This is in addition to the elegance and relative ease with which many results belonging to the body of the quantum theory of angular momentum can be derived.

Two important features of the Schwinger construction are economy and completeness. By these we mean that the unitary representation (UR) of SU(2) that is obtained by exponentiating the generators contains upon reduction every unitary irreducible representation (UIR) of SU(2) exactly once, omitting none. The feature of economy, i.e., simple reducibility, is lost when one considers the natural generalisation of the Schwinger construction from SU(2) to SU(3): indeed in a minimal oscillator construction that ensures completeness, every SU(3) UIR occurs with infinite multiplicity [8]. An explicit construction of a complete and multiplicity-free representation of SU(3), via harmonic functions on the sphere $S^5$, and oscillator construction of the same representation are given in [8].

In the present work we abstract the two special features of the Schwinger SU(2) construction mentioned above, and make them the basis of the definition of what we shall call the Schwinger Representation (SR) for an interesting class of groups. The groups we shall mainly consider are compact Lie groups with simple Lie algebras, while our considerations remain meaningful for finite groups as well. Both of these are of considerable importance in the general framework of quantum mechanics. The precise definition of the SR is given in the next Section. Here we may stress that on account of the two properties of economy and completeness it may be regarded as a ‘generating representation’ of the group concerned. While these two features are retained, what is given up in general is any elementary construction in terms of oscillator operators.

A related concept of ‘model representations’ has been introduced and studied by Gelfand et al. [10]. However the focus there has been on the families of classical noncompact simple Lie groups, and moreover on the nonunitary finite dimensional representations of these groups. As mentioned above, our motivations lie in possible applications of our concept in problems arising within the framework of quantum mechanics, where unitarity of group representations has a special significance.
The material of this paper is arranged as follows. In Section II we introduce the notion of the SR of a group and discuss its consequences for compact Lie groups and non compact Abelian groups $R^n$. Further, we show that while the original Schwinger SU(2) representation, and that for SU(3) permit interpretation in terms of particular induced representations, this ceases to be the case for SU(n) beyond $n = 3$. In section III, we discuss the SU(2) SR in a manner that anticipates generalisation later and bring out the salient features of the carrier space thus obtained. Section IV contains application of the construction developed in Section III to recover the Schwinger oscillator construction for SU(2) and Majorana’s representation for a spin $j$ system by sets of points on $S^2$. In section V, we develop the SO(3) SR and contrast it with the way this is done conventionally. In Section VI we show how the formalism developed in Section III for the SU(2) case naturally leads to the SU(n) SR for any $n$. The significance of the SR in the context of the Wigner-Weyl isomorphism for Lie groups developed by the present authors is brought out in Section VII. Section VIII contains concluding remarks and some open questions which merit further investigation.

Throughout this paper we shall adopt the usual quantum mechanical usage and denote unitary Lie group representation generators by hermitian operators.

II. THE SCHWINGER REPRESENTATION OF A GROUP

We consider a compact Lie group $G$ with simple Lie algebra $G$. (However many of the ideas developed below are meaningful also for finite groups.) Then, as is well known, every representation of $G$, and in particular every irreducible representation, may be assumed to be unitary. We shall use a notation for the UIR’s which generalises the notation familiar for SU(2) and SO(3) in quantum angular momentum theory. We label the various mutually inequivalent UIR’s of $G$ by a symbol or index $j$, standing in general for a collection of independent quantum numbers. (For SU(2), $j$ is a single numerical label taking values $0, 1/2, 1, 3/2, \ldots$). Within the $j^{th}$ UIR, realised on a Hilbert space $\mathcal{H}(j)$ of finite dimension $N_j$, we shall write $D_{mn}(g)$ for the unitary matrices representing elements $g \in G$ in a suitable orthonormal basis. The row and column indices $m', m$ are generalisations of the magnetic quantum number in angular momentum theory; like $j$, they too in general stand for collections of independent quantum numbers. (For SU(2), $N_j = 2j + 1$ and $m = j, j-1, \ldots, -j$). In terms of a normalised translation invariant volume element $dg$ and associated invariant delta function $\delta(g)$ on $G$, these matrices obey the orthogonality and completeness conditions

$$\int_G \, dg \, D_{mn}^j(g) \, D_{m'n'}^j(g)^* = \delta_{jj'}\delta_{mm'}\delta_{nn'}/N_j,$$

$$\sum_{jmn} N_j \, D_{mn}^j(g) \, D_{m'n'}(g')^* = \delta (g^{-1}g'). \tag{2.1}$$

We now define the SR of $G$ to be the simply reducible UR

$$D_0 = \sum_j \oplus D^j \tag{2.2}$$

acting on the direct sum Hilbert space

$$\mathcal{H}_0 = \sum_j \oplus \mathcal{H}^{(j)}, \tag{2.3}$$

the $j^{th}$ UIR $D^j$ acting on the subspace $\mathcal{H}^{(j)}$ of $\mathcal{H}_0$. Thus every UIR $D^j$ of $G$ occurs exactly once in this UR. For the Lie group case, $\mathcal{H}_0$ is of infinite dimension; while if $G$ is a finite group, $\mathcal{H}_0$ is of finite dimension. We can set up orthonormal bases within each $\mathcal{H}^{(j)}$, constituting all together an orthonormal basis for $\mathcal{H}_0$, as follows:

$$\mathcal{H}^{(j)} = \text{Sp}(|jm\rangle | j \text{ fixed, } m \text{ varying}),$$

$$\mathcal{H}_0 = \text{Sp}(|jm\rangle | j \text{ varying, } m \text{ fixed}),$$

$$\langle j'm'|jm \rangle = \delta_{jj'}\delta_{m'm}, \tag{2.4}$$

so that we have

$$\langle j'm'|D_0(g)|jm \rangle = \delta_{jj'} D_{m'm}^j(g). \tag{2.5}$$

We give now some immediate consequences of this definition, as well as some familiar examples.
(i) If $G$ is abelian, each UIR is one dimensional, $N_j = 1$, and the $SR$ is the same as the regular representation acting in the usual way (by left or by right translations which coincide) on square integrable functions on $G$. For nonabelian $G$ the $SR$ is always ‘leaner’ than the regular representation since there are always some UIR’s with $N_j > 1$. From this point of view, the case of simple $G$ is the exact opposite of abelian $G$: no subgroup is normal in the former, every one is normal in the latter. Thus for simple $G$ we expect qualitatively that the $SR$ will be ‘much smaller’ than the regular representation.

(ii) When $G$ is a compact simple Lie group, we can characterize the $SR$ in an interesting way. In every UR of $G$, the generators are hermitian operators obeying the commutation relations corresponding to the Lie algebra $G$ of $G$. In any individual UIR, apart from the commutation relations, the generators also obey some algebraic (symmetric polynomial) relations characteristic of that UIR. In $D_0$ however no such algebraic relations are obeyed since every UIR is present. In other words the generators of the $SR D_0$ on $H_0$ provide in a sense a minimal faithful representation of the enveloping algebra of $G$: they are not subject to any algebraic relations beyond the commutation relations.

(iii) The simple reducibility of $D_0$ implies that the commutant of $D_0$ is particularly simple: any operator $A$ on $H_0$ commuting with $D_0(g)$ for all $g$ is necessarily block diagonal, with each entry being some numerical multiple of the unit operator:

$$A \ D_0(g) = D_0(g)A, \text{ all } g \in G \Rightarrow$$

$$A = \sum_j \oplus A_j,$$

$$A_j = c_j 1_j,$$

$$1_j = \text{ unit operator on } H^{(j)}.$$  \hspace{1cm} (2.6)

This follows from Schur’s Lemma and the Wigner-Eckart theorem. Thus this commutant is commutative.

(iv) The $SR$ concept can be extended heuristically to the noncompact case $G = R^n$, leading to an interesting perspective relevant to quantum mechanics. For a quantum system with Cartesian configuration space $Q = R^n$, corresponding to $n$ canonical Heisenberg pairs of hermitian operators $\hat{q}_r, \hat{p}_r, r = 1, 2, \ldots, n$, among whom the only nonzero commutators are

$$[\hat{q}_r, \hat{p}_s] = i \delta_{rs},$$ \hspace{1cm} (2.7)

the Stone-von-Neumann theorem tells us that up to unitary equivalence there is only one irreducible representation of these relations. The Hilbert space can be described via coordinate space wave functions $\psi(q)$ or via momentum space wave functions $\phi(p)$:

$$H = L^2(R^n) = \left\{ \psi(q) \in C | \| \psi \|^2 = \int_{R^n} d^n q |\psi(q)|^2 < \infty \right\}$$

$$= \left\{ \phi(p) \in C | \| \phi \|^2 = \int_{R^n} d^n p |\phi(p)|^2 < \infty \right\},$$

$$\phi(p) = (2\pi)^{-n/2} \int_{R^n} d^n q e^{-i p \cdot q} \psi(q),$$

$$\| \phi \| = \| \psi \|,$$

$$(\hat{q}_r \psi) (q) = q_r \psi(q), \quad (\hat{p}_r \psi) (q) = -i \frac{\partial}{\partial q_r} \psi(q);$$

$$(\hat{q}_r \phi)(p) = i \frac{\partial}{\partial p_r} \phi(p), \quad (\hat{p}_r \phi)(p) = p_r \phi(p).$$ \hspace{1cm} (2.8)

In this context, these operator actions are usually viewed as providing us after exponentiation with the (unique) Stone-von Neumann UIR of the $(2n + 1)$ dimensional nonabelian Heisenberg-Weyl group of phase space displacements, the generators being $\hat{q}_r, \hat{p}_r$ and the unit operator on $H$. However the situation can now be viewed in an alternative manner: each real numerical $n$-dimensional momentum vector $p$ corresponds to a one-dimensional UIR of the abelian group of configuration space translations $G = R^n : q \rightarrow q + g$; as $p$ ranges over all of momentum space $R^n$, each such UIR is present in $H$ exactly once. (Another way of expressing this is the statement that the Cartesian momenta $\hat{p}_r$ form a complete commuting set). Thus we can view the kinematics of $n$-dimensional Cartesian quantum mechanics.
in two ways: we have the unique Stone-von Neumann UIR of the \((2n + 1)\) dimensional nonabelian Heisenberg-Weyl group, or equally well we have the \(SR\) of the abelian group \(G = R^n\) of configuration space displacements.

(v) The original Schwinger oscillator construction of \(SU(2)\) leads upon exponentiation to the \(SR\) of \(SU(2)\) in the sense defined above. (The \(SU(2)\) notational details will be taken up in Section III). Each UIR of \(SU(2)\) for \(j = 0, 1/2, 1, \ldots\) appears exactly once. In the case of \(SO(3) = SU(2)/Z_2\), the distinct UIR’s are usually labelled by \(\ell = 0, 1, 2, \ldots\); these are the integer \(j\) UIR’s of \(SU(2)\). The familiar UIR of \(SO(3)\) on square integrable functions on \(S^2\), with the simple geometric action of \(SO(3)\) elements, is a realisation of the \(SR\) of \(SO(3)\). The reduction into UIR’s in a multiplicity-free manner is achieved, as is familiar, by using the orthonormal basis provided by the spherical harmonics on \(S^2\).

In Sections III and IV we describe other ways of constructing the \(SR\)’s of \(SU(2)\) and \(SO(3)\) respectively.

After these immediate properties and examples, we make some general remarks. Purely from the representation theory point of view, the \(SR\) \(D_0\) of \(G\) is completely defined by the statement in \((2.2)\) of its UIR content. However, from the point of view of possible applications in the framework of quantum mechanics, considerable interest attaches to various ways in which this UR may be realised, with corresponding carrier spaces and group actions. A general way to construct UR’s of a group \(G\) is by the process of induction starting from UIR’s of some subgroup \(H\). Let \(H \subset G\) be some subgroup, and \(D_0\) be a UIR of \(H\). Then by an elegantly simple construction one arrives at an induced UR \(D^{(ind,H)}_{UH}\) of \(G\): the notation indicates the roles of \(H, D_0\) and the inducing procedure. Once this UR of \(G\) has been obtained, one can ask for its UIR content. Here the main result is the reciprocity theorem. The UR \(D^{(ind,H)}_{UH}\) of \(G\) contains the UIR \(D^j\) of \(G\) as many times as \(D^j\) contains \(D_0\) upon restriction from \(G\) to \(H\). One can now ask whether the \(SR\) of \(G\) arises as a particular induced UR corresponding to some carefully chosen \(H\) and \(D_0\).

In the case of \(SU(2)\), a natural subgroup choice is \(H = U(1)\) generated by \(J_3\) in the usual notation, with eigenvalues being the magnetic quantum number \(m\). However as a quick analysis using the reciprocity theorem shows, we find the result:

\[
D_0\text{ for } SU(2) = D^{(ind,0)}_{U(1)} \oplus D^{(ind,1/2)}_{U(1)}
\]

(Here the superscripts 0 and 1/2 on the right hand side indicate the \(m\) values determining the \(U(1)\) UIR’s used in the inducing process). The first term on the right accounts for all the integer \(j\) UIR’s of \(SU(2)\), while the second term accounts for the remaining half odd integer \(j\) UIR’s. In the case of \(SO(3)\) we may choose \(H = SO(2)\) and then we have

\[
D_0\text{ for } SO(3) = D^{(ind,0)}_{SO(2)}
\]

So in this case the \(SR\) is indeed a particular induced representation.

For \(SU(3)\) this situation continues to hold \([12]\). Each UIR of \(SU(3)\) is labelled by a pair of independent nonnegative integers, as \((p, q)\). It is a fact that every UIR \((p, q)\) contains the trivial (one-dimensional) UIR of the canonical \(SU(2)\) subgroup exactly once. Thus from the reciprocity theorem we see that

\[
D_0\text{ for } SU(3) = D^{(ind,0)}_{SU(2)},
\]

where the zero in the superscript on the right stands for the UIR of \(SU(2)\).

However this trend does not continue for \(SU(n)\) beyond \(n = 3\) \([13]\). In fact we show in Section VI that the \(SR\) of \(SU(n)\) for \(n \geq 4\) is not an induced UR corresponding to any choice of UIR of the canonical \(SU(n-1)\) subgroup of \(SU(n)\). There is thus a need to develop an alternative method to construct the \(SR\) of \(SU(n)\) which works uniformly for all \(n \geq 2\). This will be done for \(SU(2)\) in the next Section, for \(SO(3)\) in Section V, and for \(SU(n)\) in Section VI.

**III. THE SU(2) SCHWINGER REPRESENTATION**

To set notations we begin by recalling the defining UIR and Euler angle parametrisation of \(SU(2)\) \([14]\). An element \(g \in SU(2)\) is a \(2 \times 2\) unitary unimodular matrix

\[
g = \begin{pmatrix} \xi & -\eta^* \\ \eta & \xi^* \end{pmatrix}, \quad \xi, \eta \in \mathbb{C},
\]

\[
|\xi|^2 + |\eta|^2 = 1.
\]

(3.1)
The hermitian generators are \( \frac{1}{2}\sigma_r \), where \( \sigma_r \) for \( r = 1, 2, 3 \) are the Pauli matrices. The commutation relations are

\[
\left[ \frac{1}{2}\sigma_r, \frac{1}{2}\sigma_s \right] = i \epsilon_{rst} \frac{1}{2}\sigma_t. \tag{3.2}
\]

In the Euler angle parametrisation we express \( g \) as a product of three factors:

\[
g(\alpha, \beta, \gamma) = e^{-i\alpha\sigma_3/2} e^{-i\beta\sigma_2/2} e^{-i\gamma\sigma_3/2}
\]

\[
= \left( e^{-i(\alpha+\gamma)/2} \cos \beta/2 - e^{-i(\alpha-\gamma)/2} \sin \beta/2 \right)
\]

\[
\left( e^{i(\alpha-\gamma)/2} \sin \beta/2 e^{i(\alpha+\gamma)/2} \cos \beta/2 \right),
\]

i.e. \( \xi = e^{-i(\alpha+\gamma)/2} \cos \beta/2 \), \( \eta = e^{i(\alpha-\gamma)/2} \sin \beta/2 \). \tag{3.3}

The ranges for \( \alpha, \beta, \gamma \) are determined by the condition that (except possibly on a set of measure zero) each element \((3.1)\) must occur just once. Then one finds \[15\]:

\[
0 \leq |\xi| \leq 1 \Leftrightarrow 0 \leq \beta \leq \pi;
\]

\[
0 \leq \arg \xi, \arg \eta \leq 2\pi \Leftrightarrow 0 \leq \alpha \leq 2\pi, \ 0 \leq \gamma \leq 4\pi. \tag{3.4}
\]

The elements \( g(0, 0, \gamma) \) for \( 0 \leq \gamma \leq 4\pi \) constitute the diagonal \( U(1) \) subgroup of \( SU(2) \). Since \( \alpha \) and \( \beta \) can be interpreted as azimuthal and polar angles on \( S^2 \), the form for \( g(\alpha, \beta, \gamma) \) in \((3.3)\) is in manifest agreement with the statement \( SU(2)/U(1) = S^2 \). The normalised invariant volume element is

\[
dg = d\alpha \sin \beta d\beta \cdot d\gamma/16\pi^2. \tag{3.5}
\]

The unitary representation matrices in the \( j \)th UIR are, as is familiar \[16\]:

\[
< jm| D^j(\alpha, \beta, \gamma)|jn> = D^j_{mn}(\alpha, \beta, \gamma)
\]

\[
= e^{-im\alpha-in\gamma}d^j_{mn}(\beta) \tag{3.6}
\]

with \( d^j_{mn}(\beta) \) real. In verifying the orthogonality relation

\[
\int_{SU(2)} dg \ D^j_{mn}(\alpha, \beta, \gamma) D^{j'}_{m'n'}(\alpha, \beta, \gamma)^* = \delta_{jj'}\delta_{nn'}\delta_{mm'}/(2j+1), \tag{3.7}
\]

it is necessary to keep in mind the asymmetry between \( \alpha \) and \( \gamma \) in \((3.1)\). Thus it is simplest to first carry out the \( \gamma \) integration producing the factor \( \delta_{nn'} \). This implies that \( j' - j \) and \( m' - m \) are both integral. Then doing the \( \alpha \) integration second leads to \( \delta_{nn'} \); and finally the \( \beta \) integration produces \( \delta_{jj'} \).

The two regular representations of \( SU(2) \) act on the Hilbert space \( \mathcal{H} \) of square integrable functions on \( SU(2) \) \[17\]:

\[
\mathcal{H} = \{ \psi(\alpha, \beta, \gamma) \in \mathcal{C} \mid \| \psi \|^2 = \frac{1}{16\pi^2} \int_0^{4\pi} d\gamma \int_0^{2\pi} d\alpha \int_0^{\pi} \sin \beta d\beta \ |\psi(\alpha, \beta, \gamma)|^2 < \infty \}. \tag{3.8}
\]

When convenient we write \( \psi(g) \) . . . instead of \( \psi(\alpha, \beta, \gamma) \). The left regular representation of \( SU(2) \) is given by unitary operators \( U(g'), g' \in SU(2) \), acting on \( \psi \) as

\[
(U(g')\psi)(g) = \psi (g^{-1}g'). \tag{3.9}
\]

Similarly the right regular representation is given by unitary operators \( \tilde{U}(g') \):

\[
(\tilde{U}(g')\psi)(g) = \psi (gg'). \tag{3.10}
\]

They obey

\[
U(g')U(g) = U(g'g), \]

\[
\tilde{U}(g')U(g) = U(g'g), \]

\[
\tilde{U}(g')U(g) = U(g)\tilde{U}(g'). \tag{3.11}
\]
The generators $J_r$ of $U(g)$ such that
\[ U(g(\alpha, \beta, \gamma)) = e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3} \] (3.12)
are
\[ J_1 = i \left( \cos \alpha \cot \beta \frac{\partial}{\partial \alpha} + \sin \alpha \frac{\partial}{\partial \beta} - \cos \alpha \frac{\partial}{\partial \gamma} \right), \]
\[ J_2 = i \left( \sin \alpha \cot \beta \frac{\partial}{\partial \alpha} - \cos \alpha \frac{\partial}{\partial \beta} - \sin \alpha \frac{\partial}{\partial \gamma} \right), \]
\[ J_3 = -i \frac{\partial}{\partial \alpha}. \] (3.13)
Similarly the generators $\tilde{J}_r$ of $\tilde{U}(g)$ are
\[ \tilde{J}_1 = i \left( -\cos \gamma \frac{\partial}{\partial \alpha} + \sin \gamma \frac{\partial}{\partial \beta} + \cos \gamma \cot \beta \frac{\partial}{\partial \gamma} \right), \]
\[ \tilde{J}_2 = i \left( \sin \gamma \frac{\partial}{\partial \alpha} + \cos \gamma \frac{\partial}{\partial \beta} - \sin \gamma \cot \beta \frac{\partial}{\partial \gamma} \right), \]
\[ \tilde{J}_3 = i \frac{\partial}{\partial \gamma}. \] (3.14)
The complete set of commutation relations among them is
\[ [J_r, J_s] = i \varepsilon_{rst} J_t, \]
\[ [\tilde{J}_r, \tilde{J}_s] = i \varepsilon_{rst} \tilde{J}_t, \]
\[ [J_r, \tilde{J}_s] = 0. \] (3.15)
Thus the left representation generators are right translation invariant and vice versa. As is well known these two sets of generators share a common Casimir invariant, and are related by the adjoint UIR of $SU(2)$, namely the defining representation of $SO(3)$:
\[ J^2 = J_r J_s = \tilde{J}_r \tilde{J}_s, \]
\[ \tilde{J}_r = -R_{sr}(\alpha, \beta, \gamma) J_s. \] (3.16)
Acting on $D_\alpha^j (\alpha, \beta, \gamma)$ we have:
\[ J_3 D_\alpha^j (\alpha, \beta, \gamma) = -m D_\alpha^j (\alpha, \beta, \gamma), \]
\[ \tilde{J}_3 D_\alpha^j (\alpha, \beta, \gamma) = n D_\alpha^j (\alpha, \beta, \gamma), \]
\[ J^2 D_\alpha^j (\alpha, \beta, \gamma) = j(j+1) D_\alpha^j (\alpha, \beta, \gamma). \] (3.17)
We now develop a method to extract the $SR$ of $SU(2)$ from the (left) regular representation, in a way which generalises to all $SU(n)$. The functions $(2j + 1)^{1/2} D_{mn}^j (\alpha, \beta, \gamma)$ for all $j, m, n$ form an orthonormal basis for $\mathcal{H}$ in which the two commuting UR’s $U(g), \tilde{U}(g)$ are simultaneously reduced into UIR’s. In the UR $U(g)$ each UIR $j$ of $SU(2)$ occurs $(2j+1)$ times, and the quantum number $n$, eigenvalue of $\tilde{J}_3$, acts as a multiplicity index. (Conversely, $m$ plays this role for the reduction of $\tilde{U}(g)$). We can then see that if we restrict ourselves to the subset of basis functions $D_{mj}^j (\alpha, \beta, \gamma)$ with maximum possible value $j$ for the eigenvalue $n$ of $\tilde{J}_3$, and to the subspace of $\mathcal{H}$ spanned by these functions, we pick up each UIR of $SU(2)$ exactly once from the reduction of $U(g)$. This leads to the identification of a subspace $\mathcal{H}_0 \subset \mathcal{H}$ by the definition
\[ \mathcal{H}_0 = \left\{ \psi(\alpha, \beta, \gamma) \in \mathcal{H} \mid \Bigl( \tilde{J}_1 + i \tilde{J}_2 \Bigr) \psi(\alpha, \beta, \gamma) = 0 \right\} \] (3.18)
(Strictly speaking, wave functions in the domain of and annihilated by $\tilde{J}_1 + i \tilde{J}_2$ form a dense set in $\mathcal{H}_0$, which upon completion gives $\mathcal{H}_0$). On the other hand we know in advance that
\[ \mathcal{H}_0 = \text{Sp} \left\{ (2j + 1)^{1/2} D_{mj}^j (\alpha, \beta, \gamma), \right. \]
\[ \left. j = 0, 1/2, 1, \ldots, m = j, j - 1, \ldots, -j \right\} \] (3.19)
The equivalence of (3.18) and (3.19) can be directly established as follows. The condition defining wave functions in $H_0$ reads

$$\left(\frac{i\partial}{\partial \gamma} - \tan \beta \frac{\partial}{\partial \beta} - \frac{i}{\cos \beta} \frac{\partial}{\partial \alpha}\right) \psi(\alpha, \beta, \gamma) = 0.$$  

(3.20)

This is a complex first order partial differential equation whereas $\alpha \beta \gamma$ are all real. Therefore we cannot conclude that $\psi(\alpha, \beta, \gamma)$ is effectively reduced to a function of two independent real combinations of $\alpha \beta \gamma$. Essentially, this is like imposing the Cauchy-Riemann equations - $\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) f(x, y) = 0$ - on a complex function of two real variables. The result is that $f(x, y)$ has to be an analytic function of the complex combination $z = x + iy$. Considering first combinations of $\alpha$ and $\beta$, and then of $\gamma$ and $\beta$, which obey (3.20), we find that $\psi(\alpha, \beta, \gamma)$ can be any analytic function of $e^{i\alpha} \tan \beta/2$ and $e^{-i\gamma} \sin \beta$. (The analyticity condition arises because the complex conjugate combinations $e^{-i\alpha} \tan \beta/2$, $e^{i\gamma} \sin \beta$ do not obey (3.20).) However this is equivalent to the statement that $\psi(\alpha, \beta, \gamma)$ must be an analytic function of $\xi, \eta$ of (3.3):

$$\psi \in H_0 \Leftrightarrow \psi(\alpha, \beta, \gamma) = f(\xi, \eta).$$  

(3.21)

On the other hand the functions $D^j_{m,j}(\alpha, \beta, \gamma)$ are known to be given by (16):

$$D^j_{m,j}(\alpha, \beta, \gamma) = \sqrt{2j}u_{jm}(\xi, \eta),$$

$$u_{jm}(\xi, \eta) = \xi^j \eta^{-m}/\sqrt{(j + m)! (j - m)!},$$

(3.22)

so the equivalence of (3.18) with (3.19) follows.

To cast the UIR’s present in $H_0$ into the standard forms of quantum angular momentum theory, we notice from (3.17) that the eigenvalue of $J_z$ is $-m$, and as a short calculation shows:

$$(J_1 + iJ_2)D^j_{m,j}(\alpha, \beta, \gamma) = -\sqrt{(j + m)(j - m + 1)} D^j_{m-1,j}(\alpha, \beta, \gamma).$$  

(3.23)

If we therefore define the family of wave functions

$$\mathcal{Y}_{jm}(\alpha, \beta, \gamma) = (-1)^j (-m) (2j + 1)^{1/2} D^j_{m,j}(\alpha, \beta, \gamma)$$

$$= \sqrt{(2j + 1)!} \eta^{j+m} (-\xi)^{j-m}/\sqrt{(j + m)! (j - m)!}$$

$$= \sqrt{(2j + 1)!} u_{jm}(\eta, -\xi),$$

$$j = 0, 1/2, 1, \ldots, \quad m = j, j-1, \ldots, -j,$$  

(3.24)

they form an orthonormal basis for $H_0$,

$$\frac{1}{16\pi^2} \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\alpha \int_0^{4\pi} d\gamma \mathcal{Y}_{jm}(\alpha, \beta, \gamma) \mathcal{Y}_{j'm'}(\alpha, \beta, \gamma)^* = \delta_{jj'} \delta_{mm'};$$  

(3.25)

and moreover for each fixed $j$, the $\mathcal{Y}_{jm}(\alpha, \beta, \gamma)$ transform under the left regular representation according to the standard form of the jth UIR of SU(2). The restriction of the left regular representation from $H$ to $H_0$ may be denoted by $D_0$, and it is a realisation of the SR of SU(2).

The following comments may be made concerning the specific way in which the carrier space above has been obtained. It is important to notice that each basis function $\mathcal{Y}_{jm}(\alpha, \beta, \gamma)$ retains a dependence on each of the three real independent arguments. This can be easily seen when verifying the orthonormality condition (3.25): doing the $\gamma$ integration first produces $\delta_{jj'}$, the $\alpha$ integration next produces $\delta_{mm'}$, while the final $\beta$ integration produces the correct normalisation. This is similar to the comments made earlier in connection with eqn. (3.7). This means that the extraction of the subspace $H_0$ within the space $H = L^2(SU(2))$ carrying the regular representations, since it involves limiting oneself to solutions of a complex differential equation, does not amount to limiting oneself to functions defined on a lower dimensional submanifold of the full ‘configuration space’ SU(2). In other words, the limitation to a subspace at the vector space level is not achieved by a limitation to any submanifold of the group manifold. This is similar to the relationships among the position, momentum and Bargmann representations of the Heisenberg canonical commutation relations in quantum mechanics. While the first two can be handled in the real realm via the concept of polarisation of a symplectic structure, the third brings in complex quantities in a novel manner.
Moreover, to further clarify the meaning of the functions \( \mathcal{Y}_{jm}(\alpha, \beta, \gamma) \), namely that they essentially depend on the three variables, and that obtaining the SR from the left regular representation does not require to quotient the group manifold, it is possible to study their relations with the properties of the generalised coherent states for the group \( SU(2) \). As it is well known \([18]\), if the fiducial vector in each finite dimensional UIR of \( SU(2) \) is chosen to be the highest weight in the Cartan-Weyl setting, then the coherent states are in correspondence with points of a 2-sphere \( S^2 \sim SU(2)/U(1) \), where, with the standard identification, \( \gamma \) has been quotiented away:

\[
\langle j, m | \alpha \beta \rangle = D^j_{mj}(\alpha, \beta, \gamma = 0)
\]

(3.26)

So that the functions \( \mathcal{Y}_{jm}(\alpha, \beta, \gamma) \) are, by a direct check:

\[
\mathcal{Y}_{jm}(\alpha, \beta, \gamma) = e^{-\gamma j} \langle j, m | \alpha \beta \rangle
\]

(3.27)

This shows, once more, that \( \mathcal{Y}_{jm} \) functions do depend on the three variables, so obtaining the SR from the left regular does not require to quotient the group manifold of \( SU(2) \).

Secondly in this carrier space each basis function is a single term expression, a monomial, rather than a sum of several distinct terms, which is the case for a general \( D^j_{mn}(\alpha, \beta, \gamma) \) and for the usual spherical harmonics on \( S^2 \). In the next Section we exploit these features to connect this form of the \( SU(2) \) SR to other known results.

### IV. APPLICATIONS OF SU(2) SCHWINGER REPRESENTATION

In this Section we use the construction of the previous Section to link up to the original Schwinger oscillator operator construction for \( SU(2) \), and to the Majorana theorem on the geometrical representation of pure states for a spin \( j \) system for any \( j \).

(a) The Schwinger Oscillator construction

The orthonormality relation \( (3.26) \) for the basis functions \( \mathcal{Y}_{jm}(\alpha, \beta, \gamma) \) of \( \mathcal{H}_0 \) can be exhibited in an alternative form suggesting interesting generalisation. Introduce two independent complex variables \( z_1, z_2 \) proportional to \( \eta, -\xi \):

\[
\begin{align*}
z_1 &= \rho \eta = \rho e^{i(\alpha-\gamma)/2} \sin \beta/2, \\
z_2 &= -\rho \xi = -\rho e^{-i(\alpha+\gamma)/2} \cos \beta/2, \\
|z_1|^2 + |z_2|^2 &= \rho^2, \quad 0 \leq \rho < \infty.
\end{align*}
\]

(4.1)

The uniform integration measure over the two complex planes is

\[
d^2z_1 d^2z_2 \equiv |z_1| |z_2| d|z_1| d|z_2| d\arg z_1 d\arg z_2 = \pi^2 \mathcal{g} \rho^2 d\rho^2,
\]

(4.2)

where \( \mathcal{g} \) is given in \((3.3)\). Then \((3.25)\) takes the form

\[
(2j + 1)! \int \int \frac{d^2z_1}{\pi} \frac{d^2z_2}{\pi} \delta(\rho^2 - 1) u_{jm}(z_1, z_2) u_{j'm'}(z_1, z_2)^* = \delta_{jj'} \delta_{mm'}.
\]

(4.3)

Remembering that the last two factors of the integrand are actually \( \rho \)-independent, and that the result on the right hand side really arises from the integration over \( SU(2) \) with measure \( \mathcal{g} \), we see that we can replace \( \delta(\rho^2 - 1) \) by any \( (\text{real positive}) \) function \( f_j(\rho^2) \) subject to

\[
\int_0^\infty d\rho^2 \rho^2 f_j(\rho^2) = 1,
\]

(4.4)

and then \((4.3)\) will remain valid in the form

\[
(2j + 1)! \int \int \frac{d^2z_1}{\pi} \frac{d^2z_2}{\pi} f_j(\rho^2)(\rho^2 - 2j) u_{jm}(z_1, z_2) u_{j'm'}(z_1, z_2)^* = \delta_{jj'} \delta_{mm'}.
\]

(4.5)

An easy and suggestive choice consistent with \((4.4)\) is

\[
f_j(\rho^2) = (\rho^2)^{2j} e^{-\rho^2}/(2j + 1)!,
\]

(4.6)
which leads to
\[
\int \int \frac{d^2z_1}{\pi} \frac{d^2z_2}{\pi} e^{-|z_1|^2-|z_2|^2} u_{jm}(z_1, z_2) u_{j'm'}(z_1, z_2)^* = \delta_{jj'} \delta_{mm'}.
\] (4.7)

This is recognised to be just the Bargmann entire function realisation of the Schwinger oscillator operator construction for $SU(2)$, with the familiar complete system of basis functions
\[
u_{jm}(z_1, z_2) = z_1^{j+m} z_2^{-m} / \sqrt{(j+m)!(j-m)!}
\] (4.8)
forming an orthonormal basis in the Bargmann Hilbert space $\mathcal{H}$. The oscillator operators $a_1$, $a_2$ correspond to multiplication by $z_1$, $z_2$, while the measure in $\mathcal{H}$ is such that $a_1$ and $a_2$ act as $\frac{\partial}{\partial z_1}$, $\frac{\partial}{\partial z_2}$ respectively.

It is in this way that the original Schwinger oscillator operator construction for $SU(2)$ can be recovered from the SR of $SU(2)$ in the form realised in the previous Section.

(b) The Majorana representation for spin $j$

It is very well known from the theory of the Poincaré-Bloch sphere that each pure state of a spin 1/2 system (two level quantum system) can be represented in a unique fashion by a point on $S^2$. Majorana’s theorem generalises this to pure states of a spin $j$ system for any $j$. We show how this result can be obtained immediately and transparently from the work of the previous Section.

The orthonormal basis functions for the spin $j$ UIR contained within the SR $\mathcal{D}_0$ of $SU(2)$, given in [10], are expressible in the form
\[
\mathcal{Y}_{jm}(\alpha, \beta, \gamma) = (-1)^{j-m}(2j+1)!/2 D^j_{-m,j}(\alpha, \beta, \gamma)
\]
\[
= \sqrt{(j+m)!(j-m)!} \left( e^{-i(\alpha+\gamma)/2} \cos \beta/2 \right)^{j-m} \left( e^{i(\alpha-\gamma)/2} \sin \beta/2 \right)^{j+m}
\]
\[
= \sqrt{(j+m)!(j-m)!} \cdot \xi^{2j} \cdot (-1)^{j-m} \zeta^{j+m},
\]
\[
\zeta = \frac{\eta}{\xi} = e^{i\alpha} \tan \beta/2.
\] (4.9)

The variable $\zeta$, which can take any value in the complex plane since $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \pi$, is the result of stereographic projection applied to the sphere $S^2$, with the South pole as vertex, and onto the plane tangent to $S^2$, at the North pole. Thus each $\zeta$ corresponds to a unique point on $S^2$, the North and South poles being mapped onto $\zeta = 0$ and $\infty$ respectively. A general vector $\psi$ within the spin $j$ UIR in $\mathcal{D}_0$ is thus of the form
\[
\psi = \sum_{m=-j}^{+j} C_m \mathcal{Y}_{jm}(\alpha, \beta, \gamma)
\]
\[
= \sqrt{(2j+1)!} \cdot \xi^{2j} \cdot \sum_{m=-j}^{+j} \frac{(-1)^{j-m}}{(j+m)!(j-m)!} C_m \zeta^{j+m}
\] (4.10)

As it stands, this wave function is a common standard format times a polynomial of degree $\leq 2j$ in the complex variable $\zeta$. In the generic case with all $C_m \neq 0$, we have a polynomial of degree $2j$, so $\psi$ can be uniquely factored into the form
\[
\psi = \sqrt{(2j+1)!} \cdot \xi^{2j} \cdot C_j \cdot (\zeta - \zeta_1)(\zeta - \zeta_2) \ldots (\zeta - \zeta_{2j}).
\] (4.11)

The (unordered) set of points $\zeta_1, \zeta_2, \ldots, \zeta_{2j}$ (some of which may coincide) corresponds to an (unordered) set of points on $S^2$, which set determines $\psi$ uniquely and vice versa (upto overall normalisation of $\psi$). This is the celebrated Majorana result obtained transparently from the way the SR of $SU(2)$ was constructed in Section III.

In particular the importance of each $\mathcal{Y}_{jm}(\alpha, \beta, \gamma)$ being a single term expression should be appreciated.

In the generic case above with all $C_m \neq 0$, none of the points $\zeta_1, \zeta_2, \ldots, \zeta_{2j}$ can either vanish or be infinite. In the most general case, if $m_1 \geq m_2$ are the largest and smallest $m$ values for which $C_m \neq 0$, i.e., $C_j = C_{j-1} = \ldots = C_{m_1+1} = 0, C_{m_1} \neq 0, \ldots, C_{m_2} \neq 0, C_{m_2-1} = C_{m_2-2} = \ldots = C_{-j} = 0$, the wave function $\psi$ has the form
\[
\psi = \sqrt{(2j+1)!} \cdot \xi^{2j} \cdot (-1)^{j-m_1} \frac{C_{m_1} z^{m_1-m_2}}{(2j+1)!(j+m_1)!(j-m_1)!}.
\]
The Euler angles now have the ranges $0 \leq R$ with coincidences permitted.

The identification of orthonormal basis functions transforming in the standard way under the left regular action by $D$ is (compare (3.15)). The expressions for their generators, $L$, reductions of both regular representations, are (2 relations too are repetitions of (3.15)). The complete set of orthonormal basis functions, realising the complete and $-\pi$, so the normalised volume element is

$$d R = \frac{1}{8\pi^2} d\alpha \sin \beta d\beta d\gamma.$$  
(5.2)

The left and right regular representations of $SO(3)$, denoted again by $\mathcal{H}$, is

$$\mathcal{H} = \left\{ \psi(\alpha, \beta, \gamma) \in \mathcal{C} \mid \|\psi\|^2 = \frac{1}{8\pi^2} \int_0^{2\pi} d\gamma \int_0^{2\pi} d\alpha \int_0^{\pi} \sin \beta d\beta \mid \psi(\alpha, \beta, \gamma)\mid^2 < \infty \right\}$$  
(5.3)

The left and right regular representations of $SO(3)$ are defined in ways analogous to (3.13,3.14) and need not be repeated. The expressions for their generators, $L_r$ and $L_l$, say, are the same as in (3.13,3.14), and the commutation relations too are repetitions of (3.15). The complete set of orthonormal basis functions, realising the complete reductions of both regular representations, are $(2\ell + 1)^{1/2} D_{m\ell}(\alpha, \beta, \gamma) : \ell = 0, 1, 2, \ldots, m$ and $n = \ell, \ell - 1, \ldots, -\ell$; and $-m, n$ are eigenvalues of $L_3, \tilde{L}_3$ respectively.

Following the same procedure as with $SU(2)$, we can isolate a subspace $\mathcal{H}_0 \subset \mathcal{H}$ carrying a realisation of the $SR D_0$ of $SO(3)$ by

$$\mathcal{H}_0 = \left\{ \psi(\alpha, \beta, \gamma) \in \mathcal{H} \mid \left( \tilde{L}_1 + i \tilde{L}_2 \right) \psi(\alpha, \beta, \gamma) = 0 \right\}$$

$$= \text{Sp} \left\{ (2\ell + 1)^{1/2} D_{m\ell}(\alpha, \beta, \gamma), \ell = 0, 1, 2, \ldots, m = \ell, \ell - 1, \ldots, -\ell \right\}$$  
(5.4)

The identification of orthonormal basis functions transforming in the standard way under the left regular action by $SO(3)$ is (compare (4.9)):

$$\chi_{\ell m}(\alpha, \beta, \gamma) = (-1)^{\ell-m}(2\ell + 1)^{1/2} D_{m\ell}(\alpha, \beta, \gamma)$$

$$= \sqrt{(2\ell + 1)! / (\ell + m)! (\ell - m)!} \left( e^{-i(\alpha + \gamma)} \cos^2(\beta/2) \right) \left( e^{i\alpha} \tan(\beta/2) \right)^{\ell + m}.$$  
(5.5)

The single term structure of these basis functions and the dependences on all three Euler angles should again be noted.
We have pointed out in Section II that the more familiar way of realising the SR of \(SO(3)\) is via the usual kinematical action of rotations on square integrable functions on \(S^2\), namely on functions \(\psi(\alpha, \beta)\) with spherical harmonics \(Y_{\ell m}(\beta, \alpha)\) as basis functions; and that this is the induced UR \(D_{SO(2)}^{\text{ind}(0)}\). While this realisation is fully equivalent in the sense of representation theory to the realisation given above, one sees that the actual carrier spaces and basis functions are quite different in the two cases. The realisation on \(L^2(S^2)\) is appropriate for discussing the orbital angular momentum of a spinless quantum mechanical particle; that developed in this Section is appropriate for describing the subset of states of a rigid body in quantum mechanics in which the third component of the angular momentum referred to body axes always has maximal value.

It is important to note that the Schwinger oscillator operator construction for the group \(SO(3)\) can be obtained from that of \(SU(2)\) outlined in the previous section.

Restricting the basis system in (4.8) to the set of even functions:

\[
u_{jm}(-z_1, -z_2) = u_{jm}(z_1, z_2)
\]

is equivalent to allow only for integer values of \(j\), so to define a space supporting a realisation of \(SO(3)\) Lie algebra in terms of oscillators. This means that the Schwinger oscillator construction for \(SU(2)\) goes through for \(SO(3)\).

## VI. THE SCHWINGER REPRESENTATION FOR \(SU(n)\)

We now show how the \(SU(2)\) procedure developed in Section III can be extended to the entire family of unitary unimodular groups \(SU(n)\). We begin with preliminaries about \(SU(n)\), then prove that for \(n \geq 4\) the SR of \(SU(n)\) cannot be obtained by the inducing construction from any UIR of the canonical \(SU(n-1)\) subgroup. We then sketch the generalisation of the \(SU(2)\) procedure to general \(SU(n)\), and give details in the \(SU(3)\) case.

In the so-called tensor notation the Lie algebra \(SU(n)\) of \(SU(n)\) consists of operators \(A^\lambda_{\mu}\), \(\lambda, \mu = 1, 2, \ldots, n\), obeying the commutation, conjugation and algebraic relations [21]:

\[
[A^\lambda_{\mu}, A^\rho_{\sigma}] = \delta^\lambda_{\rho} A^\lambda_{\sigma} - \delta^\rho_{\sigma} A^\rho_{\mu},
\]

\[
(A^\lambda_{\mu})^\dagger = A^\mu_{\lambda},
\]

\[
A^\lambda_{\lambda} = 0.
\]

The subset of commuting hermitian generators which can be assumed to be simultaneously diagonal in any UR of \(SU(n)\) may be taken to be (upto overall multiplicative factors):

\[
A^1_{-1} - A^2_{-2}, \ A^1_{1} + A^2_{-2} - 2A^3_{3}, \ldots,
\]

\[
A^1_{1} + A^2_{-2} + \ldots + A^{n-1}{_{n-1}} - (n-1)A^n_{-n} = -n A^n_{n}.
\]

Since \(SU(n)\) has rank \((n-1)\), there are \((n-1)\) fundamental UIR’s; a general UIR is obtained by forming the direct product of several copies of each fundamental UIR and then isolating the ‘largest’ irreducible piece. The fundamental UIR’s are the defining \(n\)-dimensional UIR consisting of \(n \times n\) unitary unimodular matrices, followed by antisymmetric tensor representations of successive ranks \(2, 3, \ldots, (n-1)\) over the defining UIR. For brevity denote the fundamental UIR of \(SU(n)\) given by antisymmetric tensors of rank \(p\) by \(\mathfrak{p}^{(n)}\), for \(p = 1, 2, \ldots, n-1\). Under complex conjugation we have

\[
\mathfrak{p}^{(n)*} = (n-p)^{(n)}.
\]

Then the reduction of each fundamental UIR under the canonical \(SU(n-1)\) subgroup is easily seen to have the two-term structure

\[
\mathfrak{p}^{(n)} = \mathfrak{p}^{(n-1)} \oplus (p-1)^{(n-1)}, \ p = 1, 2, \ldots, n-1.
\]

One sees from this that for \(n \geq 4\), there is no single UIR of \(SU(n-1)\) which occurs exactly once in each fundamental UIR of \(SU(n)\), hence also none which appears exactly once in each UIR of \(SU(n)\). For example, when \(n = 4\), we have in terms of dimensionalities \(1^{(4)} = 4, 2^{(4)} = 6, 3^{(4)} = 4^*\); their \(SU(3)\) contents are

\[
4 = 3 \oplus 1,
\]

\[
6 = 3^* \oplus 3,
\]

\[
4^* = 1 \oplus 3^*.
\]
where \( 1^{(3)} = 3, \ 2^{(3)} = 3^* \); and the statement made above is seen to be true. For the \( SU(3) \to SU(2) \) case we have in contrast

\[
\begin{align*}
3 &= 2 \oplus 1, \\
3^* &= 2 \oplus 1, \\
\end{align*}
\]

(6.6)

and in fact, as mentioned in Section II, each UIR of \( SU(3) \) does contain exactly one \( SU(2) \) invariant state. From the reciprocity theorem we conclude that for \( n \geq 4 \), the \( SR \) of \( SU(n) \) cannot be obtained by the inducing construction starting from any UIR of \( SU(n-1) \).

The method used for \( SU(2) \) in Section III, however, does work for all \( SU(n) \). The Hilbert space carrying the two commuting regular representations of \( SU(n) \) is \( \mathcal{H} = L^2(SU(n)) \):

\[
\mathcal{H} = \{ \psi(g) \in C|g \in SU(n), \| \psi \|^2 = \int dg|\psi(g)|^2 < \infty \}.
\]

(6.7)

Here \( dg \) is the normalised invariant volume element on \( SU(n) \), and the left and right regular representation operators \( U(g), \bar{U}(g) \) are defined exactly as in \( SU(3) \). Let us denote their generators by \( A_\lambda^\mu, \bar{A}_\lambda^\mu \): each set obeys eqns. (6.1), and they mutually commute. Then the subspace \( \mathcal{H}_0 \) supporting a \( SR \) \( \mathcal{D}_0 \) of \( SU(n) \) is identified by

\[
\mathcal{H}_0 = \left\{ \psi(g) \in \mathcal{H}|\bar{A}_\lambda^\mu \psi = 0, \ \lambda < \mu \right\} = \left\{ \psi(g) \in \mathcal{H}| \bar{A}_{\lambda+1}^{\lambda+1} \psi = 0, \ \lambda = 1, 2, \ldots, n-1 \right\}.
\]

(6.8)

Here we use the fact that the \( \frac{1}{2}n(n-1) \) nonhermitian operators \( \bar{A}_\lambda^\mu \) for \( \lambda < \mu \) close under commutation, so we can consistently look for their common null space. (In the defining UIR of \( SU(n) \), these are lower triangular matrices. Since \( \left[ \bar{A}_\lambda^\mu, \bar{A}_{\lambda+1}^{\lambda+1} \right] = \bar{A}_\lambda^{\lambda+1} \) etc., we can adopt the more economical definition in the second line of (6.8).

These conditions have the following effect: out of the many appearances of each \( SU(n) \) UIR in the reduction of the left regular representation \( U(g) \) on \( \mathcal{H} \), exactly one is picked up corresponding to the highest weight with respect to the right regular representation \( \bar{U}(g) \). Then the UR \( U(g) \) on \( \mathcal{H} \), when restricted to \( \mathcal{H}_0 \), gives a realisation of the \( SR \) \( \mathcal{D}_0 \) of \( SU(n) \).

We spell out the details in the \( SU(3) \) case \[22\]. The \( SU(2) \) subgroup is taken to be generated by \( A_2^1, A_2^1 - A_2^2 \). In the standard isospin notation we have:

\[
I_3 = A_2^1 - A_2^2, \ I_+ = \sqrt{2} \ A_2^1, \ I_- = \sqrt{2} \ A_2^1.
\]

(6.9)

A general \( SU(3) \) UIR is denoted by \( (p, q) \), with \( p \) and \( q \) independent nonnegative integers. \((1, 0) = 3 = \) defining representation, \((0, 1) = 3^* \). Within this UIR, whose dimension is \( N_{p,q} = \frac{1}{2}(p+1)(q+1)(p+q+2) \), an orthonormal basis is written as

\[
|p, q; I, I_3, Y \rangle,
\]

(6.10)

where \( I, I_3 \) are the usual \( SU(2) \) UIR quantum numbers, and the hypercharge \( Y \) is the eigenvalue of \(-A_3^3 \). The \('I-Y' multiplets'\) contained in the UIR \((p, q)\) are given by the rules:

\[
\begin{align*}
I &= \frac{1}{2}(r+s), \ I_3 = I, \ I - 1, \ldots, -I, \\
Y &= r - s + \frac{2}{3}(q-p), \\
r &= 0, 1, 2, \ldots, p, \quad s = 0, 1, 2, \ldots, q.
\end{align*}
\]

(6.11)

(Thus by taking \( r = s = 0 \) we see that an \( SU(2) \) singlet state with \( I = I_3 = 0 \) is always present once). The nonhermitian generators \( A_1^2, A_3^1, A_2^3 \) cause the following changes in the 'magnetic quantum numbers' \( I, I_3, Y \) of the basis states (6.10):

\[
\begin{align*}
A_1^2 &: I, I_3, Y \to I, I_3 + 1, Y, \\
A_3^1 &: I, I_3, Y \to I \pm 1/2, I_3 + 1/2, Y + 1, \\
A_2^3 &: I, I_3, Y \to I \pm 1/2, I_3 - 1/2, Y + 1.
\end{align*}
\]

(6.12)
Thus either $Y$ is increased by unity, or $Y$ is unchanged but $I_3$ is increased by unity. The unique basis state within $(p, q)$ annihilated by $A^1_{12}$ and $A^2_{3}$ (hence also by $A^1_{13}$) is then seen to be for $r = p, s = 0$:

$$|p, q; \frac{1}{2}p, \frac{1}{2}p, \frac{1}{3}(p + 2q)\rangle$$

(6.13)

With appropriate conventions this is the highest weight state in the UIR; it has the highest possible hypercharge value, and for this hypercharge it has the highest possible eigenvalue for $I_3$.

Now we use this information about UIR’s of $SU(3)$ to analyse the regular representations. These UR’s are realised on $L^2(SU(3))$, and an orthonormal basis is given in an obvious notation by the collection of all unitary representation matrices:

$$\sqrt{N_{p,q}} \ D_{I_{13} Y; I_{13} Y}^{(p,q)} (g).$$

(6.14)

The subspace $\mathcal{H}_0$ identified in (6.8) is thus seen to be spanned by those basis functions for which $\tilde{I} = I_3 = \frac{1}{2}p$, $\tilde{Y} = \frac{1}{3}(p + 2q)$:

$$\mathcal{H}_0 = \text{null space of } \tilde{A}_1 \ 2, \ \tilde{A}_2 \ 3 \ (\text{and } \tilde{A}_1 \ 3)$$

$$= \text{Sp} \left\{ \sqrt{N_{p,q}} \ D_{I_{13} Y; I_{13} Y}^{(p,q)} (g) \right\},$$

(6.15)

and we see explicitly that with respect to the left action each UIR of $SU(3)$ occurs exactly once. Thus the $SR \ D_0$ of $SU(3)$ is realised on $\mathcal{H}_0$.

To exhibit a basis $Y_{p,q; I_{13} Y} (g)$ for $\mathcal{H}_0$ which is orthonormal and transforms in the standard ‘Biedenharn’ manner under $SU(3)$ action, equations analogous to (7.2) have to be set up, but we omit the details.

VII. APPLICATION TO THE WIGNER-WEYL ISOMORPHISM

The Wigner-Weyl isomorphism (WWI) is a method to express states and operators in the traditional Hilbert space formulation of quantum mechanics in a classical phase space language [22]. Thus density matrices and general dynamical variables are represented by corresponding $c$-number functions on phase space, their Weyl symbols, while quantum mechanical expectation values are calculated as integrals of products of Weyl symbols over phase space in the manner of classical statistical mechanics. The WWI has been studied most extensively in the case of Cartesian quantum mechanics when, as mentioned in Section II, the configuration space is $Q = R^n$ and phase space is $R^{2n}$.

It has been shown elsewhere that if we consider the configuration space to be a (compact simple) Lie group $G$, the kinematical structure of quantum mechanics shows striking new features absent in the Cartesian case, so the WWI also exhibits unexpected features [21]. Interestingly the $SR$ of $G$ plays a role in this context, and this will be outlined here.

The Hilbert space of wave functions is in an obvious notation

$$\mathcal{H} = \left\{ \psi(g) \in C|g \in G, \| \psi \|^2 = \int_G dg |\psi(g)|^2 < \infty \right\}.$$  

(7.1)

The left and right regular UR’s act as in (5.10, 5.11, 5.14) reinterpreted as referring to $G$. A density operator $\hat{\rho}$ and a general dynamical variable $A$ are represented by their integral kernels

$$\hat{\rho} \rightarrow \langle g'|\hat{\rho}|g \rangle, \quad A \rightarrow \langle g'|\hat{A}|g \rangle.$$

(7.2)

where the ideal kets $|g >$ for $g \in G$ are introduced such that

$$\psi(g) = \langle g|\psi >,$$

$$\langle g'|g > = \delta(g^{-1}g'),$$

$$\int_G dg |g < g| = 1 \text{ on } \mathcal{H}.$$  

(7.3)

This allows us to express the actions of $U(g), \hat{U}(g)$ in the succinct forms

$$U(g)|g' >= |gg' >, \quad \hat{U}(g)|g' >= |g'g^{-1} >.$$

(7.4)
The trace orthonormality of these unitary operators is then immediate:

$$\text{Tr}(U(g')U(g)) = \text{Tr}(\hat{U}(g')\tilde{U}(g)) = \delta(g',g).$$  \hspace{1cm} (7.5)

The complementary ‘momentum’ basis for $\mathcal{H}$ in which both regular representations are simultaneously completely reduced into UIR’s is determined by the $D$-functions as

$$|jmn> = N_{j}^{1/2} \int dg \, D_{mm'}^{j}(g) \, |g> \hspace{1cm} (7.6)$$

with the basic properties

$$<j'm'n|jmn> = \delta_{ij'} \delta_{mm'} \delta_{nn'}, \hspace{1cm} U(g) |jmn> = \sum_{m'} D_{mm'}^{j}(g^{-1}) |jm'n>,$$

$$\tilde{U}(g) |jmn> = \sum_{n'} D_{n'n}^{j}(g) |jm'n>. \hspace{1cm} (7.7)$$

In the reduction of either regular representation each UIR $j$ of $G$ occurs $N_{j}$ times. In this basis $\hat{\rho}$ and $\hat{A}$ are represented by ‘matrices’

$$\hat{\rho} \rightarrow \langle j'm'n|\hat{\rho}|jmn>, \hspace{1cm} \hat{A} \rightarrow \langle j'm'n|\hat{A}|jmn>.$$  \hspace{1cm} (7.8)

In this scheme the WWI can be set up in two equally good ways. We describe both at this point even though only the second one will be used later.

**Option I**

With an operator $\hat{A}$ described by kernel [2] or matrix [3] we associate the Weyl symbol

$$W_{\hat{A}}(g; jmm') = \int \int dg' dg'' \langle g''|\hat{A}|g'\rangle \, D_{mm'}^{j}(g'g'') \delta(g^{-1}s(g',g''))$$

$$= \int \int dg' dg'' \langle g''|\tilde{U}(g)\hat{A}\tilde{U}(g)^{-1}|g'\rangle \, D_{mm'}^{j}(g'g'') \delta(s(g',g'')). \hspace{1cm} (7.9)$$

This symbol depends on a group element $g$ (coordinate variable) and on the discrete UIR labels $jmm'$ (momentum variable). It involves the function $s(g',g'') \in G$ dependent on two arguments, having the properties

$$s(g',g'') = s(g'',g'),$$
$$s(g',g') = g'',$$
$$s(g_{1}g'g_{2}, g_{1}g''g_{2}) = g_{1} s(g',g'') g_{2}. \hspace{1cm} (7.10)$$

A possible choice for $s(g',g'')$ is the ‘midpoint’ of the geodesic in $G$ from $g'$ to $g''$. Using [4] this solution can be written as

$$s(g',g'') = g' s_{0}(g'^{-1}g''), \hspace{1cm} (7.11)$$

where $s_{0}(g)$ is the ‘midpoint’ of the one-parameter subgroup connecting the identity $e \in G$ to $g$.

With this option we have under conjugation of $\hat{A}$ by $\tilde{U}, U$:

$$\hat{A}' = \tilde{U}(g_{1})\hat{A}\tilde{U}(g_{1})^{-1} \Rightarrow \hspace{1cm} W_{\hat{A}'}(g; jmm') = W_{\hat{A}}(gg_{1}; jmm');$$

$$\hat{A}'' = U(g_{2})^{-1}\hat{A} U(g_{2}) \Rightarrow \hspace{1cm} W_{\hat{A}''}(g; jmm') = \sum_{m_{1},m'_{1}} D_{mm_{1}}^{j}(g_{2}^{-1}) \, W_{\hat{A}}(g_{2}g_{1}; jm_{1}m'_{1}) \, D_{m_{1}m'_{1}}^{j}(g_{2}). \hspace{1cm} (7.12)$$

Finally for two operators $\hat{A}, \hat{B}$ on $\mathcal{H}$ we find:

$$\text{Tr} (\hat{A}\hat{B}) = \int dg \sum_{jmm'} N_{j} W_{\hat{A}}(g; jmm')W_{\hat{B}}(g; jm'm). \hspace{1cm} (7.13)$$
We now associate the Weyl symbol
\[ W_\hat{A}(g; jnn') = \int \int dg' dg'' \left( (\hat{A}|g') D_{n'n}^i(g'' g') \delta(g^{-1} s(g', g'')) \right) \]
\[ = \int \int dg' dg'' \left( (U(g)^{-1} \hat{A} U(g)|g') D_{n'n}^i(g'' g') \delta(s(g', g'')) \right). \]
(7.14)

Under conjugation of \( \hat{A} \) we now have:
\[ \hat{A}' = U(g_1) \hat{A} U(g_1)^{-1} \Rightarrow \]
\[ W_{\hat{A}'}(g; jnn') = \sum_{n_1, n_1'} D_{n_1 n_1'}^i(g_1^{-1}) W_{\hat{A}}(gg_1; jn_1 n_1') D_{n_1' n_1}^i(g_1); \]
\[ \hat{A}'' = U(g_2)^{-1} \hat{A} U(g_2) \Rightarrow \]
\[ W_{\hat{A}''}(g; jnn') = W_{\hat{A}}(g_2 g; jnn'). \]
(7.15)

For the trace over \( \mathcal{H} \):
\[ \text{Tr}(\hat{A} \hat{B}) = \int dg \sum_{jnn'} N_j W_{\hat{A}}(g; jnn') W_{\hat{B}}(g; jn'n). \]
(7.16)

We stress that \( \{7.4, 7.5, 7.6, 7.7\} \) hold with Option I, while \( \{7.8, 7.9, 7.10\} \) with Option II. The major differences are in the behaviours under conjugation of \( \hat{A} \).

Let us hereafter choose to work with Option II. The structure of the ‘momentum variables’ in \( W_{\hat{A}}(g; jnn') \) suggests that we bring in the \( SR \ \mathcal{D}_0(g) \) of \( G \) acting on \( \mathcal{H}_0 \), as set up in \( \{2.2, 2.3, 2.4, 2.5\} \). We can then represent the Weyl symbol of \( \hat{A} \) more compactly as simultaneously a function of \( g \) and a block diagonal operator on \( \mathcal{H}_0 \):
\[ \hat{A} \rightarrow W_{\hat{A}}(g; jnn') \rightarrow \hat{A}(g) = \sum_j \oplus \hat{A}_j(g), \]
\[ \hat{A}_j(g) = \sum_{n, n'} W_{\hat{A}}(g; jnn') |jn')(jn|. \]
(7.17)

Each \( \hat{A}_j(g) \) acts on the subspace \( \mathcal{H}^{(j)} \subset \mathcal{H}_0 \), and \( \hat{A}(g) \) acts in a block diagonal manner on \( \mathcal{H}_0 \). For two operators \( \hat{A} \) and \( \hat{B} \), traces within \( \mathcal{H}^{(j)} \) give
\[ \text{tr} \left( \hat{A}_j(g) \hat{B}_j(g) \right) = \sum_{n, n'} W_{\hat{A}}(g; jnn') W_{\hat{B}}(g; jn'n), \]
(7.18)
so the general trace formula (7.16) has the form
\[ \text{Tr}(\hat{A} \hat{B}) = \int dg \sum_j N_j \text{tr} \left( \hat{A}_j(g) \hat{B}_j(g) \right). \]
(7.19)

It is important to recognise that the trace operation on the right hand side is not over \( \mathcal{H}_0 \), because of the presence of the dimensionality factors \( N_j \). We come back to this point later.

We can now ask for the conditions on \( \hat{A} \) which make its Weyl symbol \( W_{\hat{A}}(g; jnn') \) independent of ‘coordinate’ \( g \) and dependent only on ‘momenta’ \( jnn' \) \( \{25\} \). From \( \{7.4\} \) we see that \( \hat{A} \) must belong to the commutant of the operators \( U(g) \) of the left regular representation. This means that it should be built up exclusively from the operators \( \hat{U}(g) \) of the right regular representation. After elementary calculations we can state this as a series of two-way implications:
\[ W_{\hat{A}}(g; jnn') = \text{independent of } g \Leftrightarrow \]
\[ U(g) \hat{A} = \hat{A} U(g), \text{ all } g \Leftrightarrow \]
\[ \langle g''| \hat{A}|g' \rangle = f(g''^{-1}g'), \text{ some } f \Leftrightarrow \]
\[ \hat{A} = \int dg \, f(g) \hat{U}(g) \Leftrightarrow \]
\[ \langle j'm'n'| \hat{A}|jmn \rangle = N_j^{-1/2} \delta_{jj'} \delta_{mm'} \int_{n'n}^j, \]
\[ f_{n'n}^j = N_j^{1/2} \int dg \, f(g) D_{n'n}^j(g), \]
\[ f(g) = \sum_{jmn'} N_j^{1/2} f_{n'n}^j D_{n'n}^j(g^{-1}). \] (7.20)

For such special operators \( \hat{A} \) we in fact find:
\[ W_{\hat{A}}(g; jmn') = N_j^{-1/2} f_{n'n}^j, \]
\[ \langle j'm'n'| \hat{A}|jmn \rangle = \delta_{jj'} \delta_{mm'} W_{\hat{A}}(\cdot; jmn'). \] (7.21)

When the Weyl symbol of such an \( \hat{A} \) is represented as a block diagonal operator on \( \mathcal{H}_0 \) according to (7.17), we have:
\[ \hat{A} = \int dg \, f(g) \hat{U}(g) \Leftrightarrow \]
\[ \hat{A}(g) = g - \text{ independent} = \int dg' \, f(g') \mathcal{D}_0(g'). \] (7.22)

Therefore when \( \hat{A} \) on \( \mathcal{H} \) is built up exclusively from the operators of the right regular representation \( \hat{U}(g) \), its Weyl symbol is the corresponding operator, in the sense of (7.22), in the \( SR \) of \( G \), stripping away the degeneracy of the regular representation. At the generator level we can say that if \( \hat{A} \) is a function only of the generators \( J_r \) of \( \hat{U}(g) \), then \( \hat{A} \) is identically the same function of the generators of the \( SR \) \( \mathcal{D}_0 \) on \( \mathcal{H}_0 \). The block diagonality of \( \hat{A} \) is of course assured.

This shows the important role of the \( SR \) in the WWI for quantum mechanics on a (compact simple) Lie group.

We return to the comment made after (7.19) and ask whether the definition of \( \hat{A}_j(g) \) for given \( \hat{A} \) could have been altered so as to absorb the factors \( N_j \) appearing on the right in that equation. In that case that right hand side would be expressible in terms of a trace over \( \mathcal{H}_0 \), which would make that relation more attractive. However a careful analysis shows that in that case the simplicity of the correspondence (7.22) would be lost, and therewith the direct relevance of the \( SR \). Therefore to secure (7.22) we have to retain (7.19) as it stands. Ultimately this situation can be traced to the following source. While the way in which the delta function in the trace relation (7.19) appears is extremely elementary, when we express it as in (7.19) in terms of the irreducible representation matrices of \( G \) the dimensionality factors \( N_j \) are essential.

\section*{VIII. CONCLUDING COMMENTS}

The method by which the \( SR \) has been isolated within the regular representation in the case of the group \( SU(n) \) readily generalises to all the other compact simple Lie group families, namely \( SO(2n) \), \( SO(2n+1) \), \( USp(2n) \) and even the five exceptional groups. This is because in each case the concept of highest weight in each \( UIR \) is unambiguously defined, and moreover the Lie algebra can be exhibited in the Cartan form, made up of ‘shift’ or ‘raising’ and ‘lowering’ generators in the directions of the distinct root vectors. An interesting question is how to effect a similar extraction of the \( SR \) from the regular representation in the case of finite groups, say the permutation groups \( S_N \). This presents interesting algebraic problems as generators, shifts along root vectors etc. are no longer available. The construction of the Schwinger representation for the permutation groups \( S_n \) has attracted attention in the mathematical literature: see, for instance [20].

Two other general questions suggest themselves bearing in mind the basic properties of the \( SR \) : simple reducibility and completeness: How are these properties reflected in the ‘classical limit’, can one give some differential-geometric or manifold-theoretic characterisations at the level of the coadjoint orbit space of the Lie group? If one next takes the direct product of the SR with itself, the simple reducibility aspect is likely to change, yet one can ask if any
simplifying features remain. We hope to return to some of these questions elsewhere.

References and footnotes


[13] In the work on ‘model representations’, ref.[10], the inducing construction does lead to all such representations for the noncompact groups considered.


[15] It is to be noted that in J.Schwinger, ref.[1], eqn.(2.61), the ranges chosen are $0 \leq \alpha, \gamma \leq 4\pi$, $0 \leq \beta \leq \pi$, which amounts to covering SU(2) twice. In ref.[12], the ranges chosen are $0 \leq \alpha, \gamma \leq 2\pi$, $0 \leq \beta \leq \pi$ and $2\pi \leq \beta \leq 3\pi$.

[16] Ref. [1], p.45 - 47.

[17] For the following details, see Ref.[12], p.57-65.


[25] This leads to interesting consequences and structures which are completely absent in the Cartesian case.